

Modified Maximum Likelihood Estimators for One- Way Repeated Measurements Model

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Abstract

In this paper, we will study estimation of variance components in the one-way repeated measurements model (one-way- RMM),by maximization without use numerical methods, there are many methods to estimate parameters of analysis of variance. The difficulty of this estimation increases in unbalanced repeated measurements designs. One of these methods that are frequently used in estimation is the maximum likelihood method, because of the difficulty in finding the roots of the likelihood equations, a modified method has been used for one-way- RMM in the case of two and three levels.

Keywords: one-way-RMM, maximum likelihood function, estimator of variance.

Introduction

Repeated measurements is a term used to describe data in which the response variable for each experimental units is observed on multiple occasions and possible under different experimental conditions. Repeated measures is a common data structure with multiple measurements on a single unit repeated over time. Repeated measurements analysis is widely used in many fields, for example , in the health and life science, epidemiology, biomedical, agricultural, psychological, educational researches and so on[2] Repeated measurements occur frequently in observational studies which are longitudinal in nature, and in experimental studies incorporating repeated measures designs. Vonesh and Chinchilli (1997) discussed linear and nonlinear models for the analysis of repeated measurements . Al-Mouel (2004) studied the multivariate repeated measures models and comparison of estimators. Al-Mouel and Wang (2004) presented the sphericity test for one –way multivariate repeated measurements analysis of variance model. They studied the asymptotic expansion of the sphericity test for one –way multivariate repeated measurements analysis of variance model[2]. Mohaisen and Swadi (2014) studied the Bayesian Estimators of the one- way repeated measurements model[4]. In this paper, we will study estimation of variance components in the one-way repeated measurements model (one-way- RMM),by maximization without use numerical methods, there are many methods to estimate parameters of analysis of variance. The difficulty of this estimation increases in unbalanced repeated measurements designs. One of these methods that are frequently used in estimation is the maximum likelihood method, because of the difficulty in finding the roots of the likelihood equations, a modified method has been used for one-way- RMM in the case of two and three levels.

Maximum likelihood function

Consider the model

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \Upsilon_k + e_{ijk} \quad (1)$$

where

i= 1,2,……,n is an index for experimental unit with group j,
j=1,2,……,q is an index for levels of the between-units factor (Group),
k=1,2,……, p_i is an index for levels of the within- units factor (Time),
y_{ijk} is the response measurement at time k for unit i within group j ,
μ is the overall mean ,
τ_j is the added effect for treatment group j ,
$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,
Υ_k is the added effect for time k ,
e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$$\sum_{J=1}^{q} \tau_j = 0 \quad , \quad \sum_{k=1}^{p} \Upsilon_k = 0$$

And we assume that e_{ijk} 's and $\delta_{i(j)}$'s are independent with

$$e_{ijk} \sim \text{i.i.d } N(0, \sigma_e^2) \ , \quad \delta_{i(j)} \sim \text{i.i.d } N(0, \sigma_\delta^2) \ , \quad (2)$$

The following equation describes the variance of y_{ijk} and the covariance

$$\mathrm{Cov}(y_{ijk}, y_{i'j'k'}) = \begin{cases} \sigma_\delta^2+\sigma_e^2 & \text{for } i = i', j = j', k = k' \\ \sigma_\delta^2 & \text{for } i = i', j = j', k \neq k' \\ 0 & \text{for } i \neq i' \end{cases} \qquad (3)$$

The variance and the covariance matrix are

$$\left.\begin{array}{l} \sigma_e^2 I_{P_i}+\sigma_\delta^2 J_{P_i} = \sigma_e^2(I_{P_i} + \alpha J_{P_i}) \\ \alpha=\frac{\sigma_\delta^2}{\sigma_e^2}\ , \quad J_{P_i} = 1_{P_i}1'_{P_i} \end{array}\right\} \qquad (4)$$

Therefore the level

$$y_i \sim N\big(\mu 1 + \tau 1_q + Y1_p\ , \sigma_e^2(I_{P_i} + \alpha J_{P_i})\big)$$

Then

$$|\sigma_e^2(I_{P_i} + \alpha J_{P_i})| = (\sigma_e^2\ )^{P_i}(1 + P_i\alpha)$$

$$[\ \sigma_e^2(I_{P_i} +\ \alpha J_{P_i}]^{-1} = \frac{1}{\sigma_e^2}(I_{P_i} - \frac{\alpha}{1+P_i\,\alpha}J_{P_i})$$

$\because\ y_i \sim N\big(\mu 1 + \tau 1_q + Y1_p\ , \sigma_e^2(I_{P_i} + \alpha J_{P_i})\big)$

So the maximum likelihood function of the variable y_i is

$$f_i\big(y_i, \mu 1, \tau 1_q, Y1_p, V_i\big)$$

$$= \frac{1}{(\sqrt{2\pi})^{P_i}|V_i|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(y_i - \mu 1 - \tau 1_q - Y1_p)V_i^{-1}(y_i - \mu 1 - \tau 1_q - Y1_p)\right]$$

$$= \frac{1}{(\sqrt{2\pi})^{P_i}\left[(\sigma_e^2\ )^{P_i}(1+P_i\alpha)\right]^{\frac{1}{2}}}$$

$$\exp\left[-\frac{1}{2\sigma_e^2}(y_i - \mu 1 - \tau 1_q - Y1_p)'(I_{P_i} - \frac{\alpha}{1+P_i\,\alpha}J_{P_i})(y_i - \mu 1 - \tau 1_q - Y1_p)\right] \qquad (5)$$

suppose that i=1 the maximum likelihood function of the Y_i can be written as

$$f_1\big(y_1, \mu 1, \tau 1_q, Y1_p, V_1\big) = \frac{1}{(\sqrt{2\pi})^{P_1}\left[(\sigma_e^2\ )^{P_1}(1+P_1\alpha)\right]^{\frac{1}{2}}}$$

$$\exp\left[-\frac{1}{2\sigma_e^2}\sum_{k=1}^{P_1}(y_{ijk} - \bar{y}_{1.})^2 + (\bar{y}_{1.} - \mu)^2(\frac{P_1}{1+P_1\alpha})\right] \qquad (6)$$

if we have $y_1, y_2, \dots, y_n$, the maximum likelihood function of the y_i as

$$L = \frac{1}{(\sqrt{2\pi})^{P_i}\left[(\sigma_e^2\ )^{P_i}(1+P_i\alpha)\right]^{\frac{1}{2}}}$$

$$\exp\left[-\frac{1}{2\sigma_e^2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{P_1}(y_{ijk} - \bar{y}_{i.})^2 + \sum_{i=1}^{n}(\frac{P_i}{1+P_i\alpha})\,(\bar{y}_{i.} - \mu)^2\right] \qquad (7)$$

Estimation of variance components by using modified method

1) When the model contains two levels n=2,

Suppose that $W_i = \frac{P_i}{1+P_i\alpha}$

Therefore the maximum likelihood function of two levels take the following form

$$L = \frac{1}{(\sqrt{2\pi})^{P_1+P_2}\left[(\sigma_e^2\ )^{P_1+P_2}(1+P_1\alpha)+(1+P_2\alpha)\right]^{\frac{1}{2}}}$$

$$\exp\left[-\frac{1}{2\sigma_e^2}\left\{\sum_{i=1}^{2}\sum_{j=1}^{q}\sum_{k=1}^{P_i}(y_{ijk} - \bar{y}_{i.})^2 + \sum_{i=1}^{2}W_i\,(\bar{y}_{i.} - \mu)^2\right\}\right] \qquad (8)$$

Now by taking the log of the function and partial derivation with respect to μ and equal to zero we get

$$\hat{\mu} = \frac{\sum_{i=1}^{2}W_i\,\bar{y}_{i.}}{\sum_{i=1}^{2}W_i} = \frac{W_1\,\bar{y}_1 + W_2\,\bar{y}_2}{W_1+W_2} \qquad (9)$$

Compensate for the value of $\hat{\mu}$ in the function after taking the log of its produce

$$\ln L = -(P_1 + P_2)\ln\sqrt{2\pi} - \frac{P_1+P_2}{2}\ln(\sigma_e^2) - \frac{1}{2}\ln(1 + P_1\alpha) - \ln(1 + P_2\alpha) - Q$$

$$-\frac{1}{2\sigma_e^2}\left[W_1\left(\bar{y}_{1.}\frac{W_1\,\bar{y}_{1.}+W_2\,\bar{y}_{2.}}{W_1+W_2}\right)^2 + W_2\left(\bar{y}_{2.}\frac{W_1\,\bar{y}_{1.}+W_2\,\bar{y}_{2.}}{W_1+W_2}\right)^2\right] \qquad (10)$$

where $Q = -\frac{1}{2\sigma_e^2}\sum_{i=1}^{2}\sum_{j=1}^{q}\sum_{k=1}^{P_i}(y_{ijk} - \bar{y}_{i.})^2$

By simplification of the last factor of the equation produce

$$-\frac{1}{2\sigma_e^2}\frac{W_1\,W_2\,(\bar{y}_{i.}-\bar{y}_{2.})^2}{W_1+W_2} \qquad (11)$$

Note that output is a difference between two means $\bar{y}_{i.} - \bar{y}_{2.}$

Also if simplification $\frac{W_1\,W_2}{\sigma_e^2(W_1+W_2)}$

Equal to the inverse of the variance difference between two means as follows

$$V^{-1}\,(\bar{y}_{i.}-\bar{y}_{2.})=\frac{1}{\sigma_e^2\,(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha)} \quad (12)$$

This is consistent with the known result

$$(\bar{y}_{i.}-\bar{y}_{2.})\ \sim N\left(0\,,\sigma_e^2\ \left(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right)\right) \quad (13)$$

From this result we can say that the modified likelihood function equal to the likelihood function to the difference between two means as follows

$$L\,(\bar{y}_{i.}-\bar{y}_{2.})=\frac{1}{\sqrt{2\pi\,\sigma_e^2\left|\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right|^{\frac{1}{2}}}}\exp\left\{-\frac{1}{2\sigma_e^2\,(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha)}(\bar{y}_{i.}-\bar{y}_{2.})^2\right\} \quad (14)$$

By taking the log of the function and partial derivation with respect to α and equal to zero we get

$$\hat{\alpha}=\left(\frac{(\bar{y}_{1.}-\bar{y}_{2.})}{2\sigma_e^2}-\frac{1}{2P_1}-\frac{1}{2P_2}\right) \quad (15)$$

Properties of Estimator

The possibility that the estimator negative,

Then $\frac{(\bar{y}_{i.}-\bar{y}_{2.})^2}{2\sigma_e^2\left(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right)}\sim\chi^2(1)$

From this the estimator value is negative if

$$F<\frac{\frac{1}{P_1}+\frac{1}{P_2}}{\left(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right)} \quad (16)$$

This shows that if F calculated from the analysis of variance table smaller than the estimator on the right, than the estimator value is negative. Note this possibility is small value because it depends on $\frac{1}{P_i}$ and whenever P_i big approached amount of zero, Meaning it is the possibility that the estimator negative small.

The expected value of the estimator

We can found The expected value of the estimator by using a Taylor series approximations. Taylor series which are as follows

$$f(x,y)=f(\mu_1,\mu_2)+(x-\mu_1)\frac{\partial f(x,y)}{\partial x}\Big|_{\substack{x=\mu_1\\ y=\mu_2}}+(y-\mu_2)\frac{\partial f(x,y)}{\partial y}\Big|_{\substack{x=\mu_1\\ y=\mu_2}} \quad (17)$$

And the expected value of this function is

$$E[f(x,y)]=f(\mu_1,\mu_2) \quad (18)$$

We can represent the amount values of

$$\hat{\alpha}=\left(\frac{(\bar{y}_1-\bar{y}_2)^2}{2\sigma_e^2}-\frac{1}{2P_1}-\frac{1}{2P_2}\right)$$

In values of the equation (17) as follows

$f(x,y)=\hat{\alpha}$, $x=(\bar{y}_1-\bar{y}_2)^2$, $y=\sigma_e^2$

$$f(x,y)=\frac{1}{2}\left(\frac{x}{y}-\frac{1}{P_1}-\frac{1}{P_2}\right) \quad (19)$$

And calculates the required values into the equation (17) can be found Taylor series of the function f(x,y)as will as it is the expected value as in (18) as follows

$$E[f(x,y)]=f(\mu_1,\mu_2)=\frac{1}{2}\left(\frac{\mu_1}{\mu_2}-\frac{1}{P_1}-\frac{1}{P_2}\right)=\frac{1}{2}\left(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha-\frac{1}{P_1}+\frac{1}{P_2}\right)=\alpha$$

$$\therefore\ E(\hat{\alpha})=E[f(x,y)]=\alpha \quad (20)$$

Estimation of variance

by using a Taylor series we can calculated the value of $\hat{\alpha}$ as follows

$$V[f(x,y)]=V(X)\left[\frac{\partial}{\partial x}f\Big|_{\substack{x=\mu_1\\ y=\mu_2}}\right]^2+V(y)\left[\frac{\partial}{\partial y}f\Big|_{\substack{x=\mu_1\\ y=\mu_2}}\right]^2+2cov(x,y)\left[\frac{\partial}{\partial x}f\Big|_{\substack{x=\mu_1\\ y=\mu_2}}\right]\left[\frac{\partial}{\partial y}f\Big|_{\substack{x=\mu_1\\ y=\mu_2}}\right] \quad (21)$$

By using assumptions in equation (19) be the result of variance as follows

$$V[f(x,y)]=\left[\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right]\frac{1}{4\sigma_e^2}+\frac{\mathbf{1}}{\mathbf{2(N-n)}}-\left(\frac{1}{P_1}+\frac{1}{P_2}+2\alpha\right)^{\mathbf{2}} \quad (22)$$

Test of Hypothesis

In this case that we will not find an approximate value of F but you can used Z statistics after it is to find the value of the estimators as follows

$$Z=\frac{\hat{\alpha}-\alpha_0}{\sqrt{\hat{V}(\alpha)}} \tag{23}$$

Estimator output with analysis of variance estimator compared

This is done by comparing the expected value for both the estimators.
The expected value of the modified maximum likelihood estimator by using Taylor series from the equation 11 is α and assum that the analysis variance estimator is $\hat{\alpha}^*$
Which defined as follows

$$\hat{\alpha}^* = \frac{MS_{U(G)}}{h\, MS_e} \quad , h=\frac{\left[N-\frac{\sum_{i=1}^{n} P_i^2}{N}\right]}{(n-1)}$$

To calculated the expected value of $\hat{\alpha}^*$ as it in equation (17) also to calculated the regular values as follows

$$\left.\begin{array}{l} f(x,y)=\hat{\alpha}^* \quad , x=MS_{U(G)} \quad , y=MS_e \\ f(x,y)=\frac{x}{hy}-\frac{1}{h} \end{array}\right\} \tag{24}$$

now can be found Taylor series of the function f(x,y) and the expected value as us in equation 18,

$$f(\mu_1,\mu_2)=\frac{\mu_1}{h\mu_2}-\frac{1}{h}$$

$$E[f(x,y)]=\frac{h\sigma_\delta^2+\sigma_e^2}{h\sigma_e^2}$$

Therefore

$$E(\hat{\alpha}^*)=E[f(x,y)]=\alpha \tag{25}$$

Note of the results that have been obtained that the expected value of the output is destined by using modified maximum likelihood method approximately equal to the expected value of the output estimator by using analysis variance method.

2)when the model contains three levels n=3,

This case is doen by maximum likelihood type estimators, then we can use Helmert matrix to estimate the variance components when n=3 then the size of H-matrix is (3×3) as follows

$$H=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \tag{26}$$

By using this matrix we can defined the modified maximum likelihood function which contains two variables Z_1, Z_2 follow a normal distribution,

$$Z_1=\frac{\bar{y}_{1.}-\bar{y}_{2.}}{\sqrt{2}} \quad , \quad Z_2=\frac{\bar{y}_{1.}+\bar{y}_{2.}-2\bar{y}_{3.}}{\sqrt{6}} \quad \text{and} \quad E(Z_1)=E(Z_2)=0$$

Where the variance of Z_1 , Z_2 respectively

$$\sigma_e^2\left(\frac{1}{2P_1}+\frac{1}{2P_2}+\alpha\right) \quad , \sigma_e^2\left(\frac{1}{6P_1}+\frac{1}{6P_2}+\frac{4}{6P_3}+\alpha\right)$$

Also the covariance between Z_1 , Z_2 are

$$Cov(Z_1, Z_2)=E(Z_1, Z_2)=E\left[\left(\frac{\bar{y}_{1.}-\bar{y}_{2.}}{\sqrt{2}}\right)\left(\frac{\bar{y}_{1.}+\bar{y}_{2.}-\overline{2y}_{3.}}{\sqrt{6}}\right)\right]$$
$$=\frac{1}{\sqrt{12}}\sigma_e^2\left(\frac{1}{P_1}-\frac{1}{P_2}\right)$$

If n is large can be negligted the covariance or add and examine the difference between the following two cases,

Case 1: In the case of the negligence of the covariance the modified maximum likelihood function in this case is the multiply between Z_1 and Z_2 as follows

$$L=\frac{1}{(2\pi)^{\frac{1}{2}}[V_1 V_2]^{\frac{1}{2}}}\exp\left[\frac{Z_1^2}{V_1}+\frac{Z_2^2}{V_2}\right]$$

$$=\frac{1}{\sigma_e^2\,(2\pi)^{\frac{1}{2}}\left[\left(\frac{1}{2P_1}+\frac{1}{2P_2}+\alpha\right)\left(\frac{1}{6P_1}+\frac{1}{6P_2}+\frac{4}{6P_3}+\alpha\right)\right]^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{Z_1^2}{\sigma_e^2\left(\frac{1}{2P_1}+\frac{1}{2P_2}+\alpha\right)}+\frac{Z_2^2}{\sigma_e^2\left(\frac{1}{6P_1}+\frac{1}{6P_2}+\frac{4}{6P_3}+\alpha\right)}\right)\right]$$

$$L = \frac{1}{\sigma_e^2\ (2\pi)^{\frac{1}{2}}[(t_1+\alpha)(t_2+\alpha)]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\frac{Z_1^2}{\sigma_e^2\ (t_1+\alpha)} + \frac{Z_2^2}{\sigma_e^2\ (t_2+\alpha)}\right)\right], \quad (27)$$

Where $t_1 = \frac{1}{2P_1} + \frac{1}{2P_2}$, $t_2 = \frac{1}{6P_1} + \frac{1}{6P_2} + \frac{4}{6P_3}$

By taking the log of the function and partial derivation with respect to α and equal to zero we get

$$\frac{1}{2(t_1+\hat{\alpha})} - \frac{1}{2(t_2+\hat{\alpha})} + \frac{Z_1^2}{2\hat{\sigma}_e^2\ (t_1+\hat{\alpha})^2} + \frac{Z_2^2}{2\hat{\sigma}_e^2\ (t_2+\hat{\alpha})^2} = 0 \quad (28)$$

After simplifying the equation we get the following

$$\hat{\alpha}^3 + \left[\frac{3}{2}(t_1+t_2) - \frac{1}{2}\left(\frac{Z_1^2}{\hat{\sigma}_e^2} + \frac{Z_2^2}{\hat{\sigma}_e^2}\right)\right]\hat{\alpha}^2 + \left[2t_1t_2 + \frac{1}{2}t_2^2 + \frac{1}{2}t_1^2 - \frac{Z_1^2}{\hat{\sigma}_e^2}t_2 - \frac{Z_2^2}{\hat{\sigma}_e^2}t_1\right]\hat{\alpha} +$$
$$\left[\frac{1}{2}t_1^2\,t_2 + \frac{1}{2}t_2^2\,t_1 - \frac{Z_1^2}{2\hat{\sigma}_e^2}t_2^2 - \frac{Z_2^2}{2\hat{\sigma}_e^2}t_1^2\right] = 0 \quad (29)$$

This equation of the third degree means the estimators $\hat{\alpha}$ has three roots,
So there are four possible solutions as follows:

i. All the roots be real different,
ii. One real and another pair complex,
iii. Be real root repeater,
iv. Be real roots, one of which duplicate twice and another different,

Case 2 :

In the case of a covariance we can write the covariance matrix of Z_1, Z_2 as follows

$$\Sigma = \begin{bmatrix} \sigma_e^2\left(\frac{1}{2P_1} + \frac{1}{2P_2} + \alpha\right) & \frac{1}{\sqrt{12}}\sigma_e^2\left(\frac{1}{P_1} - \frac{1}{P_2}\right) \\ 0 & \sigma_e^2\left(\frac{1}{6P_1} + \frac{1}{6P_2} + \frac{4}{6P_3} + \alpha\right) \end{bmatrix} \quad (30)$$

And the likelihood function of Z_1, Z_2 as follows

$$L = \frac{1}{(\sqrt{2\pi})^2|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2|\Sigma|}(z_1^2V(Z_2) + (z_2^2V(Z_1) - 2Z_1Z_2\mathrm{cov}(Z_1Z_2)\right]$$

If we suppose that

$t_1 = \frac{1}{2P_1} + \frac{1}{2P_2}$, $t_2 = \frac{1}{6P_1} + \frac{1}{6P_2} + \frac{4}{6P_3}$, $t_3 = \frac{1}{2P_1} - \frac{1}{2P_2}$,

The likelihood function as follows

$$L = \frac{1}{(\sqrt{2\pi})^2[\sigma_e^4\ (t_1+\alpha)(t_2+\alpha) - \frac{\sigma_e^4}{12}t_3^2} \exp\left[-\frac{\sigma_e^2}{2|\Sigma|}\left\{Z_1^2(t_2+\alpha) + Z_2^2(t_1+\alpha) - \frac{2}{\sqrt{12}}Z_1Z_2t_3\right\}\right] \quad (31)$$

By taking the log of the function and partial derivation with respect to α and equal to zero and assum that $Q = \left[(t_1+\alpha)(t_2+\alpha) - \frac{1}{12}t_3^2\right]$ we get

$$-\frac{(t_1+\hat{\alpha})+(t_2+\hat{\alpha})}{2Q} - \frac{\sigma_e^2}{2\sigma_e^4\ Q^2}\left\{[Q(z_1^2+z_2^2] - \left[z_1^2(t_2+\hat{\alpha}) + z_2^2(t_1+\hat{\alpha}) - \frac{2}{\sqrt{12}}Z_1Z_2t_3\right]\right\}$$
$$(t_1+\hat{\alpha}) + (t_2+\hat{\alpha}) = 0 \quad (32)$$

After simplifying the equation produces equation of the third degree but it is more complexity than the previous case.

2) When the model contains four levels n=4,

when n=4 then the size of H-matrix is (4×4) as follows

$$H = \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} \end{bmatrix}$$

The resulting function has a third variables Z_3 in addition to the previous two variables Z_1 and Z_2,

$Z_3 = \frac{\bar{y}_{1.}+\bar{y}_{2.}+\bar{y}_{3.}-3\bar{y}_{4.}}{\sqrt{12}}$

Where Z_3 is random variable it follows anormal distribution with mean zero and variance ($\frac{1}{12P_1} + \frac{1}{12P_2} + \frac{1}{12P_3} + \frac{9}{12P_4} + \alpha$),

The expected and covariance Z_1 , Z_2 remains the same as the pervious case with neglecting the covariance .
the modified maximum likelihood function as follows

$$L=f(Z_1, 0, V(Z_1))f(Z_2,0,V(Z_2))\, f(Z_3,0,V(Z_3)) = \frac{1}{(\sqrt{2\pi})^3[V_1(Z_1)V_2(Z_2)V_3(Z_3)]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left\{\frac{Z_1^2}{V(Z_1)}+\frac{Z_2^2}{V(Z_2)}+\frac{Z_3^2}{V(Z_3)}\right\}\right]$$

$$= \frac{1}{(\sqrt{2\pi})^3\hat{\sigma}_e^4\,[(t_1+\alpha)(t_2+\alpha)(t_3+\alpha)]^{\frac{1}{2}}} \exp\left[-\frac{1}{2\hat{\sigma}_e^2}\left\{\frac{Z_1^2}{(t_1+\alpha)}+\frac{Z_2^2}{(t_2+\alpha)}+\frac{Z_3^2}{(t_3+\alpha)}\right\}\right]$$

$$t_3=\left(\frac{1}{12P_1}+\frac{1}{12P_2}+\frac{1}{12P_3}+\frac{9}{12P_4}\right), \quad (33)$$

By taking the log of the function and partial derivation with respect to α and equal to zero we get

$$(t_1+\hat{\alpha})(t_2+\hat{\alpha})^2(t_3+\hat{\alpha})^2+(t_1+\hat{\alpha})^2(t_2+\hat{\alpha})(t_3+\hat{\alpha})^2+(t_1+\hat{\alpha})^2(t_2+\hat{\alpha})^2(t_3+\hat{\alpha})$$

$$-\frac{Z_1^2}{\hat{\sigma}_e^2}(t_2+\hat{\alpha})^2(t_3+\hat{\alpha})^2-\frac{Z_2^2}{\hat{\sigma}_e^2}(t_1+\hat{\alpha})^2(t_3+\hat{\alpha})^2)-\frac{Z_3^2}{\hat{\sigma}_e^2}(t_1+\hat{\alpha})^2(t_2+\hat{\alpha})^2)=0 \quad (34)$$

After simplifying the equation produces equation of the fifth degree ,then there is five roots of the estimator α . But the solution of this equation by numerical methods (frequently methods),so if we continue to add other levels this will increase the degree of resulting equation two degrees from the previous equation that contains the levels less than one .

5. Conclusion

The conclusions which are obtained throughout this work are given as follows :

1.The solution of the likelihood equations, a modified method has been used for one-way unbalanced repeated measurements model (one-way- RMM),by maximization without use numerical methods,

2. When the model contains two levels n=2,be the equation of a first degree, and when the model contains three levels n=3, be the equation of a second degree. and so on.

3.If we continue to add other levels this will increase the degree of resulting equation two degrees from the previous equation that contains the levels less than one .

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