

Common Fixed Point Theorems for Meir -Keeler Type Contraction Condition

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Abstract: - In this article is to prove some more common fixed point theorem in complete metric space by using Meir-Keeler type contractive conditions.

Keywords- Fixed point, common fixed point, Meir-Keeler type contractive mapping.

1. INTRODUCTION AND PRELIMINARIES

It is a fact that the fixed point theory has applications not only in many areas of Mathematics but also in many branches of quantitative sciences such as Economics and Computer Sciences. The most famous result in this field is known as the Banach Contraction Principle [3] which states that each contraction T on a complete metric space (X, d) has a unique fixed point. Here d denotes a given metric on X. A self mapping $T: X \to X$ is called a contraction if there exist a constant $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y).$$

In the literature one of the elegant generalizations of the Banach Contraction Principal is called the Meir-Keeler contraction principal [6]. Meir-keeler contraction has many extensions studied by many authors in the area (see[1,2,5,7,8]). In this paper we prove some more common fixed point theorem satisfying Meir-Keeler type contractive conditions. Infact we prove following common fixed point theorem.

Theorem 1. Let T be a self -mappings on a complete metric space X.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$\varepsilon < M(x,y) < \varepsilon + \delta$ implies $d(Su, Tv) \leq \varepsilon$ For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$ (1)

Theorem 2.1: Let S, T be self- mappings of a metric space (X, d) such that given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X, (x \neq y)$

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } d(Su, Tv) \leq \varepsilon$$

and

$$d(Su, Tv) \leq \alpha \max \left\{ \begin{matrix} \frac{d^2(u,Sw)+d^2(u,v)}{1+d(u,Sw)+d(u,v)}, \frac{d^2(v,Tt)+d^2(Sw,Tt)}{1+d(v,Tt)+d(Sw,Tt)}, \\ \alpha_3\sqrt{d(v,Sw).d(u,Tt)}, d(u,v) \end{matrix} \right\} \quad 2.1(i)$$

For all $u, v, w, t \in X$ and where $0 \leq \alpha < 1$ then S, T have a unique common fixed point.

Proof: Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows

$$Sx_{n-1} = x_n, Tx_n = x_{n+1}, Sx_{n+1} = x_{n+2}, Tx_{n+2} = x_{n+3}, \ldots$$

and $TSx_{n-1} = x_{n+1}, STx_n = x_{n+2}, TSx_{n+1} = x_{n+3}, STx_{n+2} = x_{n+4}, \ldots$

Where $n = 1, 2, 3, \ldots$

Now putting $u = Ty, v = Sx, w = x$ and $t = y$ in 2.1(i) then we have

$$d(STy, TSx) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(Ty,Sx)+d^2(Ty,Sx)}{1+d(Ty,Sx)+d(Ty,Sx)}, \\ \frac{d^2(Sx,Ty)+d^2(Sx,Ty)}{1+d(Sx,Ty)+d(Sx,Ty)}, \\ \sqrt{d(Sx,Sx).d(Ty,Ty)}, d(Ty,Sx) \end{array} \right\}$$

$$d(STy, TSx) \leq \alpha d(Sx, Ty) \qquad 2.1(ii)$$

Now putting $x = x_{n-1}$ and $y = x_n$ in 2.1(ii) then we have

$$d(STx_n, TSx_{n-1}) \leq \alpha \max \left\{ \begin{array}{c} d(Sx_{n-1}, Tx_n), \\ d(Sx_{n-1}, Tx_n), d(Sx_{n-1}, Tx_n) \end{array} \right\}$$

$$d(x_{n+2}, x_{n+1}) \leq \alpha d(x_n, x_{n+1}) \qquad 2.1(iii)$$

From 2.1(iii) we conclude that $d(x_{n-1}, x_n)$ decreases with n.

i.e., $d(x_{n-1}, x_n) \to d(x_0, x_1)$ when $n \to \infty$

If possible let $d(x_0, x_1) > 0$ and taking limit $n \to \infty$ on 2.1(iii) then we have

$$d(x_0, x_1) \leq \alpha d(x_0, x_1)$$

Which is not possible hence $d(x_0, x_1) = 0$

Next we shall show that $\{x_n\}$ is Cauchy sequence.

Now $\qquad d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n)$

$$\Rightarrow d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + d(Sx_n, Tx_m) \qquad 2.1(iv)$$

By putting $u = x_n, v = x_m, w = x_{m-1}, t = x_{n-1}$ in 2.1(i) then we have

$$d(Sx_n, Tx_m) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(x_n,Sx_{m-1})+d^2(x_n,x_m)}{1+d(x_n,Sx_{m-1})+d(x_n,x_m)}, \\ \frac{d^2(x_m,Tx_{n-1})+d^2(Sx_{m-1},Tx_{n-1})}{1+d(x_m,Tx_{n-1})+d(Sx_{m-1},Tx_{n-1})}, \\ \sqrt{d(x_m,Sx_{m-1}).d(x_n,Tx_{n-1})}, \\ d(x_n,x_m) \end{array} \right\}$$

$$= \alpha \max \left\{ \begin{array}{c} \frac{d^2(x_n,x_m)+d^2(x_n,x_m)}{1+d(x_n,x_m)+d(x_n,x_m)}, \\ \frac{d(x_m,x_n)+d(x_m,x_n)}{1+d^2(x_m,x_n)+d^2(x_m,x_n)}, \\ \sqrt{d(x_m,x_m).d(x_n,x_n)}, \\ d(x_n,x_m) \end{array} \right\}$$

$$= \alpha d(x_n, x_m)$$

$$d(Sx_n, Tx_m) \leq \alpha d(x_n, x_m) \qquad 2.1(v)$$

From 2.1(iv) and 2.1(v) we have

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + \alpha d(x_n, x_m)$$

Letting $m, n \to \infty$ then $d(x_n, x_m) \to 0$ as $2\alpha_1 + 2\alpha_2 + \alpha_4 < 1$

Hence $\{x_n\}$ is a Cauchy sequence.

Now we prove z is a common fixed point of S, T.

By putting $u = z$, $v = x_{n-1}$, $w = z$ and $t = x_{n-2}$ in 2.1(i) we have

$$d(Sz, Tx_{n-1}) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,Sz)+d^2(z,x_{n-1})}{1+d(z,Sz)+d(z,x_{n-1})}, \frac{d^2(x_{n-1},Tx_{n-2})+d^2(Sz,Tx_{n-2})}{1+d(x_{n-1},Tx_{n-2})+d(Sz,Tx_{n-2})}, \\ \sqrt{d(x_{n-1}, Sz).\, d(z, Tx_{n-2})}, d(z, x_{n-1}) \end{array} \right\}$$

$$d(Sz, x_n) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,Sz)+d^2(z,x_{n-1})}{1+d(z,Sz)+d(z,x_{n-1})}, \frac{d^2(x_{n-1},x_{n-1})+d^2(Sz,x_{n-1})}{1+d(x_{n-1},x_{n-1})+d(Sz,x_{n-1})}, \\ \sqrt{d(x_{n-1}, Sz).\, d(z, x_{n-1})}, d(z, x_{n-1}) \end{array} \right\}$$

Letting $n \to \infty$ then we have

$$d(Sz, z) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,Sz)+d^2(z,z)}{1+d(z,Sz)+d(z,z)}, \frac{d^2(z,z)+d^2(Sz,z)}{1+d(z,z)+d(Sz,z)}, \\ \sqrt{d(z, Sz).\, d(z, z)}, d(z, z) \end{array} \right\}$$

$$d(Sz, z) \leq \alpha\, d(Sz, z)$$

$$d(Sz, z) < d(Sz, z)$$

Since $\alpha < 1$ which gives $d(Sz, z) = 0 \Rightarrow Sz = z$

Thus z is a fixed point of S.

Similarly we can show that z is a fixed point of T.

Hence z is a common fixed point of S, T.

We are taking an another point q which is not equal to z such that

$$Sq = q = Tq$$

By putting $u = z, v = q, w = q, t = z$ in 2.1(i) then we have

$$d(Sz, Tq) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,Sq)+d^2(z,q)}{1+d(z,Sq)+d(z,q)}, \frac{d^2(q,Tz)+d^2(Sq,Tz)}{1+d(q,Tz)+d(Sq,Tz)}, \\ \sqrt{d(q, Sq).\, d(z, Tz)}, d(z, q) \end{array} \right\}$$

$$d(z, q) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,q)+d^2(z,q)}{1+d(z,q)+d(z,q)}, \frac{d^2(q,z)+d^2(q,z)}{1+d(q,z)+d(q,z)}, \\ \sqrt{d(q, q).\, d(z, z)}, d(z, q) \end{array} \right\}$$

$$d(z,q) \le \alpha d(z,q)$$

$d(z,q) < d(z,q)$ Since $\alpha < 1$

Which gives $d(z,q) = 0 \Rightarrow z = q$

Hence z is unique.

This completes the proof of the theorem.

Theorem 2.2: Let S, T, R be any three self mappings of a complete metric space X for given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X, (x \neq y)$

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } d(SRu, TRv) \le \varepsilon$$

and
$$d(SRu, TRv) \le \alpha \max \left\{ \begin{matrix} \frac{d^2(u,SRw)+d^2(u,TRt)+d^2(u,SRw)}{1+d(u,SRw)+d(u,TRt)+d(u,SRw)}, \\ \frac{d^2(v,SRw)+d^2(u,TRt)+d^2(v,TRt)}{1+d(v,SRw)+d(u,TRt)+d(v,TRt)}, \\ \sqrt{d(v,SRw)d(u,TRt)}, \\ d(SRw,TRt), d(u,v) \end{matrix} \right\} \qquad 2.2(i)$$

For $u, v, w, t \in X$ and $0 \le \alpha < 1$ then SR, TR have a unique common fixed point.

Proof: Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows

$$SRx_{n-1} = x_n, TRx_n = x_{n+1}, SRx_{n+1} = x_{n+2,} TRx_{n+2} = x_{n+3,\dots\dots}$$

and $TRSRx_{n-1} = x_{n+1}, SRTRx_n = x_{n+2}, TRSRx_{n+1} = x_{n+3,}$

$$SRTRx_{n+2} = x_{n+4,\dots\dots}$$

Where $n = 1, 2, 3, \dots$

Now putting $u = TRy, v = SRx, w = x$ and $t = y$ in 2.2(i), then we have

$$d(SRTRy, TRSRx) \le \alpha \max \left\{ \begin{matrix} \frac{d^2(TRy,SRx)+d^2(TRy,TRy)+d^2(TRy,SRx)}{1+d(TRy,SRx)+d(TRy,TRy)+d(TRy,SRx)}, \\ \frac{d^2(SRx,SRx)+d^2(TRy,TRy)+d^2(SRx,TRy)}{1+d(SRx,SRx)+d(TRy,TRy)+d(SRx,TRy)}, \\ \sqrt{d(SRx,SRx).d(TRy,TRy)}, \\ d(SRx,TRy), d(TRy,SRx) \end{matrix} \right\}$$

$$d(SRTRy, TRSRx) \le \alpha\, d(SRx, TRy) \qquad 2.2(ii)$$

Now putting $x = x_{n-1}$ and $y = x_n$ in 2.2(ii) then we have

$$d(SRTRx_n, TRSRx_{n-1}) \le \alpha\, d(SRx_{n-1}, TRx_n)$$

$$d(x_{n+2}, x_{n+1}) \le \alpha\, d(x_n, x_{n+1}) \qquad 2.2(iii)$$

From 2.2(iii) we conclude that $d(x_{n-1}, x_n)$ decreases with n

i.e. $\quad d(x_{n-1}, x_n) \to d(x_0, x_1)$ when $n \to \infty$

If possible let $d(x_0, x_1) > 0$ and Taking limit $n \to \infty$ on 2.2(iii) then we have

$$d(x_0, x_1) \leq (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) d(x_0, x_1)$$

$$d(x_0, x_1) < d(x_0, x_1)$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$

$$d(x_0, x_1) = 0$$

Next we shall show that $\{x_n\}$ is Cauchy sequence.

Now $$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n)$$

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + d(SRx_n, TRx_m) \qquad 2.2(iv)$$

By putting $u = x_n, v = x_m, w = x_{m-1}, t = x_{n-1}$ in 2.2(i) then we have

$$d(Sx_n, Tx_m) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(x_n,SRx_{m-1})+d^2(x_n,TRx_{n-1})+d^2(x_n,SRx_{m-1})}{1+d(x_n,SRx_{m-1})+d(x_n,TRx_{n-1})+d(x_n,SRx_{m-1})}, \\ \frac{d^2(x_m,SRx_{m-1})+d^2(x_n,TRx_{n-1})+d^2(x_m,TRx_{n-1})}{1+d(x_m,SRx_{m-1})+d(x_n,TRx_{n-1})+d(x_m,TRx_{n-1})}, \\ \sqrt{d(x_m, SRx_{m-1}).\, d(x_n, TRx_{n-1})}, \\ d(SRx_{m-1}, TRx_{n-1}) \end{array} \right\}$$

$$= \alpha \max \left\{ \begin{array}{c} \frac{d^2(x_n,x_m)+d^2(x_n,x_n)+d^2(x_n,x_m)}{1+d(x_n,x_m)+d(x_n,x_n)+d(x_n,x_m)}, \\ \frac{d^2(x_m,x_m)+d^2(x_n,x_n)+d^2(x_m,x_n)}{1+d(x_m,x_m)+d(x_n,x_n)+d(x_m,x_n)}, \\ \sqrt{d(x_m, x_m).\, d(x_n, x_n)}, \\ d(x_m, x_n), d(x_n, x_m) \end{array} \right\}$$

$$= \alpha\, d(x_n, x_m)$$

$$\Rightarrow d(Sx_n, Tx_m) \leq \alpha\, d(x_n, x_m) \qquad 2.2(v)$$

From 2.2(iv) and 2.2(v) we have

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + \alpha\, d(x_m, x_n)$$

Letting $m, n \to \infty$ then $d(x_n, x_m) \to 0$ as $\alpha < 1$

Hence $\{x_n\}$ is a Cauchy sequence.

Now we prove z is a common fixed point of SR, TR.

By putting $u = z, v = x_{n-1}, w = z$ and $t = x_{n-2}$ in 2.2(i) we have

$$d(SRz, TRx_{n-1}) \leq \alpha \max \left\{ \begin{array}{c} \frac{d^2(z,SRz)+d^2(z,TRx_{n-2})+d^2(z,SRz)}{1+d(z,SRz)+d(z,TRx_{n-2})+d(z,SRz)}, \\ \frac{d^2(x_{n-1},SRz)+d^2(z,TRx_{n-2})+d^2(x_{n-1},TRx_{n-2})}{1+d(x_{n-1},SRz)+d(z,TRx_{n-2})+d(x_{n-1},TRx_{n-2})}, \\ \sqrt{d(x_{n-1}, SRz).\, d(z, TRx_{n-2})}, \\ d(SRz, TRx_{n-2}), d(z, x_{n-1}) \end{array} \right\}$$

letting $n \to \infty$ then we have

$$d(SRz, z) \leq \alpha \max \begin{Bmatrix} \frac{d^2(z,SRz)+d^2(z,z)+d^2(z,SRz)}{1+d(z,SRz)+d(z,z)+d(z,SRz)}, \\ \frac{d^2(z,SRz)+d^2(z,z)+d^2(z,z)}{1+d(z,SRz)+d(z,z)+d(z,z)}, \\ \sqrt{d(z,SRz).d(z,z)}, \\ d(SRz,z), d(z,z) \end{Bmatrix}$$

$$d(SRz, z) \leq \alpha\, d(SRz, z)$$

$$d(SRz, z) < d(SRz, z)$$

Since $\alpha < 1$ which gives $d(SRz, z) = 0 \Rightarrow SRz = z$

Thus z is a fixed point of SR.

Similarly we can show that z is a fixed point of TR.

Hence z is a common fixed point of SR, TR.

Now we are taking an another point q which is not equal to z such that $SRq = q = TRq$

By putting $u = z, v = q, w = q, t = z$ in 2.2(i) then we have

$$d(SRz, TRq) \leq \alpha \max \begin{Bmatrix} \frac{d^2(z,SRq)+d^2(z,TRz)+d^2(z,SRq)}{1+d(z,SRq)+d(z,TRz)+d(z,SRq)}, \\ \frac{d^2(q,Sq)+d^2(q,Tz)+d^2(Sq,Tz)}{1+d(q,Sq)+d(q,Tz)+d(Sq,Tz)}, \\ \sqrt{d(q,SRq).d(z,TRz)}, \\ d(SRq,TRz), d(z,q) \end{Bmatrix}$$

$$d(z, q) \leq \alpha \max \begin{Bmatrix} \frac{d^2(z,q)+d^2(z,z)+d^2(z,q)}{1+d(z,q)+d(z,z)+d(z,q)}, \\ \frac{d^2(q,q)+d^2(z,z)+d^2(q,z)}{1+d(q,q)+d(z,z)+d(q,z)}, \\ \sqrt{d(q,q).d(z,z)}, \\ d(q,z), d(z,q) \end{Bmatrix}$$

$$d(z, q) \leq \alpha\, d(z, q)$$

$d(z, q) < d(z, q)$ Since $\alpha < 1$

Which gives $d(z, q) = 0 \Rightarrow z = q$

Hence z is unique.

This completes the proof of theorem.

Conclusion

In this present article we prove some more common fixed point theorem in complete metric space by using Meir-Keeler type contractive conditions. In fact our main result is more general then other previous known results

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