Remarks on Monte Carlo Method in Simulating Financial Problems

Oyelami Benjamin Oyediran
National Mathematical Centre, PMB 118 Garki PO Abuja, Nigeria
* E-mail of the corresponding author: boyelami2000@yahoo.com

Abstract

In this paper, an exposition is made on the use of Monto Carlo method in simulation of financial problems. Some selected problems in financial economics such as pricing of plain vanilla options driven by continuous and jump stochastic processes are simulated and results obtained.

Key words and Phrases: Monte Carlo Method, models, Simulation and Stochastic Differential Equations.

1. Introduction

Von Neumann gave the code name Monte Carlo for his military project at Los Almos. The city of Monte Carlo is known for gambling (roulette wheel) activities. Monte Carlo (MC) methods, to be precise, involve all the statistical simulation methods for obtaining numerical solution of problems in probability theory and other areas. Simulation can be said to be imitation of events in prescribed stochastic (random) experiment, therefore, MC simulation implies simulation that are being influenced by random events(Francis,1987) .

Monte Carlo methods are conceptual integration tool very useful for finding numerical approximations to higher dimensional integrals.Its elegance in Finance hinges on the fact that many financial problems can be expressed in integral or expectation forms, hence, we can easily design MC estimators for them.

Monte Carlo methods were first introduced to finance in 1964 by David B. Hertz through his article discussing their application in Corporate Finance. Phelim Boyle pioneered the use of simulation in derivative valuation in his seminar paper published in the Journal of Financial Economics 1977. In the Finance and financial engineering MC are applied for pricing derivatives, deriving risk-neutral probabilities and replicating strategies like hedging, estimation of loss probability and value at risk, usefulness in evaluating Greeks, variance reduction, Antithetic paths and Control variate method and so on (Boyle,1977;Deffie,1995;Francis,1987;Hammersly,1960 & Jeremy,2002).

The long held assumptions that risk assessment metrics and risk premiums don't change much have been dramatically challenged. The global meltdown of financial markets and the real economy in 2008 and 2009 has shown that risk premiums can change quickly in both developed and emerging markets .A Monte Carlo simulation analysis uses several thousands of iterations but may be perceived as being more analytically robust. However, invalid statistical assumptions and distributions can create invalid results, leading to inappropriate results. The potential disadvantage of the method is the computational burden; very large numbers of sample paths are needed to be generated to have accurate results

Furthermore, it is advisable to have a firm understanding of the concept of uncertainty and statistical analysis before attempting Monte Carlo simulation for financial modeling and analysis

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However, the Monte Carlo Simulation methods that are used in the field of finance are quite hard to use with respect to American Options. This is because the Monte Carlo Simulation technique may be estimated with respect to the option value that arises for the starting point only. But in many cases relating to American Options, the option value at the various intermediary times are also required, which lies between the starting time and end time of Monte Carlo Simulation. The simulation do not run backward from the

expiry times of various options and thus, getting information by way of Monte Carlo becomes hard to obtain (Jeremy, 2002; Merton, 1976).

Advantages of MC methods over numerical solutions of partial differential equations for pricing of derivative securities are that PDE models may poise difficulty if the asset price dynamics are sufficiently complex to solve or the solution do not exist as in some cases; if payoff of the security is path dependant not on terminal dates and for multi- factors PDE models the numerical solutions are often difficult to evaluate.

There are two sources of error in the generation of sample paths for MC methods and these are sampling error and discretization error .The sampling error is due to random nature in MC method and can be reduced by variance reduction method, while from discretization approach, the Euler method increases the number of sample paths (replication) and reduces the discretization error (David, 1979; Jeremy, 2002 & Paul, 2004).

1. Statement of the Problem and Preliminary(Notes)

The following notations would be used throughout this paper:

We will call the probability system $(\Omega, \sum, F, p)$ a .filtered probability space Ω being the set of points with events, which is a $\delta -$ algebra of subsets of Ω such that $\Omega \varepsilon F$ and p denotes the probability measure. The .filtration $F_t \subset F$ is the information available up to time t such that if s < t then $F_s \subset F_t$ .We also denote the expectation of a random variable x by E[x] and the norm of the vector y by $|y|$ and make use of the standard Euclidean norm $|y| = \left(\sum_{i=1}^{n} y_i^{2} \right)^{1/2}$ .N (0; 1) is the standard normal variate and w (t) the standard Brownian process.

Definition 1

We say that a discretization X has strong order of convergence $\beta > 0$ if

$$E\left[| X(nh) - X(T)|\right] \le ch^{\beta}$$

For some constant c and sufficiently small h. X has weak order of convergence β if

$$|\left[Ef(X(nh))\right] - E[f(X(T))\right] \le ch^{\beta}$$

For some constant c all sufficiently small h, for all f in a set

$$C_p^{2\beta+2} = \left\{ g : g \varepsilon C^{(2\beta+2)}, | g(x) | \le (1 + |x|^{q}) \right\}$$

For some constants k and q and $x \varepsilon R^{d}$.

A derivative is contract between two persons or parties that defines rights and obligations of the persons or parties. Options are agreement where one party or person pays the other for the right to something in future. Stock option where one party pays for the right to buy a stock at future date at a pre specify price. If the right can be excised at expiration date (maturity period) the option is said to be European option. if option is called in between the starting date of the contract and the maturity date the option is said to be American option. Options that are traded in standard organized market are called Plain Vanilla options. Options traded "out of counter" in a nonstandard markets are called Exotic Options.

Now consider a random sample X={X_1, X_2, X_3…X_n} of size n which are independent, identically distributed and the statistical average $X_n = \frac{\sum_{i=1}^{n} x_i}{n}$ such that $E(x_i) = X$ and $\text{var}(x_i) = \sigma_X^2 < \infty$, then by strong law of large numbers $X_n \to \hat{X}$ as $n \to \infty$ with probability 1.

Therefore, MC requires that $E(X_n) \to \widehat{X} \pm Z_{\delta/2} \frac{S_X}{\sqrt{n}}$ where S_X is the sample standard deviation, $1-\alpha$ quartile and $z_{\delta/2}$ belongs to the standard normal distribution, $N(0,1)$ .The rate of convergence of MC is of order $n^{-\frac{1}{2}}$ and does not depend on the dimension of the problem being solved. It also provides the confidence interval for the solution for which it is being use for (Oyelami,2011a,b).

Most probabilistic financial modeling software are based on Monte Carlo analysis, a risk analysis tool used for managing uncertainty .Many financial simulation packages today contain Financial Tools with Monte Carlo Simulation facilities but our discussion will not be on them. We will also discuss the underlying principle of MC with particular emphasis on European Style of Pricing of Options. The basic assumptions often used for simulating financial models using MC methods are:

2.1 Basic Assumptions

1. Derivative Security can be perfectly replicated (Hedge);
2. Prices are expectation of discounted payoffs under Martingale measure;
3. The market is complete, no transaction costs are charged.

The system protocol for designing MC methods is outlined in the .figure 1.

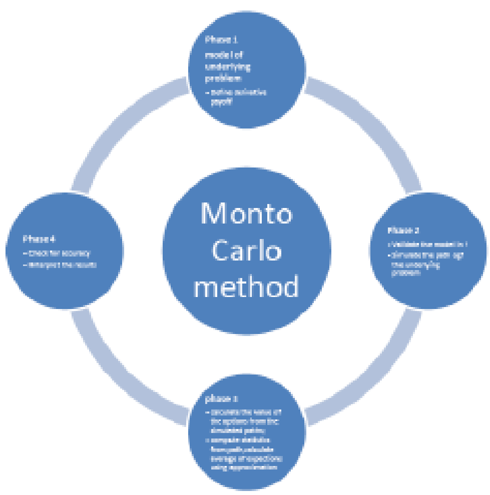


Figure 1: Schematic diagram of Monte Carlo Simulation

The fundamental equation for pricing of asset under arbitrage free assumption is the following stochastic differential equation

$$dS(t) = rS(t) + \sigma dw(t) \qquad (1)$$

Where r = mean rate of return (interest rate) σ is the volatility of the stock, K is strike or exercise price, T is the maturity or expiration date and $w(T) \sim N(0,1)$.

The pay off for the model is

$$E(e^{-rT}\max(S(T)-K,0))$$

This can be obtained using MC method. Pricing of Stocks using Black Scholes equation or the binomial model etc., the underlying pricing mechanism of MC is to generate the solution to the model using the following algorithm (Paul, 2004):

1.2 Algorithm 1

For i=1, 2….n generate $Z_i \sim N(0,1)$

Set: $S_i(T) = S(0)\exp[(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_i]$ (2)

Set: $C_i = e^{-rT}\max(S_i(T) - K, 0)$

Set $\tilde{C}_i = \frac{\sum C_i}{n}$

For any $n \geq 1$ the estimator $\tilde{C}_i$ is unbiased and by the strong law of numbers $\hat{C}_i \to \hat{C}$ as n $\to \infty$ with probability 1.

Therefore, the Call Price is $C \pm z_{\delta/2}\frac{S_C}{\sqrt{n}}$ Where S_C is the sample standard deviation for the sample $C=\{C_1, C_2, C_3, \ldots, C_n\}$ and N(0,1) is the standard normal distribution. The parity equation can be used to obtain the Put price by replacing C_i by $P_i = e^{-rt}\max(K - S_i(T), 0)$.

The Monte Carlo Method approach takes N number of trials as input, where N could be 1,000 to 1,000,000 large depending on the accuracy required for the result. The method of reduction of variance can be used to speed up the simulation time (run time)

The pseudo code for one individual experiment is shown below:

// A Sequential Monte Carlo simulation of the Black-Scholes model

1. For i=0 to N-1 do
2. $t := S * \exp(r - 0.5 * \sigma \wedge 2) * T + \sqrt{T} * randomNumber()), T \geq t$ is temporary value
3. $trial[i] := \exp(-r * T) * \max\{t - E, 0\}$
4. End for
5. $mean := mean(trial)$
6. $stddev := stdev(trial, mean)$

Where

S: asset value function

E: exercise price

r: continuously compounded interest rate

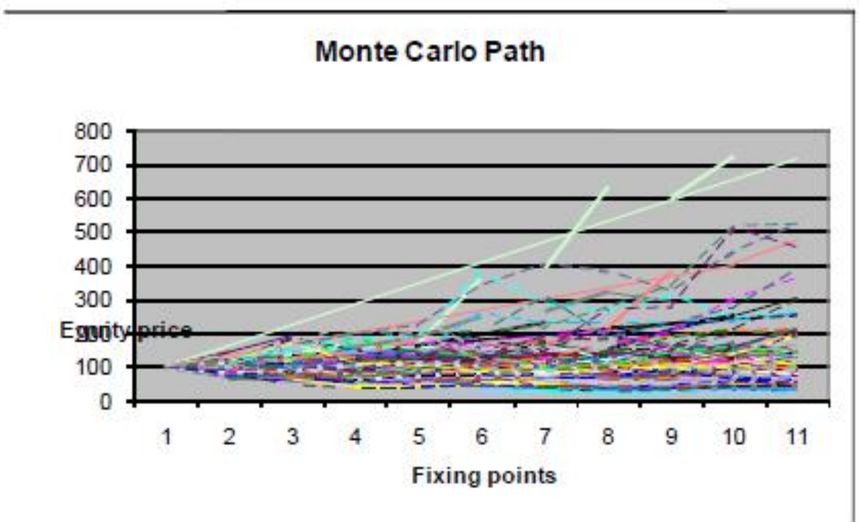
σ : Volatility of the asset

T : expiry time

N : number of trials

3. Pricing of assets which are Path Dependent

If an asset depends not only on the initial and maturity dates but also on intermediate dates, e.g. Asian option it is said to be path dependent (Jeremy,2002;Paulo;2006 & Paul,2004).For Path dependent which considers averaging of the stock at various



maturities, the MC can be modeled by setting $0 < t_0 < t_1 < t_2 < ... < t_n = T$,replace $S_i(T)$ in eq. (1) by the Euler iteration scheme as

$$S(t_{j+1}) = S(t_j)\exp([r - \frac{1}{2}\delta^2](t_{j-1} - t_j) + \sigma Z_j\sqrt{(t_{j-1} - t_j)}, Z_j \sim N(0,1), j = 1,2,3,....n \qquad (3)$$

And repeat the MC algorithm. The error bound for this Euler iteration scheme see [9] is

$$E[|S(nh) - S(T)|] \leq ch^{-\frac{1}{2}} .$$

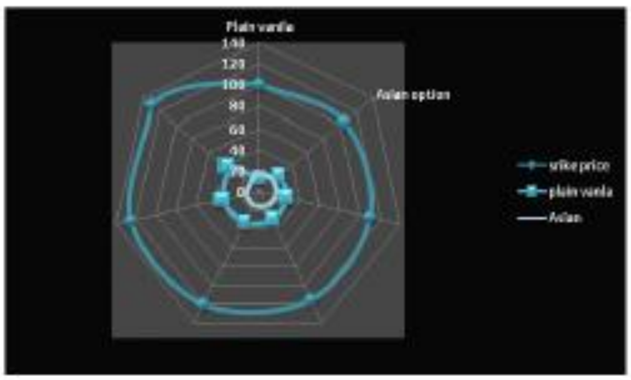
c>0, k and q are some constants such that $|S(T)| \leq k(1 + |T|^q)$ .We implement the MC algorithm for Plain vanilla, constant rate r = 0:03; volatility σ = 0:35; maturity T = 5; Strike price K = 100 naira.

Asian option constant rate r = 0:02; volatility σ = 0:40; maturity T =2;Strike price K = 100 naira. Without loss of generality assume that the asset pays no dividend for simplicity sake.

We obtained the following result in Table 1:

Strike price	Plain vanilla Put Price	Asian option put price
100	12.8050	18.8282
105	24.3944	17.1643
110	27.0408	16.6215
115	29.7814	15.1853
120	32.6109	13.8772
125	35.5236	12.6861
130	38.5144	11.6017

Table 1: Simulation of Put prices of Plain vanilla and Asian option



2.2 Pricing of assets with jumps

Merton introduced the jump-diffusion model for pricing derivative securities (Merton,1976) .Other jump models are useful in modeling the dynamics of market indices such as exchange rates, commodities prices and interest rates. Pricing a European call option with jump models are often assume that the asset can jump from positive to negative value or zero(Paulo,2006;Paul,2004;Steven,2004& Sato,1999) .The stochastic differential equation describing assets with jumps is

$$dS(t) = (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dw(t) + S(t)dQ(t) \qquad (4)$$

Where S(t) is stock price at time t, α is the mean rate of return, S(t_) accounts for the jump at time t_ which is continuous at the right.

$$N(t) = \sum_{m=1}^{M} N_m(t),$$

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t) = \sum_{m=1}^{N(t)} Y_m$$

$N(t)$ is Poisson process with intensity $\lambda = \sum_{i=1}^{m} \lambda_i$ and Q (t) is compound Poisson process.

Suppose $\{Y_1, Y_2, \ldots, Y_n\}$ is a sample which is independent and identically distributed with probability

$$P\{Y_1 = y_n\} = P(y_n) = \frac{\lambda_m}{\lambda,}$$

And

$$\beta = E(Y_1) = \frac{\sum \lambda_i}{\lambda}$$

And let

$$S_i(t_{j+1}) = S_i(t_j)\exp\{(\sigma Z_j \sqrt{t_j - t_{j-1}} - \alpha - \beta\lambda - \tfrac{1}{2}\sigma^2)t_j\}\prod_{i=1}^{N(t)}(1 + Y_i) \qquad (5)$$

$Z_j \sim N(0,1),\ j = 1,2,3,\ldots,n$. This model is very useful in interest rate modeling with jump behavior.

Replace C_i by $S_i(t_{j+1})$ in Algorithm 1 to generate the Sample Path for the MC then when n become very large the result obtained for the MC for the solution to equation (5) will be very accurate, Alternatively,

We can simulate equation (5) by setting $X(t) = \log S(t)$ to get

$$X(t_{k+1}) = X(t_k) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)(t_{k+1} - t_{k+1})$$
$$+ \sigma(z(t_{k+1}) - z(t_k)) + \sum_{j=N(t_k)}^{N(t_{k+1})} \log Y_i$$

We can simulate the jump times from $S(\tau_{k+1}) = S(\tau_k -)Y_{k+1}$

3. Conclusion

Monte Carlo methods are essential tools for simulating several .financial problems such as pricing of derivatives and in risk management. There are the Quasi-Monte Carlo(QMC) methods which are MC methods obtained generation of random numbers using various effective methods which are discussed in this paper. Both the MC and QMC offer powerful tools for simulation of several .financial problems and further exploration on their applications in .finance should be exploited.

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Professor Benjamin Oyediran Oyelami was born in Jos in Nigeria on 11th May 1964..He attended University of Jos, Jos, Nigeria (Bsc,Mathematics,1988),University of Ibadan, Ibadan, Nigeria(Msc,Mathematics,1991) and obtained PhD in Mathematics from Abubakar Tafawa Balewa University of Technology Bauchi ,Nigeria in 1999. A foreign member of the American Mathematical Society. He was invited speaker in many International and National Conferences. Author of substantial articles including monographs and co-editor of some Conference Proceedings. Co developer of B-transform method .Current Research interests, Modelling and Simulation of Impulsive Systems and Financial problems