Two New Predictor-Corrector Iterative Methods with Third- and Ninth-Order Convergence for Solving Nonlinear Equations

Noori Yasir Abdul-Hassan
Department of Mathematics, College of Education for Pure Science, University of Basrah, Basrah, IRAQ.

Abstract

In this paper, we suggest and analyze two new predictor-corrector iterative methods with third and ninth-order convergence for solving nonlinear equations. The first method is a development of [M. A. Noor, K. I. Noor and K. Aftab, Some New Iterative Methods for Solving Nonlinear Equations, World Applied Science Journal, 20(6),(2012):870-874.] based on the trapezoidal integration rule and the centroid mean. The second method is an improvement of the first new proposed method by using the technique of updating the solution. The order of convergence and corresponding error equations of new proposed methods are proved. Several numerical examples are given to illustrate the efficiency and performance of these new methods and compared them with the Newton's method and other relevant iterative methods.

Keywords: Nonlinear equations, Predictor–corrector methods, Trapezoidal integral rule, Centroid mean, Technique of updating the solution; Order of convergence.

1. Introduction

The fundamental problem, which arise in various fields of pure and applied sciences, is the exact solution of the nonlinear equation of the form:

$$f(x) = 0 \tag{1}$$

Where $f{:}I\subseteq R\rightarrow R$ for an open interval I. In recent years, many authors developed several iterative methods for solving nonlinear equation (1) by using some numerical techniques as Taylor's series, quadrature formulas, homotopy, decomposition or predictor-corrector technique [1-8, 10-27, 29-59]. Taylor's series expansion of $f(x)$ around a given initial point $x = x_0$, yields the important methods. The famous Newton's method (N for simplicity) is a one of these methods that used to solve equation (1) by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots \tag{2}$$

This method has quadratic convergence in some neighborhood of a simple root α of f. Moreover, it has efficiency index is 1.41421; (see [10]). In addition, using Taylor expansion of the second order for the function $f(x)$, gives the well-known iterative method, defined by

$$x_{n+1} = x_n - \frac{2f(x_n)\,f'(x_n)}{2f'^2(x_n) - f(x_n)\,f''(x_n)}, \quad n = 0, 1, \dots \tag{3}$$

This is known as Halley's (H) method [15, 16, 19, 22, 43], which has cubic convergence and its efficiency index equals 1.44225. Newton's method and Halley's method are members of a family of one-step (explicit) iterative methods.

The quadrature formulae such as quadratic spline, cubic spline, midpoint integral rule, trapezoidal integral rule and Simpson's integral rule are best techniques to provide various numerical methods, see [20, 25, 45, 53-57, 59]. The first study of quadrature formulae was by Weerakoon and Fernando [57] who studied new variant of Newton's method based on trapezoidal integral rule. Özban [54] considered of Weerakoon and Fernando method by using arithmetic mean, called it the Arithmetic mean Newton's (AN) method, which has third-order convergence with efficiency index equals 1.44225, defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{(f'(x_n)+f'(y_n))/2}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n = 0,1,\ldots \tag{4}$$

Further, he proposed a variant of Newton's method by replacing the arithmetic mean with Harmonic mean in denominator of equation (4), which called it Harmonic mean Newton's (HN) method, It has third-order convergence with efficiency index equals 1.44225, and defined by

$$x_{n+1}=x_n-\frac{f(x_n)\ (f'(x_n)+f'(y_n))}{2\,f'(x_n)\,f'(y_n)}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n = 0,1,\ldots \tag{5}$$

Moreover, he used the midpoint integration formula instead of the trapezoidal integral rule to give another variant of Newton's method; called it Midpoint Newton's (MN) method, and given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\frac{x_n+y_n}{2}\right)}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n = 0,1,\ldots \tag{6}$$

In [25] Hasanov and et al. modified Newton’s method with a third order convergent method by using Simpson’s rule, which denoted it by (SN) method and defined by

$$x_{n+1}=x_n-\frac{6\,f(x_n)}{f'(x_n)+4f'\left(\frac{x_n+y_n}{2}\right)+f'(y_n)}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n= 0,1,\ldots \tag{7}$$

This method has third-order convergence and it efficiency index equals 1.31607. Tibor lukic and et al. [42] modified Newton’s method by replacing arithmetic mean with Geometric mean in equation (4), called it Geometric mean Newton’s (GN) method. This method converges cubically and its efficiency index equals 1.44225, and given by

$$x_{n+1}=x_n-\frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_n)\,f'(y_n)}}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n= 0,1,\ldots \tag{8}$$

Ababneh [1] suggested a modified Newton’s method based on Contra-Harmonic mean instead of Arithmetic mean, called it Contra Harmonic Newton’s (CHN) method, and has cubic convergence with efficiency index equals 1.44225, and defined by

$$x_{n+1}=x_n-\frac{f(x_n)\ (f'(x_n)+f'(y_n))}{f'^2(x_n)+f'^2(y_n)}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n= 0,1,\ldots \tag{9}$$

Noor and et al. [45] derived some new variants of Newton’s method by using the trapezoidal rule for integration, one of these variants is known as (NR1) method, given by

$$x_{n+1}=x_n-2\frac{f(x_n)}{f'(x_n)}-(y_n-x_n)\frac{f'(y_n)}{f'(x_n)}, y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n = 0,1,\ldots \tag{10}$$

This method has second-order of convergence and its efficiency index equals 1.41421.

To updating the numerical solution of nonlinear equation (1), several authors are concerned with the multi-step predictor-corrector technique. The forms that presented in equations (4)-(10) are good examples of two-step predictor-corrector iterative methods. About three-step iterative methods, Hafiz and al-Goria [21] proposed new higher-order iterative method based on a Halley iterative method and the weight combination of mid-point with Simpson quadrature formulas and using predictor–corrector technique, which called it the Predictor-Corrector Halley's (PCH) method and has ninth-order convergence with efficiency index that is 1.36874 and defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \frac{12 f(x_n)}{f'(x_n) + 10 f'(w_n) + f'(y_n)}, \quad w_n = \frac{(x_n + y_n)}{2},$$

$$x_{n+1} = z_n - \frac{2 f(z_n) f'(z_n)}{2 f'^2(z_n) - f(z_n) f''(z_n)}, \quad n = 0, 1, \ldots \tag{11}$$

Bahgat and Hafiz [8], proposed a modification of Newton's method with high-order iterative method for solving nonlinear equations. This method based on a Halley and Householder iterative method and using predictor-corrector technique. Therefore, this method has eighteenth-order convergence and it efficiency index equals 1.43519. Which called it the Predictor-Corrector Newton-Halley (PCNH) method, and defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \frac{2 f(y_n) f'(y_n)}{2 f'^2(y_n) - f(y_n) f''(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f^2(z_n) f''(z_n)}{2 f'^3(z_n)}, \quad n = 0, 1, \ldots \tag{12}$$

In this paper, we will first develop the (NR1) method depending on the correction term of the arithmetic mean Newton's (AN) method and using the centroid mean instead of the arithmetic mean. This idea leads to the new two-step predictor-corrector iterative method with third order convergence. Secondly, by using the technique of updating the solution, we suggest a new three-step predictor-corrector iterative method with ninth-order convergence for solving the nonlinear equation (1). This method based on the new proposed two-step iterative method as a predictor and Halley's method as a corrector. Per iteration, the first new proposed method requires one evaluation of the function and two evaluations of its first derivative, but the second new proposed method requires two evaluations of the function, three of its first derivative and one of its second derivatives. Therefore, these methods have the same efficiency index which equals 1.44225. Several numerical examples are given to show the efficiency and the performance of two new proposed methods.

2. Preliminaries

Definition 2.1 (see [20, 28, 55]): Let $\alpha \in R$, $x_n \in R$, $n = 0, 1, 2, \ldots$. Then the sequence $\{x_n\}$ is said to converge to α if $\lim_{n\to\infty} | x_n - \alpha | = 0$. If, in addition, there exist a constant $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n > n_0$, $| x_{n+1} - \alpha | \leq c \, | x_n - \alpha |^P$, then $\{x_n\}$ is said to be convergence to α with convergence order at least p. If p = 2 or 3, the convergence is said to be quadratic or cubic respectively.

Notation 2.1 (see [2, 5, 11, 15]): Let $e_n = x_n - \alpha$ is the error in the n^{th} iteration. Then the relation

$$e_{n+1} = c\, e_n^p + o(e_n^{p+1}) \tag{13}$$

is called the *error equation* for the method. By substituting $e_n = x_n - \alpha$ for all n in any iterative method and simplifying, we obtain the error equation for that method. The value of p obtained is called *order of convergence* of this method, which produces the sequence $\{x_n\}$.

Definition 2.2 (see [4, 11]): *Efficiency index* is simply defined as

$$E.I.=p^{1/m} \tag{14}$$

Where p is the order of the method and m is the number of functions evaluations required by the method (units of work per iteration).

Notation 2.2 (see [9, 30, 33, 34, 56]): For given positive scalars a and b, some other well-known means are defined as

$$\text{Arithmetic mean(AM)} = \frac{a+b}{2} \tag{15}$$

$$\text{Centroid mean(CM)} = \frac{2(a^2+a.b+b^2)}{3\,(a+b)} \tag{16}$$

$$\text{Harmonic mean(HM)} = \frac{2\,a.b}{a+b} \tag{17}$$

$$\text{Contra Harmonic mean(CHM)} = \frac{a^2+b^2}{a+b} \tag{18}$$

$$\text{Geometric mean(CM)} = \sqrt{a.b} \tag{19}$$

3. Construction of the new methods

Assume that α be a simple root of a sufficiently differentiable function $f\,(x)$. Consider the numerical solution of the nonlinear equation (1). Then, from Newton's theorem, we have

$$f\,(x) = f\,(x_n) + \int_{x_n}^{x} f'\,(t)\;dt \tag{20}$$

By using trapezoidal integration rule to approximate the indefinite integral in equation (20), we get

$$f\,(x) = f\,(x_n) + \frac{(x-x_n)}{2}\;\{f'\,(x) + f'\,(x_n)\} \tag{21}$$

From equation (1), replace x by x_{n+1}, equation (21) can be approximate as

$$f\,(x_n) + \frac{(x_{n+1}-x_n)}{2}\;\{f'\,(x_{n+1}) + f'\,(x_n)\} = 0 \tag{22}$$

Equation (22), can be written by two forms

$$x_{n+1} = x_n - 2\frac{f\,(x_n)}{f'\,(x_n)} - [x_{n+1} - x_n]\frac{f'\,(x_{n+1})}{f'\,(x_n)} \tag{23}$$

And,

$$x_{n+1} = x_n - \frac{f\,(x_n)}{(f'(x_{n+1})+f'\,(x_n)\,)/2} \tag{24}$$

Now, by using the centroid mean instead the arithmetic mean in the denominator of the equation (24), we have

$$x_{n+1} = x_n - \frac{3f(x_n)\ (f'(x_{n+1})+f'(x_n))}{2\,(f'^2(x_{n+1})+\ f'(x_{n+1})f'(x_n)+f'^2(x_n))} \qquad (25)$$

Further, we can be write equation (25) as

$$[x_{n+1} - x_n] = -\frac{3f(x_n)\ (f'(x_{n+1})+f'(x_n))}{2\,(f'^2(x_{n+1})+\ f'(x_{n+1})f'(x_n)+f'^2(x_n))} \qquad (26)$$

Then, by substituting the equation (26) into the bracket $[x_{n+1} - x_n]$ of equation (23), we obtain

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} - \{\frac{3f(x_n)\ (f'(x_{n+1})+f'(x_n))}{2\,(f'^2(x_{n+1})+\ f'(x_{n+1})f'(x_n)+f'^2(x_n))}\}\frac{f'(x_{n+1})}{f'(x_n)} \qquad (27)$$

Now, by simplifying equation (27) and using the predictor-corrector technique, we suggest the following new two-step iterative method for solving the nonlinear equation (1):

Algorithm (3.1): For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(x_n)}{f'(x_n)}\left\{1 - \frac{3f'(y_n)\,(f'(x_n)+f'(y_n))}{2\,(f'^2(x_n)+\ f'(x_n)f'(y_n)+f'^2(y_n))}\right\},\ n = 0,1,\ldots \qquad (28)$$

This is new variant of Newton's method based on the centroid mean when trapezoidal integral rule is used. We call it Centroid mean Trapezoidal-Newton's (CTN) method. It has third-order convergence and its efficiency index equals 1.44225.

For more refinement of Algorithm (3.1), we use the technique of updating the solution. therefore, using algorithm (3.1) as a predictor and Halley's method as a corrector to suggest the following new three-step predictor-corrector iteration method for solving the nonlinear equation (1). We call it Predictor-Corrector Centroid mean Trapezoidal –Newton's –Halley's (PCTNH) method, which has ninth-order convergence and its efficiency index is 1.44225.

Algorithm (3.2): For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)}\{1 - \frac{3f'(y_n)\,(f'(x_n)+f'(y_n))}{2\,(f'^2(x_n)+\ f'(x_n)f'(y_n)+f'^2(y_n))}\},$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'^2(z_n)-f(z_n)f''(z_n)},\ n = 0,1,\ldots \qquad (29)$$

4. Analysis of Convergence

Hereunder, we derive the convergence order and error equations of the above-suggested methods (algorithms (3.1)-(3.2)).

Theorem 4.1: Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \subseteq R \rightarrow R$ for an open interval I. If x_0 is sufficiently close to α, then the iterative method defined by Algorithm (3.1) is of order three.

Proof: Let α be a simple zero of f. Then by Taylor's expansion, we have

$$f(x_n) = f'(\alpha)\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + \dots\} \quad (30)$$

$$f'(x_n) = f'(\alpha)\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + \dots\} \quad (31)$$

$$f'^2(x_n) = f'^2(\alpha)\{1 + 4c_2 e_n + (6c_3 + 4c_2^2) e_n^2 + (12c_3c_2 + 8c_4) e_n^3 + \dots\} \quad (32)$$

Where $c_j = \frac{f^{(j)}(\alpha)}{f'(\alpha)}$, j = 2, 3, ...

From (30) and (31), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (-2c_3 + 2c_2^2) e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3) e_n^4 + \dots \quad (33)$$

From (33) and (28), we obtain

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + (-7c_3c_2 + 3c_4 + 4c_2^3) e_n^4 + \dots \quad (34)$$

Now, by Taylor's expansion, we have

$$f'(y_n) = f'(\alpha)\{1 + 2c_2^2 e_n^2 + (4c_2c_3 - 4c_2^3) e_n^3 + (-11c_3c_2^2 + 6c_2c_4 + 8c_2^4) e_n^4 + \dots\} \quad (35)$$

Then,

$$f'^2(y_n) = f'^2(\alpha)\{1 + 4c_2^2 e_n^2 + 8(c_2c_3 - c_2^3) e_n^3 + (-22c_3c_2^2 + 12c_2c_4 + 20c_2^4) e_n^4 + \dots\} \quad (36)$$

From (31) and (35), we obtain

$$f'(x_n) + f'(y_n) = f'(\alpha)\{2 + 2c_2 e_n + (3c_3 + 2c_2^2) e_n^2 + (4c_3c_2 - 4c_2^3 + 4c_4) e_n^3 + (11c_3c_2^2 + 6c_2c_4 + 8c_2^4 + 5c_5) e_n^4 + \dots\} \quad (37)$$

From (35) and (37), we have

$$3f'(y_n)\{f'(x_n) + f'(y_n)\} = f'^2(\alpha)\{6 + 6c_2 e_n + (18c_2^2 + 9c_3) e_n^2 + (36c_3c_2 + 12c_4 - 24c_2^3) e_n^3 + (-57c_3c_2^2 + 54c_2c_4 + 60c_2^4 + 15c_5) e_n^4 + \dots\} \quad (38)$$

Also, from (31), (32), (35) and (36), we have

$$2\{f'^2(x_n) + f'(x_n)f'(y_n) + f'^2(y_n)\} = f'^2(\alpha)\{6 + 12c_2 e_n + (20c_2^2 + 18c_3) e_n^2 + (24c_4 + 48c_3c_2 - 16c_2^3) e_n^3 + (30c_5 - 38c_3c_2^2 + 68c_2c_4 + 40c_2^4 + 18c_3^2) e_n^4 + \dots\} \quad (39)$$

And, from (38) and (39), we have

$$\left\{1 - \frac{3f'(y_n)\{f'(x_n) + f'(y_n)\}}{2\{f'^2(x_n) + f'(x_n)f'(y_n) + f'^2(y_n)\}}\right\} = c_2 e_n + ((-5/3)c_2^2 + (3/2)c_3) e_n^2 + ((4/3)c_2^3 - 4c_3c_2 + 2c_4) e_n^3 + ((20/9)c_2^4 - (3/2)c_3^2 - (17/3)c_2c_4 + (19/6)c_3c_2^2 + (5/2)c_5) e_n^4 + \dots \quad (40)$$

Then, from (33) and (40), we get

$$\frac{f(x_n)}{f'(x_n)}\left\{1-\frac{3f'(y_n)(f'(x_n)+f'(y_n))}{2(f'^2(x_n)+f'(x_n)f'(y_n)+f'^2(y_n))}\right\}= c_2 e_n^2+(-(8/3)c_2^2+(3/2)c_3)e_n^3+ (5c_2^3-(15/2)c_3c_2+2c_4)e_n^4+\ldots \quad (41)$$

Now, by substituting (34) and (41) in (28), we have

$$x_{n+1}= \alpha+((2/3)c_2^2+(1/2)c_3)e_n^3+(-c_2^3+(1/2)c_3c_2+c_4)e_n^4+\ldots \quad (42)$$

Thus, from (13) and (42), we obtain

$$e_{n+1}=x_{n+1}-\alpha= ((2/3)c_2^2+(1/2)c_3)e_n^3+O(e_n^4) \quad (43)$$

This means the new two-step iterative method (CTN) has third-order convergence.

Theorem 4.2: Let $\alpha\in I$ be a simple zero of a sufficiently differentiable function $f:I\subseteq R\to R$ for an open interval I. If x_0 is sufficiently close to α, then the iterative method defined by Algorithm (3.2) is of order nine.

Proof: from (34) - (41), we have

$$z_n= \alpha+((2/3)c_2^2+(1/2)c_3)e_n^3+(-c_2^3+(1/2)c_2c_3+c_4)e_n^4+(-(14/9)c_2^4-(1/2)c_3c_2^2+ (2/3)c_2c_4-(3/2)c_3^2+(3/2)c_5)e_n^5+\ldots \quad (44)$$

Then, by expanding $f(z_n)$, $f'(z_n)$ and $f''(z_n)$ in Taylor's series about α, we get

$$f(z_n)= f'(\alpha)\{((2/3)c_2^2+(1/2)c_3)e_n^3+(-c_2^3+(1/2)c_2c_3+c_4)e_n^4+(-(14/9)c_2^4- (1/2)c_3c_2^2+(2/3)c_2c_4-(3/2)c_3^2+(3/2)c_5)e_n^5+\ldots\} \quad (45)$$

$$f'(z_n)=f'(\alpha)\{1+((4/3)c_2^3+c_2c_3)e_n^3+(-c_2^4+c_2^2c_3+2c_2c_4)e_n^4+(-(28/9)c_2^5- c_3c_2^3+(4/3)c_2^2c_4-3c_2c_3^2+3c_2c_5)e_n^5+\ldots\} \quad (46)$$

$$f''(z_n)=f'(\alpha)\{2c_2+(4c_3c_2^2+3c_3^2)e_n^3+(-6c_3c_2^3+3c_2c_3^2+6c_3c_4)e_n^4+ (-(28/3)c_3c_2^4-3c_3^2c_2^2+4c_3c_2c_4-9c_3^3+9c_3c_5)e_n^5+\ldots\} \quad (47)$$

Now, from (45)-(47), we have

$$\frac{2f(z_n)f'(z_n)}{2f'^2(z_n)-f(z_n)f''(z_n)}= ((2/3)c_2^2+(1/2)c_3)e_n^3+(-c_2^3+(1/2)c_2c_3+c_4)e_n^4+ (-(14/9)c_2^4-(1/2)c_3c_2^2+(2/3)c_2c_4-(3/2)c_3^2+(3/2)c_5)e_n^5+\ldots \quad (48)$$

By substituting (44) and (48) in (29), we get

$$x_{n+1}= \alpha+(-(1/8)c_3^4+(8/27)c_2^8-(3/8)c_3^3c_2^2-(1/6)c_3^2c_2^4+(10/27)c_3c_2^6)e_n^9+\ldots \quad (49)$$

Then, from (13) and (49), we obtain

$$e_{n+1}= (-(1/8)c_3^4+(8/27)c_2^8-(3/8)c_3^3c_2^2-(1/6)c_3^2c_2^4+(10/27)c_3c_2^6)e_n^9+O(e_n^{10}) \quad (50)$$

Thus, we observe that the new three-step iterative method (PCTNH) has ninth-order convergence.

5. Numerical Examples and Results

In this section, we consider some numerical examples to demonstrate the efficiency and the performance of the new proposed predictor- corrector iterative methods. In addition, we compare these new methods with Newton's method and other relevant iterative methods. We have used the stopping criteria $|x_{n+1} - x_n| < \epsilon$, where $\epsilon = 10^{-15}$, for computer programs. All the computations are performed using Maple 13.0. The following Table 1 given the test functions and their solution x^*. The order of convergence (p),

number of evaluations of the function per iteration (m), the efficiency index (E.I.) and the error equation (e_{n+1}) of various iterative methods are given in Table 2 and the number of iterations (NITER) to find $x^*$is given in Table 3. NC in Table 3 means that the method does not converge to the root x^*.

Table 1: Different test functions and their approximate zeroes (x^*)

f(x)	x^*
$f_1(x) = xe^{x^2} - sin^2(x) + 3\cos(x) + 5$	-1.207647827130919
$f_2(x) = x^3 - x + 3$	-1.671699881657161
$f_3(x) = \cos(x) + e^{-x}$	1.746139530408013
$f_4(x) = (1 + \cos(x))(e^x - 2)$	0.693147180559945
$f_5(x) = x^3 + 4x^2 - 10$	1.365230013414097
$f_6(x) = (x-1)^3 - 1$	2
$f_7(x) = sin(x) + xcos(x)$	2.028757838110434
$f_8(x) = x^3 - 10$	2.154434690031884

Table 2: Comparisons between the methods depending on the efficiency index and errors equations

Method	p	m	E.I.= $p^{1/m}$	Error equation (e_{n+1})
N	2	2	1.41421	$c_2 e_n^2 + O(e_n^3)$
H	3	3	1.44225	$(c_2^2 - c_3) e_n^3 + O(e_n^4)$
NR1	2	3	1.25992	$- c_2 e_n^2 + O(e_n^3)$
AN	3	3	1.44225	$(c_2^2 + (1/2)c_3) e_n^3 + O(e_n^4)$
SN	3	4	1.31607	$c_2^2 e_n^3 + O(e_n^4)$
MN	3	3	1.44225	$(c_2^2 -(1/4) c_3) e_n^3 + O(e_n^4)$
HN	3	3	1.44225	$(1/2) c_3 e_n^3 + O(e_n^4)$
GN	3	3	1.44225	$(c_2^2 + c_3) e_n^3 + O(e_n^4)$
CHN	3	3	1.44225	$((1/3) c_2^2 + (1/4) c_3) e_n^3 + O(e_n^4)$
PCH	9	7	1.36874	$(1/512)(8 c_2^2 - c_3)^3 (c_2^2 - c_3) e_n^9 + O(e_n^{10})$
PCNH	18	8	1.43519	$c_2^9(2 c_2^2- c_3)(c_2^2 - c_3)^3 e_n^{18} + O(e_n^{19})$
CTN	3	3	1.44225	$((2/3) c_2^2 +(1/2) c_3) e_n^3 + O(e_n^4)$
PCTNH	9	6	1.44225	$((-1/8)c_3^4+(8/27)c_2^8-(3/8)c_3^3c_2^2-(1/6)c_3^2c_2^4+(10/27)c_3c_2^6) e_n^9 + O(e_n^{10})$

Table 3: Comparisons between the methods depending on the number of iterations (NITER)

f(x)	x_0	Number of Iterations(NITER)												
		known methods											New methods	
		1-step		2-step							3-step		2-step	3-step
		N	H	NR1	AN	SN	MN	HN	GN	CHN	PCH	PCNH	CTN	PCTNH
$f_1(x)$	-1.2	4	3	4	3	3	3	3	3	3	3	2	3	2
	-2	9	5	7	7	6	6	6	6	7	4	3	6	4
	-0.5	110	5	NC	12	9	6	5	7	18	4	4	5	3
$f_2(x)$	1	14	23	7	6	NC	NC	15	NC	NC	6	5	6	4
	2	23	17	71	NC	NC	NC	8	NC	NC	8	5	6	5
	5	42	14	8	9	8	11	10	NC	NC	12	6	7	4
$f_3(x)$	2.5	6	11	7	4	4	4	4	4	4	NC	4	4	3
	-0.5	6	15	6	5	4	4	5	5	4	NC	3	5	4
	1.5	5	8	5	4	4	4	4	4	4	7	3	4	3
$f_4(x)$	0.5	5	5	5	4	4	4	4	4	4	4	2	4	3
	1.5	6	5	5	5	4	5	5	5	5	6	3	5	3
	-0.2	7	5	7	5	4	5	5	5	5	NC	3	5	3
$f_5(x)$	-1.5	33	91	54	32	85	18	29	NC	NC	23	8	8	6
	-3	NC	40	22	NC	NC	33	NC	NC	NC	8	NC	22	5
	3.5	7	5	6	5	5	5	5	5	6	5	3	5	3
$f_6(x)$	1.5	8	5	49	6	6	6	5	5	7	5	3	5	3
	-2.5	21	14	61	8	9	10	9	8	NC	10	6	7	4
	-5	14	13	85	8	9	10	7	15	NC	6	6	8	6
$f_7(x)$	2	4	4	4	3	3	3	3	3	3	4	2	3	3
	3	6	6	7	5	4	5	5	5	5	5	3	5	4
	2.5	5	6	6	4	4	4	4	4	4	6	3	4	3
$f_8(x)$	-1.5	13	18	6	5	7	8	7	6	12	6	8	4	3
	-3	19	9	126	NC	NC	NC	18	13	8	6	5	6	5
	-2	12	6	41	8	6	5	7	6	NC	5	5	7	4

6. Conclusion

In this paper, we proposed two new predictor-corrector type iterative methods. The first one is two-step iterative method with third-order convergence and denoted it (CTN) method, which is a modification of (NR1) method based on (AN) method with the centroid mean. The second one is three-step iterative method with ninth-order convergence and denoted it (PCTNH) method, which is an improvement of (CTN) method by using the technique of updating the solution. Numerical results presented in Table 2 and Table 3 shows that:

1. The efficiency of (CTN) method is better than classical Newton's method, (NR1) method and some other methods. In addition, (CTN) method is better in terms of number of iterations to solve the nonlinear equations than Newton’s method, (NR1) method and some relevant iterative methods.
2. The (PCTNH) method has ninth- order of convergence and efficiency index equals 1.44225, which

makes it competitive with the (PCH) and (PCNH) methods that have ninth and eighteenth order of convergence while their efficiency indices equals 1.36874 and 1.43519 respectively. Therefore, (PCTNH) method is better in terms of efficiency than other methods. Moreover, it has more quickly convergence than Newton's method, (NR1) method, (CTN) method and other methods.

3. The centroid mean is a suitable choice with the trapezoidal integration rule to construction an efficient iterative methods for solving nonlinear equations.
4. Finally, the new proposed methods have better convergence efficiency and performance over those Methods in most cases.

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