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Convex Regularization Method for Solving Cauchy Problem of the Helmholtz Equation

Benedict Barnes[1], E. Osei-Frimpong[2], J. Ackora-Prah[3] and S. K. Amponsah[4]
[1,2,3,4] Kwame Nkrumah University of Science and Technology
Department of Mathematics
Kumasi-Ghana
[1] Corresponding Author
[1]bbarnes.cos@knust.edu.gh

Abstract

In this paper, we introduce the Convex Regularization Method (CRM) for regularizing the (instability) solution of the Helmholtz equation with Cauchy data. The CRM makes it possible for the solution of Helmholtz equation to depend continuously on the small perturbations in the Cauchy data. In addition, the numerical computation of the regularized Helmholtz equation with Cauchy data is stable, accurate and gives high rate of convergence of solution in Hilbert space. Undoubtedly, the error estimated analysis associated with CRM is minimal.

Mathematics Subject Classification: 44B28; 44B30

Keywords: Convex Regularization Method, ill-posed Helmholtz equation with Cauchy data, stable solution

1 Introduction

Quite recently, ill-posed problems have become the concern of mathematicians and scientists in general. The Helmholtz equation with Cauchy data comes from readings given by physical instrument, which are always limited to some level of accuracy. For this reason, the continuous dependence of solution of the Helmholtz equation with Cauchy data is not guaranteed. Thus, the numerical computation of such problem is distorted, and gives unreasonable approximation as Cauchy data is being perturbed.

Due to the pioneering work by [19], who introduced Tikhonov Regularization Method (TRM), which assumes that there is a Laplace-type linear bounded operator

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in the Helmholtz equation from a Hilbert space X to another Hilbert space Y. This method regularizes the Laplace-type operator occuring in the Helmholtz equation by combining two minimal conditions: the Gauss Least-Squares Method (LSM) and the Moore-Penrose Pseudo-inverse Matrix Method (PIMM), see [13] . The Lagrange method of undetermined multipliers is then applied to the combined condition to obtain

$$\|Aw(x,y) - f(x)\|_Y^2 + \alpha\|Cw(x,y)\|_Y^2 = \min_{x,y},$$

where $\alpha > 0$ is a regularization parameter, is called Euler-Tikhonov equation. The above equation is minimized with respect to spatial variable y to obtain the regularized solution as:

$$w_\alpha(x,y) = (A^*A + \alpha I)^{-1}A^*f(x),$$

where

$$C^*C = I,$$

I is a unit operator from a Hilbert X to another Hilbert space Y and $w_\alpha(x,y)$ is the regularized solution. The TRM has gained populace as authors in [8] have applied it to regularize the homogeneous Helmholtz equation. In [20], they dicussed the applicability of the TRM. Also, a closely related method to TRM was given by author in [26]. The authors in [22], introduced Spectral Regularization Method SRM which generalizes the TRM for different range of values of regularization parameter α. The regularization parameter in SRM depends on the product of the Laplace-type operator and minimized Laplace-type operator in the Helmholtz equation.

The Quasi-Reversibility Regularization Method (Q-RRM) assumes that the Laplace-type operator in the Helmholtz equation is bijective, but its inverse operator A^{-1} is not continuous from a Hilbert space Y to another Hilbert space X. In this method, the Helmholtz equation is regularized by subtracting a product of square of regularization parameter and a mixed four-order partial derivatives from the Laplace-type operator. That is,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \alpha^2\frac{\partial^4}{\partial x^2 \partial y^2},$$

see [10]. Again, the solution of the Helmholtz equation can be regularized by Quasi-Boundary Value Method (Q-BVM). This method assumes that the Laplace-type operator in the Helmholtz equation is self-adjoint and unbounded from a Hilbert space X into another Hilbert space Y, [21]. In Q-BVM assumes that a unique solution exists for the Helmholtz equation, but this solution does not depend continuously on the small changes in the Cauchy data. Also, the method of fundamental solution faces similar challenges [25].

The authors in [11] introduced an Alternating Iterative Regularization Method (AIRM) which regularizes the solution of the Helmholtz equation with Cauchy data. Since then a number of works have obtained different results with different iterative schemes, [2, 7, 23]. One particular result which has drawn much attention and of interest is the Iterative Regularization Method (IRM) by authors in [3]. In their work, the iterative scheme

$$\lambda = \lambda(\xi) = e^{-\sqrt{(|\xi|^2 - k^2)}} < 1,$$

where $\xi \in \mathbf{R}$ is the frequency component, and $k \in \mathbf{I}^+$ is the wave number, is introduced into an instability solution of the Helmholtz equation together with Cauchy data.

In most practices, we observed that

$$\|\xi^2 - k^2\| \to \infty,$$

in the iterative scheme, the solution of the Helmholtz equation with Cauchy data grows exponentially. This is due to the fact that as

$$\|\xi^2 - k^2\| \to \infty,$$

the iterative scheme λ approaches zero, and the regularized solution, by IRM, approaches unstable solution of the Helmholtz equation. Thus, the IRM fails to regularize unstable solution of the Helmholtz equation with Cauchy data. In addition, an IRM includes the regularization parameter α as well as the number of iterations m in its error analysis. The instability of solution of the Cauchy problem of the Helmholtz equation cannot be restored by the IRM when the modulus of the difference between the squares of the frequency component and the wave number of the iterative scheme tend to infinity.

In similar related methods, the authors in [1, 4, 9, 14, 16, 18, 20] have either introduced or applied methods of regularization for solving the Cauchy problem of the Helmholtz equation in suitable functional spaces.

In this paper, we show that the solution of the Helmholtz equation with Cauchy data is ill-posed in the sense of Hadamard. Thus, the solution of the Helmholtz equation does not depend on the small perturbations in the Cauchy data. The Cauchy problem for the Helmholtz equation is as follows:

$$\Delta w(x,y) + k^2 w(x,y) = 0, \qquad 0 < x \leq 1, \quad y \in \mathbf{R}$$

$$\begin{aligned} w(0,y) &= \phi(y), \quad y \in \mathbf{R} \\ w_x(0,y) &= 0, \quad y \in \mathbf{R} \end{aligned} \tag{1}$$

where $\phi(y)$ is the initial data and k is the wave number. By the method of separation of variables, we obtain

$$w(x,y) = \phi(y)\cosh(x\sqrt{(y^2-k^2)}) \tag{2}$$

By finding the norm of equation (2), we obtain

$$\begin{aligned} \|w(x,y)\| &= \|\phi(y)\cosh(x\sqrt{(y^2-k^2)})\| \\ \|w(x,y)\| &\le |\phi(y)|\|\cosh(x\sqrt{(y^2-k^2)})\|. \end{aligned}$$

We observe that $\|\cosh(x\sqrt{(y^2-k^2)})\|$ in the above equation increases exponentially for $y > k$. A small perturbation in the initial data $\phi(y)$ results in a large growth in the solution $w(x,y)$ without any bound. Thus, the third condition of the well-posedness according to Hadamard is violated. Hence, equation (1) together with Cauchy data is ill-posed.

2 The Main Results

In this section, we introduce the CRM for regularizing ill-posed Helmholtz equation with Cauchy data. By CRM, we assume that the unique solution exists for the Cauchy problem of the Helmholtz equation, but the solution does not depend continuously on the Cauchy data. A partial differential operator $L(w(x,y)) : \Omega \subset H \to \mathbf{R}$ is linear if,

$$L(\alpha w_1(x,y) + \beta w_2(x,y)) = \alpha L(w_1(x,y)) + \beta L(w_2(x,y)),\ \forall w_1(x,y), w_2(x,y) \in \Omega,\ \alpha, \beta \in \mathbf{R}$$

and the $L(w(x,y))$ is bounded if and only if

$$\|L(w(x,y))\| \le C\|w(x,y)\|_\Omega,\ \ \forall w(x,y) \in \Omega,\ C > 0,$$

see [17].

The bounded inverse theorem guarantees the existence of inverse partial differential operator A^{-1} from a Hilbert space H to a subHilbert space $\Omega \in H$.

Theorem 2.1 (Bounded Inverse Theorem) *Let A be a bounded linear below Laplace-type operator in the Helmholtz equation from a subspace Ω in a Hilbert space H into a Hilbert space H. Then A has a continuous inverse* *operator A^{-1} from its range $R(A)$ into Ω. Conversely, if there is a continuous inverse operator*

$$A^{-1} : R(A) \to \Omega,$$

then there is a positive constant C such that

$$\|Aw(x,y)\|_H \geq C\|w(x,y)\|_\Omega, \quad \forall \ w(x,y) \in \Omega$$

see [12].

By dint of the CRM, we reformulate the Fourier transform of equation (2) as a convex function with respect to ξ and then iterate the resulting equation m number of times to obtain the desired result. Let

$$\hat{w}(1,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi}\phi(y)\cosh(\sqrt{(y^2-k^2)})dy, \qquad i = \sqrt{-1}. \tag{3}$$

and

$$\|\phi^\delta(\xi) - \phi(\xi)\| \leq \delta, \tag{4}$$

where, $\phi(y)$ and $\phi^\delta(y)$ are the exact and measured data for equation (3), respectively. The noise level is denoted by $\delta > 0$, $\|.\|$ denotes the L^2-norm and $w(1,\xi)$ is the Fourier transform of $w(1,y)$.

Thus, we assume that $w(1,\xi) \in L^2(\mathbf{R})$ for all $0 < x \leq 1$

$$\|w(1,\xi)\| \leq E \tag{5}$$

Definition 2.2 *Let X be a convex subset of vector space V. We say that $f : X \to \mathbf{R}$ is convex, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in X$ and $\lambda \in (0,1)$, see [5].

In order to regularize unstable solution of the Helmholtz equation with Cauchy data, we modify definition (3.1) to fractional scalar as follows.

Lemma 2.3 *Let X be a convex subset of vector space V. We say that $f : X \to \mathbf{R}$ is convex, if*

$$f\Big((1-\alpha)x + (\frac{2\alpha}{1+\alpha^2})y\Big) \leq +(1-\alpha)f(x) + \frac{2\alpha}{(1+\alpha^2)}f(y)$$

for all $x, y \in X$ and $\alpha \in (0,1)$.

By the CRM, we substitute the inequality in lemma (3.2) into equation (2) as follows:

$$\begin{aligned}
\hat{w}^{\delta}(1,\xi) &= \phi((1-\alpha)\xi_1 + (\frac{2\alpha}{1+\alpha^2})\xi_2)\cosh\left(\sqrt{([(1-\alpha)\xi_1^2 + (\frac{2\alpha}{1+\alpha^2})\xi_2^2] - k^2)}\right)\\
\hat{w}^{\delta}(1,\xi) &\le (1-\alpha)\cosh(x\sqrt{|\xi_1|^2 - k^2})\phi^{\delta}(\xi_1) + \frac{2\alpha}{(1+\alpha^2)}\cosh(x\sqrt{|\xi_2|^2 - k^2})\phi^{\delta}(\xi_2)\\
\hat{w}^{\delta}(1,\xi) &= (1-\alpha)^m \hat{w}_0^{\delta}(x,\xi_1) + \sum_{j=0}^{m-1}(1 - \frac{2\alpha}{(1+\alpha^2)})^j . \frac{2\alpha}{(1+\alpha^2)}\cosh(x\sqrt{|\xi|^2 - k^2})\phi^{\delta}(\xi_2)\\
\hat{w}^{\delta}(1,\xi) &= (1-\alpha)^m \hat{w}_0^{\delta}(x,\xi_1) + (1 - (1 - \frac{2\alpha}{1+\alpha^2})^m)\cosh(x\sqrt{|\xi_2|^2 - k^2})\phi^{\delta}(\xi_2),
\end{aligned}$$

where, $\xi_1,\ \xi_2 \in \mathbf{R}$ and $\alpha \in (0,1)$ is the regularization parameter and m is the number of iterations.

Thus,

$$\hat{w}_m^{\delta}(1,\xi) = \begin{cases} \cos(\sqrt{k^2 - |\epsilon|^2})\phi(\xi), & |\xi| \le k \\ (1-\alpha)^m \hat{w}_0^{\delta}(x,\xi_1) & \\ +(1-(1-\frac{2\alpha}{1+\alpha^2})^m)\cosh(x\sqrt{|\xi|^2 - k^2})\phi^{\delta}(\xi_2) & |\xi| \ge k \ and\ m > 1 \end{cases} \tag{6}$$

where,

$$\hat{w}_0^{\delta}(1,\xi) = \phi^{\delta}(\xi)\cosh(\sqrt{|\xi|^2 - k^2}), \qquad for\ \ m = 1 \quad and \ \ \alpha = 0.$$

is the initial approximation for equation (6). We provide the estimated error associated with equations (6) and (2) as follows:

Theorem 2.4 (Error of Estimation) *Suppose that*

$$w(x,y) = \phi(y)\cosh(x\sqrt{(|\xi|^2 - k^2)})$$

is the exact solution to equation (1) *and*

$$\hat{w}_m^{\delta}(1,\xi) = \begin{cases} \cos(\sqrt{k^2 - |\epsilon|^2})\phi(\xi), & |\xi| \le k \\ (1-\alpha)^m \hat{w}_0^{\delta}(1,\xi) & \\ +(1-(1-\frac{2\alpha}{1+\alpha^2})^m)\cosh(\sqrt{|\xi|^2 - k^2})\phi^{\delta}(\xi) & |\xi| \ge k \ \ m = 2,3,\dots \end{cases}$$

where

$$\hat{w}_o^{\delta}(1,\xi) = \phi^{\delta}(\xi)\cosh(\sqrt{|\xi|^2 - k^2}), \qquad for\ \ m = 1 \quad and \ \ \alpha = 0$$

be its regularized solution. Assume further that equations (4) *and* (5) *hold, if* $\alpha = (\frac{E}{\delta})$ *is chosen as a regularization parameter, then the estimated error is*

$$\begin{aligned} \|w(1,\xi) - w_m^{\delta}(1,\xi)\| &\leq E(\frac{E^2+\delta^2-E\delta}{E^2+\delta^2})^m + (\frac{\delta-E}{\delta})^m E \\ &+ \delta^2(1-(\frac{2(E^2+\delta^2-E\delta)}{E^2+\delta^2})^m)E^{-1} \end{aligned} \tag{7}$$

Proof: Considering $|\xi| \leq 0$, we obtain

$$\begin{aligned} w_0(1,\xi) &= \cos(\sqrt{k^2-|\xi|}) \\ \|w_0(1,\xi)\| &= \|\cos(\sqrt{k^2-|\xi|})\| \\ \|w_0(1,\xi)\| &\leq 1 \end{aligned}$$

But for $|\xi| > k$, we obtain

$$\begin{aligned} \|w(1,\xi) - w_m^{\delta}(1,\xi)\| &= \|\hat{w}(1,\xi) - \hat{w}_m^{\delta}(1,\xi)\| \\ \|w(1,\xi) - w_m^{\delta}(1,\xi)\| &= \|\phi(\xi)\cosh(\eta) \\ &- [(1-\alpha)^m \hat{w}_o^{\delta}(1,\xi) + (1-(1-\frac{2\alpha}{1+\alpha^2})^m)\cosh(\sqrt{|\xi|^2-k^2})\phi^{\delta}(\xi)]\| \\ \|w(1,\xi) - w_m^{\delta}(1,\xi)\| &\leq \|\phi(\xi)\cosh(\eta)(1-\frac{2\alpha}{1+\alpha^2})^m\| + \|(1-\alpha)^m \hat{w}_0^{\delta}\| \\ &+ \|(1-(1-\frac{2\alpha}{1+\alpha^2})^m)\alpha^{-1}(\phi(\xi)-\phi^{\delta}(\xi))\| \\ \|w(1,\xi) - w_m^{\delta}(1,\xi)\| &\leq E(\frac{E^2+\delta^2-E\delta}{E^2+\delta^2})^m + (\frac{\delta-E}{\delta})^m E \\ &+ \delta^2(1-(\frac{2(E^2+\delta^2-E\delta)}{E^2+\delta^2})^m)E^{-1} \end{aligned}$$

We show that the CRM converges to a point in $((0,1]\times\mathbf{R})$. Thus, we prove that the regularized solution converges for $|\xi| \leq k$. By Parseval identity, we obtain

$$\begin{aligned} \|w_{m,1}^{\delta}(1,y) - w_{m,2}^{\delta}(1,y)\| &= \|\hat{w}_{m,1}^{\delta}(1,y) - \hat{w}_{m,2}^{\delta}(1,y)\| \\ \|w_{m,1}^{\delta}(1,y) - w_{m,2}^{\delta}(1,y)\| &= \|\cos(\sqrt{(k^2-|\xi|^2)})(\phi_1^{\delta}-\phi_2^{\delta})\| \\ |w_{m,1}^{\delta}(1,y) - w_{m,2}^{\delta}(1,y)\| &\leq \delta \end{aligned}$$

Also, we observed that when $|\xi| > k$, we obtain

$$\begin{aligned} \|w_{m,1}^{\delta}(1,y) - w_{m,2}^{\delta}(1,y)\| &= \|\hat{w}_{m,1}^{\delta}(1,y) - \hat{w}_{1,2}^{\delta}(1,y)\| \\ \|w_{m,1}^{\delta}(1,y) - w_{m,2}^{\delta}(1,y)\| &= \|[(1-\alpha)^m \hat{w}_0^{\delta}(1,\xi) + (1-(1-\frac{2\alpha}{1+\alpha^2})^m)\cosh(\eta)](\phi_{\alpha,1}^{\delta}-\phi_{\alpha,2}^{\delta})\| \\ \|w_{\alpha,1}^{\delta}(x,y) - w_{\alpha,2}^{\delta}(x,y)\| &\leq \delta A(\xi) + \delta B(\eta), \end{aligned} \tag{8}$$

where,

$$\begin{aligned} A(\xi) &= \sup_{m\geq 1} \|(1-\alpha)^m \hat{w}_0^\delta(1,\xi)\| \\ A(\xi) &\to 0 \quad and \ \ (1-\alpha)^m \to 0 \ \ as \ \ m \to \infty \end{aligned} \tag{9}$$

and

$$\begin{aligned} B(\eta) &= \|(1-(1-\frac{2\alpha}{1+\alpha^2})^m)\cosh(\eta)\| \\ B(\eta) &\to 0 \ \ and \ \ (1-(1-\frac{2\alpha}{1+\alpha^2})^m) \to 0 \ \ as \ \ m \to \infty \end{aligned} \tag{10}$$

Substituting equations (10) and (9) into equation (8), we obtain

$$\|w_{m,1}^\delta(1,y) - w_{m,2}^\delta(1,y)\| \to 0 \quad as \quad m \ \to \infty.$$

3 Results and Discussion

In this section, we the compare exact solution with solution by the IRM and finally, the solution by CRM. In figure 1, we display the quantitative solution of equation (2). We can see from figure 1, the solution decreases sharply from approximately -0.25 to zero and grows very fast from this point to approximately 0.25. This is an indication of unstability of the exact solution of the Helmholtz equation with Cauchy data

In figure 2, we display the regularized solutions by the IRM (green solid graph) and by the CRM (red asterisk graph). In addition, we observed that ξ approaches zero, the green solid graph moves further away whereas the red asterisk graph draws close to this point. This indicates that the regularized solution by CRM is more stable than the regularized solution by the IRM for $-1 \leq \xi \leq 1$. In addition, we observed that the regularized solution by IRM approaches exact solution (ill) as the frequency content ξ is equal to wave number k. Similar trend is observed when $|\xi^2 - k^2| \to \infty$. Similar regularized solutions are displayed in figure 3, for $m = 4$.

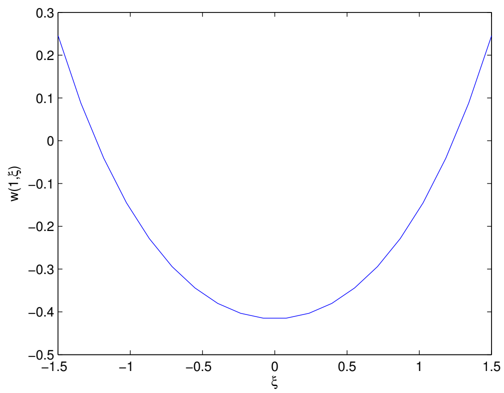

Figure 1: Solution by the classical method in one dimension

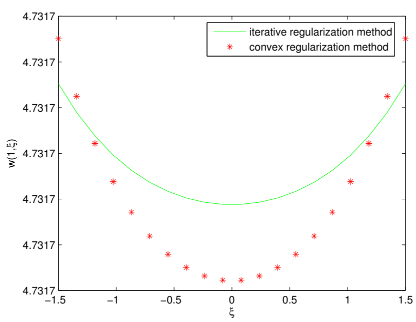

Figure 2: Comparison of IRM and CRM for $m = 5$

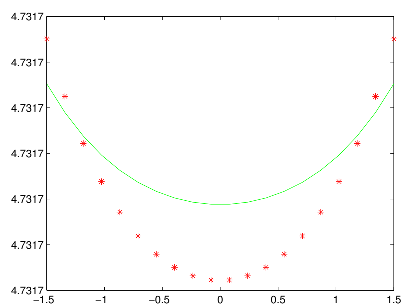

Figure 3: Comparison of IRM and CRM for $m = 4$

4 Conclusion

We observed that the regularized solution by the CRM is more stable as compared to the IRM and others. Unlike the IRM and its variants, the CRM restores the stability of the solution of the Helmholtz equation with Cauchy data as $|\xi^2 - k^2| \to \infty$ or ξ is equal to k. Last but not least, the estimated error associated with CRM is mininal as compared to other methods of regularization.

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