

Fully Fuzzy Linear System in Circuit Analysis with the Study of Weak Solution

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Abstract

In this paper, a simpler method to solve a fully fuzzy linear system (FFLS) with unrestricted coefficient matrix is discussed. FFLS is applied in circuit analysis instead of crisp linear system to reflect the real life situation much better. Arithmetic operations of triangular fuzzy number (TFN) are justified by forming FFLS in an electrical circuit with fuzzy sources and fuzzy resistors and then the system was solved by the simpler method. Finally, the case of weak solution is overcome by proposing a new definition of TFN.

Keywords: Fuzzy number, Triangular fuzzy number, Fully fuzzy linear system, Circuit analysis, Weak solution

1. Introduction

In real life applications where system of linear equations arises in various fields of engineering and sciences, there may be situation when the values of the parameters i.e. the coefficients and right hand side constants are not known exactly or cannot be stated precisely except their estimation or some bounds on them. As fuzzy numbers are the imprecise form of precise real numbers, it is natural to treat the parameters of the system as fuzzy numbers instead of real numbers in those situations. Therefore it is imperative to develop mathematical models and numerical procedures to solve fuzzy linear system. The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh, L. A. [9]. Dubois et al. [6, 7] introduced the notion of L-R fuzzy numbers and discussed system of linear fuzzy constraints. Klir, G. J. & Yuan, B. [1] discussed the solution of single fuzzy equation of the types A+X=B and A.X=B. Freidman et al. [8] were first to propose the model for solving the fuzzy systems with the crisp coefficient matrix and fuzzy right-hand side column. Later, systems where all the parameters are considered to be fuzzy were introduced. Various authors [2, 3, 4, 5] have studied these models and gave analytical as well as numerical techniques to solve such systems. Recently, FFLS is introduced in various fields of applications, one of which is circuit analysis. Rahgooy et al. [10] and Das et al. [11] applied FFLS in electrical circuit. But the case of weak solution, which is very common in real life, is not clearly discussed in literature. In this paper, we have discussed this case in details and proposed a way to take care of the situation in an easy manner.

2. Preliminaries

In this section, we review some definitions and formulae which will be required for our main presentations in the later sections.

2.1 Fuzzy numbers

A fuzzy set μ is called a fuzzy number if it is defined on the set $\mathbb{R}$ of real numbers i.e. $\mu \in \mathcal{F}(\mathbb{R})$ satisfying at least the following three properties:

(i) μ is normal that is there exist at least one $x \in \mathbb{R}$ such that $\mu(x) = 1$ or equivalently $Core\ (\mu) \neq \emptyset$

(ii) μ is convex or equivalently ${}^{\alpha}\mu$, the alpha cut of μ , is a closed interval for every $\alpha \in (0,1]$

(iii) The membership function of μ is at least piecewise continuous.

Alternatively, a fuzzy number μ can be defined as a fuzzy set whose membership function can be expressed in the general form as:

$$\mu(x) = \begin{cases} l(x), & for \quad a \le x < b \\ 1, & for \quad b \le x \le c \\ r(x), & for \quad c < x \le d \\ 0, & otherwise \end{cases} \quad (1)$$

where $l(x)$, called the *left membership function*, is a non-decreasing function from $[a, b)$ to $[0,1]$, continuous from the right and $r(x)$, called the *right membership function*, is a non-increasing function from $(c, d]$ to $[0,1]$, continuous from the left.

2.2 Triangular Fuzzy Number

When $l(x) = (x-a)/(b-a)$, $r(x) = (x-d)/(d-c)$ and $b = c$ in (1), then we call μ a triangular fuzzy number. In other words, the general form of a triangular fuzzy number T can be written as follows:

$$T(x) = \begin{cases} \frac{x-a}{b-a}, & for \quad a \le x < b \\ \frac{x-c}{b-c}, & for \quad b \le x < c \\ 0, & otherwise \end{cases} ; a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R} \quad (2)$$

The TFN T defined by (2) can be expressed as $T = \langle a, b, c \rangle$ (3) using the real numbers a, b, c as parameters. We call it the "vertex form". In this form, we must have

$$a \le b \le c \quad (4)$$

When $a = b = c$, then T becomes the fuzzy representation of the crisp real number b. Sometimes a *tilde* "$\sim$" is used over the fuzzy number T like $\tilde{T}$ to distinguish it from the real numbers. A TFN may also be expressed as:

$$T(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha}, & for \quad m - \alpha \le x < m \\ 1 - \dfrac{x-m}{\beta}, & for \quad m \le x < m + \beta \\ 0, & otherwise \end{cases} ; \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+, m \in \mathbb{R}$$

Clearly, this is just another equivalent form of (2). This is also compactly written as $T = \langle m, \alpha, \beta \rangle$. We call it the "*spread form*". In this paper, we use the vertex form defined in (3).

2.2.1 Positive, Negative and Half Positive-half negative TFN

A TFN $T = \langle a, b, c \rangle$ is said to be *positive* i.e. $T > 0$ iff $a > 0$. It is nonnegative i.e. $T \ge 0$ iff $a \ge 0$.

A TFN $T = \langle a, b, c \rangle$ is said to be *negative* i.e. $T < 0$ iff $c < 0$. It is non-positive i.e. $T \le 0$ iff $c \le 0$.

A TFN $T = \langle a, b, c \rangle$ is said to be half positive-half negative when $a < 0$ and $c > 0$.

2.2.2 Equality of two TFNs

Two TFNs $T_1 = \langle a_1, b_1, c_1 \rangle$ and $T_2 = \langle a_2, b_2, c_2 \rangle$ are said to be equal i.e. $T_1 = T_2$ iff $a_1 = a_2$; $b_1 = b_2$; $c_1 = c_2$

We see that for any triangular fuzzy number $T = \langle a, b, c \rangle$, there is only one value b for which the membership grade is 1 and on both side of this modal value b, the membership grade decreases linearly as the distance increases. This enables to expresses the human perception of uncertain quantity such as "numbers around b" or "about b". This physical interpretation of TFN will be considered in latter section.

2.2.3 Example: Consider the fuzzy number "around-2", denoted by $\widetilde{-2} = \langle -3, -2, -1 \rangle$ whose membership function is defined by

$$\widetilde{-2}(x) = \begin{cases} x+3, & for \quad -3 \le x < -2 \\ -(x+1), & for \quad -2 \le x < -1 \\ 0, & otherwise \end{cases}$$

2.3 Arithmetic Operation of TFNs

The formulae for the three standard arithmetic operations- addition, subtraction and multiplication of two TFNs are given as follows:

(i) $\langle a,b,c\rangle \oplus \langle p,q,r\rangle = \langle a+p, b+q, c+r\rangle$

(ii) $\langle a,b,c\rangle \ominus \langle p,q,r\rangle = \langle a-r, b-q, c-a\rangle$

(iii) $\langle a,b,c\rangle \odot \langle p,q,r\rangle = \langle ap, bq, cr\rangle \quad for\ \ 0 \le a < b < c, 0 \le p < q < r$

$\langle a,b,c\rangle \odot \langle p,q,r\rangle = \langle ar, bq, cp\rangle \quad for\ \ a < b < c < 0, 0 \le p < q < r$

$\langle a,b,c\rangle \odot \langle p,q,r\rangle = \langle ar, bq, cr\rangle \quad for\ \ a < 0, c > 0, 0 \le p < q < r$

2.4 Fully Fuzzy Linear System

When all the right hand side constants and all the coefficients are fuzzy numbers in a linear system, then it is called *fully fuzzy linear system* abbreviated FFLS. Such a system in n fuzzy variables $\widetilde{x_1}, \widetilde{x_2} \dots \dots \dots, \widetilde{x_n}$ and n equations can be written, in general, in the form

$$\left.\begin{array}{ccccccc} \widetilde{a_{11}} \otimes \widetilde{x_1} & \oplus & \widetilde{a_{12}} \otimes \widetilde{x_2} & \oplus & \cdots\cdots & \oplus & \widetilde{a_{1n}} \otimes \widetilde{x_n} = \widetilde{b_1} \\ \widetilde{a_{21}} \otimes \widetilde{x_1} & \oplus & \widetilde{a_{22}} \otimes \widetilde{x_2} & \oplus & \cdots\cdots & \oplus & \widetilde{a_{2n}} \otimes \widetilde{x_n} = \widetilde{b_2} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \widetilde{a_{n1}} \otimes \widetilde{x_1} & \oplus & \widetilde{a_{n2}} \otimes \widetilde{x_2} & \oplus & \cdots\cdots & \oplus & \widetilde{a_{nn}} \otimes \widetilde{x_n} = \widetilde{b_n} \end{array}\right\} \tag{5}$$

where $\oplus$ and $\otimes$ denotes fuzzy addition and fuzzy multiplication respectively and the coefficients $\widetilde{a_{ij}}\ ; i,j = 1,2,\dots,n$ and the constants $\widetilde{b_j}\ ; j = 1,2,\dots,n$ are known fuzzy numbers such as triangular, trapezoidal, Gaussian etc. In this paper, we consider FFLS where all the fuzzy numbers are triangular. We can compactly write (5) in matrix vector notation as $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$ where $\widetilde{\boldsymbol{A}}$ is an $n \times n$ fuzzy matrix and $\widetilde{\boldsymbol{x}}$ and $\widetilde{\boldsymbol{b}}$ both are $n \times 1$ fuzzy vectors.

2.5 Kirchhoff's Rules

Kirchhoff's circuit rules are two elementary laws that deal with the current and voltage in the lumped element model of electrical circuits. They were first described in 1845 by German physicist Gustav Kirchhoff. Kirchhoff's first rule is known as Kirchhoff's junction rule and the second one is known as loop rule.

2.5.1 Junction rule: At any junction in an electrical circuit, the sum of currents flowing into that node is equal to the sum of the currents flowing out of that node. Or equivalently, the algebraic sum of currents in a network of conductors meeting at a point is zero. Mathematically $\sum_{k=1}^{n} I_k = 0$

2.5.2 Loop rule: The directed sum of the electrical potential differences (voltage) around any closed network is zero. More simply, the sum of the electro motive forces in any closed loop is equivalent to the sum of the potential drops in that loop. Mathematically $\sum_{k=1}^{n} V_k = 0$

3. Solution of an FFLS with Unrestricted Coefficient Matrix

Solving an FFLS is not as straightforward as its crisp counterpart since fuzzy numbers are sets defined by special membership function. Thus, for efficiency we consider only such FFLS where all the coefficients and constants are TFNs. We also consider the fuzzy variables to be unknown TFNs though it is proved that multiplication of TFNs does not produce TFN. As each right hand side constant is a TFN, clearly the solutions should not be so. But we can approximately take the multiplication of TFNs as another TFN by the approximation formulae given in (iii) of subsection 2.3. This assumption reduces the complexity of calculations to a great extent. We consider an FFLS of the form $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$. Since in real life applications, most often positive solutions of a system are desired, we can also assume, without loss of generality, that each of the fuzzy variables $\widetilde{x_i}$ is non-negative i.e. $\widetilde{\boldsymbol{x}} \ge 0$. In literature so far, another restriction proposed on the fuzzy coefficient matrix $\widetilde{\boldsymbol{A}}$ is that $\widetilde{\boldsymbol{A}}$ has to be be non-negative i.e. each of the coefficients must be non-negative. This restriction clearly causes loss of generality and does not reflect the real scenario of various applications. Hence, it is necessary to develop methods to solve FFLS where all the coefficients need not be non-negative i.e. $\widetilde{\boldsymbol{A}}$ is unrestricted. We now illustrate a new method to solve such an FFLS.

3.1 Simple Method

To describe the method efficiently we consider a general 2×2 FFLS of the form:

$$\begin{aligned} \widetilde{a_{11}} \otimes \widetilde{x_1} \ \oplus \ \widetilde{a_{12}} \otimes \widetilde{x_2} = \widetilde{b_1} \\ \widetilde{a_{21}} \otimes \widetilde{x_1} \ \oplus \ \widetilde{a_{22}} \otimes \widetilde{x_2} = \widetilde{b_2} \end{aligned} \tag{10}$$

All the fuzzy variables are assumed to be non-negative and each fuzzy coefficient is unrestricted. Since all the fuzzy coefficients need not be of the same type, let us assume that $\widetilde{a_{11}} \geq 0, \widetilde{a_{12}} < 0, \widetilde{a_{21}} < 0$ and $\widetilde{a_{22}}$ is half positive-half negative. This assumption is just for describing the procedure clearly. Now, since all the coefficients and constants are TFNs, we must have $\widetilde{a_{ij}} = \langle f_{ij}, g_{ij}, h_{ij} \rangle$ and $\widetilde{b_i} = \langle p_i, q_i, r_i \rangle$ for each $i = 1,2,3$ and $j = 1,2,3$. Also setting $\widetilde{x_i} = \langle u_i, v_i, w_i \rangle$ for each $i = 1,2,3$ we can write (10) as

$$\begin{aligned} \langle f_{11}, g_{11}, h_{11} \rangle \otimes \langle u_1, v_1, w_1 \rangle \ \oplus \ \langle f_{12}, g_{12}, h_{12} \rangle \otimes \langle u_2, v_2, w_2 \rangle \ = \ \langle p_1, q_1, r_1 \rangle \\ \langle f_{21}, g_{21}, h_{21} \rangle \otimes \langle u_1, v_1, w_1 \rangle \ \oplus \ \langle f_{22}, g_{22}, h_{22} \rangle \otimes \langle u_2, v_2, w_2 \rangle \ = \ \langle p_2, q_2, r_2 \rangle \end{aligned} \tag{11}$$

Now, since $\widetilde{a_{11}} \geq 0$, we do the first multiplication of the first equation of (11) using the first formula of (iii). Again, since $\widetilde{a_{12}} < 0$ and $\widetilde{a_{21}} < 0$, we do the second multiplication of the first equation and the first multiplication of the second equation of (11) using the second formula of (iii). For the last multiplication, we use the last formula of (iii) as $\widetilde{a_{22}}$ is half positive-half negative. Completing these multiplications, we get from (11)

$$\begin{aligned} \langle f_{11} u_1, g_{11} v_1, h_{11} w_1 \rangle \ \oplus \ \langle f_{12} w_2, g_{12} v_2, h_{12} u_2 \rangle \ = \ \langle p_1, q_1, r_1 \rangle \\ \langle f_{21} w_1, g_{21} v_1, h_{21} u_1 \rangle \ \oplus \ \langle f_{22} w_2, g_{22} v_2, h_{22} w_2 \rangle \ = \ \langle p_2, q_2, r_2 \rangle \end{aligned} \tag{12}$$

Now performing the fuzzy additions in (12), we get,

$$\begin{aligned} \langle f_{11} u_1 + f_{12} w_2, g_{11} v_1 + g_{12} v_2, h_{11} w_1 + h_{12} u_2 \rangle \ = \ \langle p_1, q_1, r_1 \rangle \\ \langle f_{21} w_1 + f_{22} w_2, g_{21} v_1 + g_{22} v_2, h_{21} u_1 + h_{22} w_2 \rangle \ = \ \langle p_2, q_2, r_2 \rangle \end{aligned} \tag{13}$$

Using the equality of two TFNs, we get the following six crisp linear equations from (13) in the six variables u_1, v_1, w_1, u_2, v_2 and w_2

$$f_{11} u_1 + f_{12} w_2 = p_1$$

$$g_{11} v_1 + g_{12} v_2 = q_1$$

$$h_{11} w_1 + h_{12} u_2 = r_1$$

$$f_{21} w_1 + f_{22} w_2 = p_2$$

$$g_{21} v_1 + g_{22} v_2 = q_2$$

$$h_{21} u_1 + h_{22} w_2 = r_2$$

Thus the original 2×2 FFLS has now become a 6×6 crisp system. We can solve this system by any of the known methods like Cramer's Rule, Gaussian Elimination, LU-Factorization etc. to obtain the values of u_1, v_1, w_1, u_2, v_2 and w_2 which yields the solution of the FFLS as

$$\begin{bmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{bmatrix} = \begin{bmatrix} \langle u_1, v_1, w_1 \rangle \\ \langle u_2, v_2, w_2 \rangle \end{bmatrix}$$

We described the procedure for 2×2 FFLS but this can easily be generalized for $n \times n$ FFLS. In that case, we will get a $3n \times 3n$ crisp system which can then be solved usually. We now give a numerical example.

3.1.1 Example: Consider the following 2×2 FFLS:

$$\begin{aligned} \langle 2,3,4 \rangle \otimes \langle u_1, v_1, w_1 \rangle \oplus \langle -3,-2,-1 \rangle \otimes \langle u_2, v_2, w_2 \rangle &= \langle -13,-2,9 \rangle \\ \langle -3,-2,-1 \rangle \otimes \langle u_1, v_1, w_1 \rangle \oplus \langle -1,1,2 \rangle \otimes \langle u_2, v_2, w_2 \rangle &= \langle -14,0,9 \rangle \end{aligned} \tag{14}$$

Using the procedure described above, we get the following 6 crisp linear equations:

$$2u_1 - 3w_2 = -13$$

$$3v_1 - 2v_2 = -2$$

$$4w_1 - u_2 = 9$$

$$-3w_1 - w_2 = -14$$

$$-2v_1 + v_2 = 0$$

$$-u_1 + 2w_2 = 9$$

We now treat this 6 equations as a 6×6 crisp linear system $M\boldsymbol{x} = \boldsymbol{b}$ where

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}; \ \boldsymbol{x} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} -13 \\ -2 \\ 9 \\ -14 \\ 0 \\ 9 \end{bmatrix}$$

Solving this crisp system by Cramer's Rule, we get

$$u_1 = 1, v_1 = 2, w_1 = 3, u_2 = 3, v_2 = 4 \ and \ w_2 = 5$$

Thus the fuzzy solution of the given FFLS (14) is

$$\begin{bmatrix} \langle 1,2,3 \rangle \\ \langle 3,4,5 \rangle \end{bmatrix}.$$

4. Application of FFLS in Circuit Analysis

In circuit analysis, there often arises system of linear equations that we need to solve to find the value of unknown currents. But in reality, we can never be sure that a 5-volt source in a circuit always produces exactly the same voltage; there may be, in fact there is, some small ups and downs in voltage producing. Similar conditions apply for the resistors in circuit. Thus, we should consider sources, resistors etc. in a circuit to be fuzzy. Hence, it is natural and more convenient to replace the crisp parameters of a linear system in circuit analysis by fuzzy parameters. We will illustrate this application with triangular fuzzy numbers in the example below.

4.1 *Crisp version*

First consider a circuit shown in the following figure:

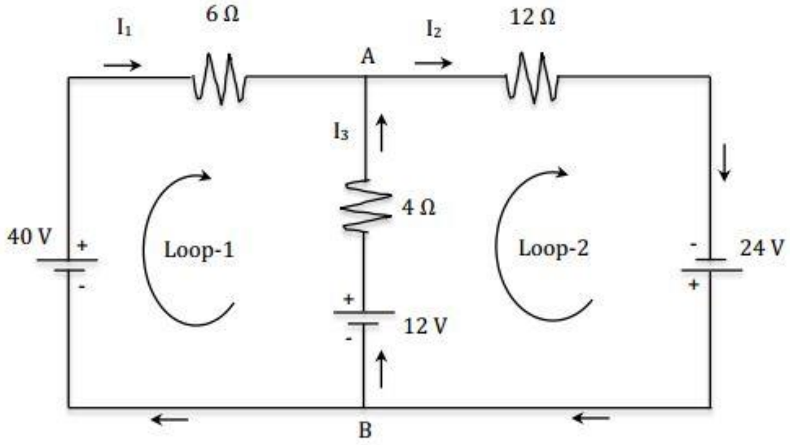

Figure 4.1 An electrical circuit.

We need to find the unknown currents I_1, I_2. Using Kirchhoff's junction rule at node A and B, we respectively get,

$$I_1 + I_3 = I_2 \ ,$$

$$I_2 = I_1 + I_2 \ .$$

Both of these equations yield

$$I_3 = I_2 - I_1 \qquad (15)$$

Now, using (15), Ohm's law i.e. $E = IR$ where E is the voltage drop for the resistor R and I is the current and using Kirchhoff's loop rule around loop-1 and loop-2, we get respectively,

$$\left.\begin{array}{r} 6 \times I_1 - 4 \times (I_2 - I_1) + 12 - 40 = 0 \\ 4 \times (I_2 - I_1) + 12 \times I_2 - 24 - 12 = 0 \end{array}\right\} . \qquad (16)$$

Simplifying above equations, we get the following 2×2 crisp linear system

$$\left.\begin{array}{r} 10 \times I_1 - 4 \times I_2 = 28 \\ -4 \times I_1 + 16 \times I_2 = 36 \end{array}\right\} . \qquad (17)$$

The solution of (17) is easily found to be (4.11111, 3.27778) using Cramer's rule.

4.2 *Fuzzy version*

If we now consider the same circuit with fuzzy sources of and fuzzy currents as we have explained earlier, we have to replace the parameters of the above system by fuzzy numbers and the variables by fuzzy variables.

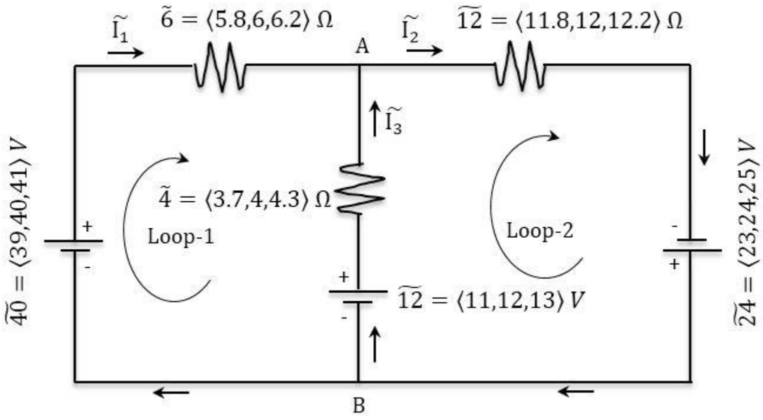

Figure 4.2 A fuzzy electrical circuit.

Here, we use triangular fuzzy numbers. Replacing the crisp values $40\ V, 12\ V$ *and* $24\ V$ of the sources by the triangular fuzzy numbers $\widetilde{40} = \langle 39,40,41\rangle\ V$, $\widetilde{12} = \langle 11,12,13\rangle\ V$ and $\widetilde{24} = \langle 23,24,25\rangle\ V$ respectively; and similarly the crisp values $6\ \Omega, 4\ \Omega$ *and* $12\ \Omega$ of the resistors by the triangular fuzzy numbers $\tilde{6} = \langle 5.8,6,6.2\rangle\ \Omega$, $\tilde{4} = \langle 3.7,4,4.3\rangle\ \Omega$ and $\widetilde{12} = \langle 11.8,12,12.2\rangle\ \Omega$ respectively, we get from (16)

$$\left.\begin{aligned} \tilde{6}\times\tilde{I_1} - \tilde{4}\times\left(\tilde{I_2}-\tilde{I_1}\right) + \widetilde{12} - \widetilde{40} = 0 \\ \tilde{4}\times\left(\tilde{I_2}-\tilde{I_1}\right) + \widetilde{12}\times\tilde{I_2} - \widetilde{24} - \widetilde{12} = 0 \end{aligned}\right\} . \tag{18}$$

Using the negation formula and rearranging the terms in (18), we get,

$$\left.\begin{aligned} \left(\tilde{6}+\tilde{4}\right)\times\tilde{I_1} - \tilde{4}\times\tilde{I_2} = \widetilde{40} - \widetilde{12} \\ -\tilde{4}\times\tilde{I_1} + (\widetilde{12}+\tilde{4})\times\tilde{I_2} = \widetilde{24} + \widetilde{12} \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} (\langle 5.8,6,6.2\rangle + \langle 3.7,4,4.3\rangle)\times\tilde{I_1} - \langle 3.7,4,4.3\rangle\times\tilde{I_2} = \langle 39,40,41\rangle - \langle 11,12,13\rangle \\ -\langle 3.7,4,4.3\rangle\times\tilde{I_1} + (\langle 11.8,12,12.2\rangle + \langle 3.7,4,4.3\rangle)\times\tilde{I_2} = \langle 23,24,25\rangle + \langle 11,12,13\rangle \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} \langle 9.5,10,10.5\rangle\times\tilde{I_1} + \langle -4.3,-4,-3.7\rangle\times\tilde{I_2} = \langle 26,28,30\rangle \\ \langle -4.3,-4,-3.7\rangle\times\tilde{I_1} + \langle 15.5,16,16.5\rangle\times\tilde{I_2} = \langle 34,36,38\rangle \end{aligned}\right\} \tag{19}$$

Thus we get the following 2×2 FFLS:

$$\left.\begin{aligned} \widetilde{10}\times\tilde{I_1} + (\widetilde{-4})\times\tilde{I_2} = \widetilde{28} \\ (\widetilde{-4})\times\tilde{I_1} + \widetilde{16}\times\tilde{I_2} = \widetilde{36} \end{aligned}\right\} . \tag{20}$$

Clearly, (20) is the fuzzy version of the crisp system (17). This justifies the arithmetic operations with TFNs. Now, treating the fuzzy variables $\tilde{I_k}$ as unknown triangular fuzzy numbers $\langle u_k, v_k, w_k\rangle$ for each $k = 1,2$ we get from (19)

$$\begin{aligned} \langle 9.5,10,10.5\rangle \otimes \langle u_1,v_1,w_1\rangle \oplus \langle -4.3,-4,-3.7\rangle \otimes \langle u_2,v_2,w_2\rangle = \langle 26,28,30\rangle \\ \langle -4.3,-4,-3.7\rangle \otimes \langle u_1,v_1,w_1\rangle \oplus \langle 15.5,16,16.5\rangle \otimes \langle u_2,v_2,w_2\rangle = \langle 34,36,38\rangle \end{aligned} \tag{21}$$

Clearly, the fuzzy matrix of the FFLS (21) is neither positive nor negative. Also the non-negativity condition of the fuzzy variables is met by the natural requirement that the values of current should not be negative. So, here we can use the method for solving an FFLS with unrestricted coefficient matrix. Using the procedure described in section 3, we get the following 6 crisp linear equations from (21)

$$9.5\ u_1 - 4.3\ w_2 = 26\ ,$$

$$10\, v_1 - 4\, v_2 = 28 \ ,$$

$$10.5\, w_1 - 3.7\, u_2 = 30 \ ,$$

$$-4.3\, w_1 + 15.5\, u_2 = 34 \ ,$$

$$-4\, v_1 + 16\, v_2 = 36 \ ,$$

$$-3.7\, u_1 + 16.5\, w_2 = 38 \ .$$

We now treat this 6 equations as a 6×6 crisp linear system $\boldsymbol{Mx} = \boldsymbol{b}$, where

$$\boldsymbol{M} = \begin{bmatrix} 9.5 & 0 & 0 & 0 & 0 & -4.3 \\ 0 & 10 & 0 & 0 & -4 & 0 \\ 0 & 0 & 10.5 & -3.7 & 0 & 0 \\ 0 & 0 & -4.3 & 15.5 & 0 & 0 \\ 0 & -4 & 0 & 0 & 16 & 0 \\ -3.7 & 0 & 0 & 0 & 0 & 16.5 \end{bmatrix} ; \quad \boldsymbol{x} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 26 \\ 28 \\ 30 \\ 34 \\ 36 \\ 38 \end{bmatrix}.$$

Now, $\det(\boldsymbol{M}) = 2.97806 \times 10^6 \neq 0$. Hence the crisp system is solvable. Using Cramer's rule, we get its solution as follows:

$$u_1 = 4.20619, v_1 = 4.11111, w_1 = 4.02343, u_2 = 3.30972, v_2 = 3.27778, w_2 = 3.24624$$

That is, the fuzzy solution of (20) is $(4.\widetilde{11111}, 3.\widetilde{30972})$ as it should be. But the problem is, though the crisp parts of the fuzzy solution i.e. (4.11111, 3.27778) matches exactly with the crisp solution of (17) as expected, inequality (4) is totally violated by this fuzzy solution as we can't take $\langle 4.20619, 4.11111, 4.02343 \rangle = 4.\widetilde{11111}$ and $\langle 3.30972, 3.27778, 3.24624 \rangle = 3.\widetilde{27778}$ by the current definition of TFN. This is called "weak solution" in literature. One way to remove this problem is that we change the fuzzy part of the right hand side constants in (18) i.e. we increase or decrease the left and right spread of the fuzzy coefficients in a way so that the solution does not come out weak. Das et al. [11] used this way. But this is both unscientific and illogical because there always is a tolerance limit of the resistors and sources in a circuit and thus the impreciseness of the right hand side constants can't vary as our wish. Another way can be that we accept the current solution by taking $\langle 4.20619, 4.11111, 4.02343 \rangle = 4.\widetilde{11111}$ and $\langle 3.30972, 3.27778, 3.24624 \rangle = 3.\widetilde{27778}$ as valid TFNs in the way that inequality (4) need not be satisfied in vertex form. To do this, we propose a slight variation in the definition of vertex form as follows:

4.2.1 New Definition

We define $\tilde{T} = \langle a, b, c \rangle$ to be a TFN in vertex form whose membership function is given by:

$$T(x) = \begin{cases} \frac{x-a}{b-a}, & for \quad \min(a,b) \le x < \max(a,b) \\ \frac{x-c}{b-c}, & for \quad \min(b,c) \le x < \max(b,c); a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R} \\ 0, & \qquad otherwise \end{cases} \tag{22}$$

Clearly this definition also works for the case when $a \le b \le c$ as the membership function by (22) then matches with that of $\tilde{T}$ by (2). Also, this definition produces the same TFN $\tilde{T}$ whenever $c \le b \le a$ which removes the problems of correcting weak solutions by rearranging the fuzzy constants. Thus, by this definition, we get the solution as:

$$\begin{bmatrix} \tilde{I_1} \\ \tilde{I_2} \end{bmatrix} = \begin{bmatrix} \langle 4.20619, 4.11111, 4.02343 \rangle \\ \langle 3.30972, 3.27778, 3.24624 \rangle \end{bmatrix}$$

The membership functions of $\tilde{I_1}$ and $\tilde{I_2}$ are given by (22) as follows:

$$\tilde{I_1}(x) = \begin{cases} \dfrac{x - 4.20619}{-0.09508}, & for \quad 4.11111 \le x < 4.20619 \\ \dfrac{x - 4.02343}{0.08786}, & for \quad 4.02343 \le x < 4.11111 \\ 0, & otherwise \end{cases}$$

$$\tilde{I_2}(x) = \begin{cases} \dfrac{x - 3.30972}{-0.03194}, & for \quad 3.27778 \le x < 3.30972 \\ \dfrac{x - 3.24624}{0.03154}, & for \quad 3.24624 \le x < 3.27778 \\ 0, & otherwise \end{cases}$$

The above fuzzy solution represents the values of the fuzzy currents in the considered circuit with fuzzy sources and fuzzy resistance. As there may be small ups and downs in voltage or resistance, the values of the currents should not be crisp; rather the values of the currents should vary between an interval with different grades of membership as expressed by the fuzzy solution. The graphs of the membership functions of $\tilde{I_1}$ and $\tilde{I_2}$ are shown in figure 4.3.

5. Conclusion

In this paper, we gave a simple method to solve a fully fuzzy linear system with unrestricted coefficient matrix. We strongly focused on the application of FFLS in circuit analysis and justified the arithmetic of TFNs by comparing crisp and fuzzy system for the same circuit. We discussed the case where the situation of weak solution arises in details. We were able to overcome the situation in an easier manner by proposing a new definition of TFN. No other author has discussed this so clearly.

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Figures

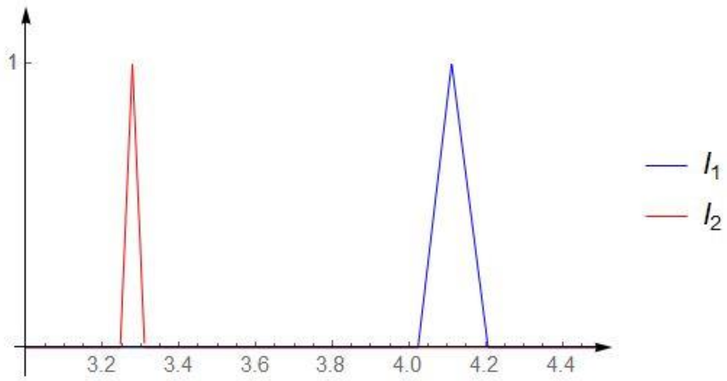

Figure 4.3: Graphs of fuzzy currents discussed in section 4.2