Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.12, 2016

Transient Analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ Queueing Model with Retrial Priority Service, Negative Arrival, Two kinds of Vacations, Breakdown, Delayed Repair, Balking, Reneging and Feedback

G.Ayyappan*, P.Thamizhselvi
Department of Mathematics, Pondicherry Engineering College,
Puducherry-605 014, India. ayyappan@pec.edu, tmlselvi972@gmail.com

Abstract

This paper deals with the analysis of batch arrival retrial queue with two classes non-preemptive priority units, negative arrival, balking as well as reneging, feedback, emergency and Bernoulli vacation for an unreliable server. Here we assume that customers arrive according to compound Poisson process in which priority customers are assigned to class one and class two customers are of a low-priority type. If the server is free at the time of any batch arrivals, the customers of this batch begins to be served immediately. The low-priority customer may join the orbit with feedback if the service is not satisfied (or) may leave the system if the service is satisfied. The priority customers that find the server busy are queued and then served in accordance with FCFS discipline. The priority customers may renege the queue if the server is not avilable in the system and there is no optional for feedback service to the priority customers. The arriving low-priority customers on finding the server busy then they are queued in the orbit in accordance with FCFS retrial policy without balking (or) may balk the orbit. While the server is serving to the customers, it faces two types of break-down there are breakdowns by the arrival of negative customer and break-down at any instant of service and server will be down for a short interval of time. Further concept of the delay time of repair is also introduced for breakdowns. We consider two different kinds of vacations, one is an emergency and the other one is Bernoulli vacation, the emergency vacation means at the time of the server serving the customer suddenly go for a vacation and the interrupted customer waits to get the remaining service and after the completion of each service, the server either goes for a vacation or may continue to serve for the next customer; if any . The retrial time, service time, vacation time, delay time and repair time are all follows general(arbitrary) distribution. Finally, we obtain some important performance measures of this model.
Keywords:
Batch arrival, priority queue, retrial queue, negative arrival, emergency and Bernoulli vacation, unreliable server, breakdown and repair.
AMSC: $60\mathbf{K}25, 60\mathbf{K}30, 90\mathbf{B}22$

1 Introduction

The study on queuing models have become an indispensable area due to its wide applicability in real life situations, all the models considered have had the property that units proceed to service on a first-come, first-served basis. This is obviously not only the manner of service, and there are many alternatives, such as last-come, first-served, selection in random order and selection by priority. In order to offer different quality of service for different kinds of customers, we often control a queueing system by priority mechanism. This phenomenon is common in pratice. For example, in telecommunication transfer protocol, for guaranteeing different layers service for different customers, priority classes control may appear in header of IP package or in ATM cell. Priority control is also widely used in production practice, transportation management.

Retrial queues are characterized by the feature that arriving customers who find the server is busy then join the retrial group to try their luck again after a time period. Queues in which customers are allowed to conduct retrials have been extensively used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processor unit.

Several authors discussed the single arrival and batch arrival retrial queueing systems with priority service. Ayyappan et al. (2014) have studied a transient behavior of $M^{[X]}/G/1$ retrial queueing model with non persistence customers, random breakdown, delaying repair and Bernoulli vacation. Atencia et al (2005) have studied a single-server retrial queue with general retrial times and Bernoulli schedule, Gautam Choudhury et al (2012) have studied a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, Gomez Corral (1999) have studied a stochastic analysis of a single server retrial queue with general retrial times, Jain et al (2008) have studied a bulk arrival retrial queue with unreliable server and priority subscribers, Chuan Ke et al (2009) have studied a modified vacation policy for $M/G/1$ retrial queue with balking and feedback, Jinbiao Wu et al (2013) have studied a single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. Madan et al (2011) have studied an $M^{[X]}/G/1$ queue with Bernoulli schedule general vacation times, general extended vacation, random breakdown, general delay times for repairs to start and general repair times, Udaya Chandrika et al (2010) and (2014) have studied a single-server retrial queueing system with two different vacation policies and batch arrival retrial G-queue and an unreliable server with delayed repair, Yang et al.(1987) and (1994) have studied a survey on retrial queue and an approximation method for the $M/G/1$ retrial queue with general retrial times.

Here we deals with the analysis of batch arrival retrial queue with two classes of non-preemptive priority units, negative arrival, balking as well as reneging, feedback, emergency and Bernoulli vacation for an unreliable server. Here we assume that customers arrive according to compound Poisson process in which priority customers are assigned to class one and class two customers are of a low-priority type. If the server is free at the time of any batch arrivals, the customers of this batch begins to be served immediately. The low-priority customer may join the orbit with feedback if the service is not satisfied(or) may leave the system if the service is satisfied. The priority customers that find the server busy are queued and then served in accordance with FCFS discipline. The priority customers may renege the queue if the server is not available in the system and there is no optional for feedback service to the priority customers. The arriving low-priority customers on finding the server busy then they are queued in the orbit in accordance with FCFS retrial policy without balking (or) may balk the orbit. While the server is serving to the customers, it faces two types of break-down there are breakdowns by the arrival of negative customer and break-down at any instant of service and server will be down for a short interval of time. Further concept of the delay time of repair is also introduced for breakdowns. We consider two different kinds of vacations, one is an emergency and the other one is Bernoulli vacation, the emergency vacation means at the time of the server serving the customer suddenly go for a vacation and the interrupted customer waits to get the remaining service and after the completion of each service, the server either goes for a vacation or may continue to serve for the next customer; if any . The retrial time, service time, vacation time, delay time and repair time are all follows general(arbitrary) distribution. Finally, we obtain some important performance measures of this model.

The rest of the paper is organized as follows: Mathematical description of our model in section (2). Definitions, equations governing of our model and the time dependent solution have been obtained in section (3) and (4). The corresponding steady state results have been derived explicitly in section (5). Stochastic decomposition and some performance measures in section (6) and (7). Average queue size and the average waiting time are computed in section (8) and (9). Some particular cases have been discussed in section (10).

2 Mathematical description of our model

(i) Priority and Low-priority units arrive at the system in batches of variable sizes in a compound Poisson process and they are provided one by one on a FCFS basis. Let $\lambda_1 c_i\ dt$ and $\lambda_2 c_i\ dt (i = 1, 2, 3, ...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt)$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda_1 > 0,\ \lambda_2 > 0$ are the average arrival rates of priority and non-priority customers and priority customers only form queue, if the server is busy. The server must serve all the priority units present in the system before taking up low-priority unit for service. In other words, there is no priority unit present in the system at the time of starting the service of a non-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more priority units arrive during the service time of a low-priority unit, the current service of a low-priority units is not stopped and a priority unit will be taken up for service only after the current service of a non-priority unit is complete.

(ii) Low-priority customers are considered as retrial customers. If the server is available, it begins the service to one of the customer immediately from the arriving batch of low-priority customers and the remaining customers leave the service area and hence join the orbit. Also upon arrival, if the customers finds the server busy or on vacation they join the orbit with probability b or balks the orbit with probability (1-b). The retrial time, that is time between successive repeated attempts of each customer in the orbit is assumed to be generally distributed with distribution function $A(x)$, density function $a(x)$. The conditional completion rate for retrials is given by $\eta(x) = \frac{a(x)}{(1-A(x))}$

(iii) Each customer under priority and low-priority service provided by a single server on a first come - first served basis. The service time for both priority and low-priority units follows general(arbitrary) distribution with distribution functions $B_i(v)$ and the density functions $b_i(v),\ i = 1, 2.$

(iv) Let $\mu_i(x)dx$ be the conditional probability of completion of the priority and low-priority unit service during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, i = 1, 2.$$

and therefore,

$$b_i(v) = \mu_i(v)e^{-\int_0^s \mu_i(x)dx}, i = 1, 2.$$

(v) Negative customers arrive singly according to a Poisson stream with rate λ^-. The arrival of a negative customer removes the customer being in service from the system and make the server breakdown. When the server fails, it stop providing service and its repairs do not start immediately and there is a delay time to start repair.

(vi) The delay time to start repair follows general (arbitrary) distribution with distribution function $D(s)$ and the density function $d(s)$. Let $\xi(x)dx$ be the conditional probability of a completion of delay time to repair during the interval $(x, x + dx]$, given that the elapsed delay time is x, so that

$$\xi(x) = \frac{d(x)}{1 - D(x)}$$

and therefore,

$$d(s) = \xi(s)e^{-\int_0^s \xi(x)dx}.$$

(vii) The Repair time follows general (arbitrary) distribution with distribution function $R(t)$ and the density function $r(t)$. Let $\beta(x)dx$ be the conditional probability of a completion of repair during the interval $(x, x + dx]$, given that the elapsed repair time is x, so that

$$\beta(x) = \frac{r(x)}{1 - R(x)}$$

and therefore,

$$r(t) = \beta(t)e^{-\int_0^s \beta(x)dx}.$$

(viii) The server may take the emergency vacation during the service time which is distributed as exponentially with rate β. When the server is in emergency vacation, the customer in service remains in the service position to complete the service. The emergency vacation time for interruptions of priority and low-priority costomers follows general(arbitrary) distributions with distribution functions $H_i(t)$ and the density functions $h_i(t),\ i = 1, 2.$ respectively.

(ix) Let $\gamma_i(y)dy$ be the conditional probability of completion of the emergency vacation during the interval $(y, y+dy]$, given that the elapsed emergency vacation time is x, so that

$$\gamma_i(y) = \frac{h_i(y)}{1 - H_i(y)}, i = 1, 2$$

and therefore,

$$h_i(t) = \gamma_i(t)e^{-\int_0^s \gamma_i(y)dy}, i = 1, 2.$$

(x) We further assume that as soon as the completion of each service the server has the option to take a vacation of random length with probability θ, in which case the vacation starts immediately or else with probability $(1-\theta)$, he may decide to continue serving the next unit present in the system, if any.

(xi) The vacation time follows general (arbitrary) distribution with distribution function $V(t)$ and the density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)}$$

and therefore,

$$v(t) = \gamma(t)e^{-\int_0^s \gamma(x)dx}.$$

(xii) The priority customer may renege the queue during the breakdown, emergency vacation, Bernoulli vacation, delay time to repair and repairing time and it follows Poisson stream with mean rate $\xi > 0$.

(xiii) As soon as the completion of a service for a non-priority customers, if they are not satisfied with their service they join the tail of the orbit as a feedback customer with probability p or may leave the system with probability q for satisfied customers.

(xiv) If the server is busy or on vacation, an arriving low-priority customers either join the orbit with probability b or balks(do not join the orbit) with probability $(1-b)$.

(xv) On returning from vacation, the server instantly starts serving the customer at the head of the queue, if any. The server stays in the system for being available if there are no customers in the queue.

(xvi) The server may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Whenever the server breaks down, its repairs do not start immediately and there is a delay time to start repair.

(xvii) The delay time to start repairs follows general (arbitrary) distribution with distribution function $D_i(t)$ and the density function $d_i(t)$, $i = 1, 2$. Let $\xi_i(x)dx$ be the conditional probability of a completion of a delay time to repair during the interval $(y, y+dy]$, given that the elapsed delay time to repair is y, so that

$$\xi_i(y) = \frac{d_i(y)}{1 - D_i(y)}, i = 1, 2$$

and therefore,

$$d_i(t) = \xi_i(t)e^{-\int_0^s \xi_i(y)dy}, i = 1, 2.$$

(xviii) The Repair time follows general (arbitrary) distribution with distribution function $R_i(t)$ and the density function $r_i(t)$, $i = 1, 2$. Let $\beta_i(y)dy$ be the conditional probability of a completion of a repair during the interval $(y, y+dy]$, given that the elapsed repair time is y, so that

$$\beta_i(y) = \frac{r_i(y)}{1 - R_i(y)}, i = 1, 2$$

and therefore,

$$r_i(t) = \beta_i(t)e^{-\int_0^s \beta_i(y)dy}, i = 1, 2.$$

(xix) Various stochastic processes involved in the system are assumed to be independent of each other.

3 Definitions and notations

We define

(i) $P^{(1)}_{m,n}(x,t)$ = Probability that at time t, the server is active providing service and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit excluding the one priority unit in service with elapsed service time for this customer is x. Accordingly, $P^{(1)}_{m,n}(t) = \int_0^\infty P^{(1)}_{m,n}(x,t)dx$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit excluding one priority unit in service without regard to the elapsed service time x of a priority unit.

(ii) $P^{(2)}_{m,n}(x,t)$ = Probability that at time t, the server is active providing service and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit excluding the one low-priority unit in service with elapsed service time for this customer is x. Accordingly, $P^{(2)}_{m,n}(t) = \int_0^\infty P^{(2)}_{m,n}(x,t)dx$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit excluding one low-priority unit in service without regard to the elapsed service time x of a non-priority unit.

(iii) $D_{m,n}(x,t)$ = Probability that at time t, the server is on delay to start repair(server inactive due to negative arrival) with elapsed delay time is x and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $D_{m,n}(t) = \int_0^\infty D_{m,n}(x,t)dx$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed delay time is x .

(iv) $R_{m,n}(x,t)$ = Probability that at time t, the server is on repair with elapsed repair time is x and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $R_{m,n}(t) = \int_0^\infty R_{m,n}(x,t)dx$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed repair time is x .

(v) $E^i_{m,n}(x,y,t)$ = Probability that at time t, the server is on emergency vacation with elapsed emergency vacation time is y and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $E^i_{m,n}(x,t) = \int_0^\infty E^i_{m,n}(x,y,t)dy$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed emergency vacation time is y.

(vi) $V_{m,n}(x,t)$ = Probability that at time t, the server is on vacation with elapsed vacation time x and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $V_{m,n}(t) = \int_0^\infty V_{m,n}(x,t)dx$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed vacation time is x .

(vii) $D^{(i)}_{m,n}(x,y,t)$ = Probability that at time t, the server is on delay to start repair(server inactive due to active breakdown) with elapsed delay time is y and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $D^{(i)}_{m,n}(x,t) = \int_0^\infty D^{(i)}_{m,n}(x,y,t)dy$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed delay time is y .

(viii) $R^{(i)}_{m,n}(x,y,t)$ = Probability that at time t, the server is on repair with elapsed repair time is y and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit. Accordingly, $R^{(i)}_{m,n}(x,t) = \int_0^\infty R^{(i)}_{m,n}(x,y,t)dy$ denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit, without regard to the elapsed repair time is y .

(ix) $I_{0,0}(t)$ = Probability that at time t, there are no priority and low-priority customers in the system and the server is idle but available in the system.

4 Equations Governing the System

The system is then governed by forward Kolmogorov equations

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_1+\lambda_2)I_{0,0}(t) + (1-\theta)\int_0^\infty P^{(1)}_{0,0}(x,t)\mu_1(x)dx$$
$$+ q(1-\theta)\int_0^\infty P^{(2)}_{0,0}(x,t)\mu_2(x)dx + \int_0^\infty R_{0,0}(x,t)\beta(x)dx + \int_0^\infty V_{0,0}(x,t)\gamma(x)dx, \quad (1)$$

$$\frac{\partial}{\partial t}I_{0,n}(x,t) + \frac{\partial}{\partial x}I_{0,n}(x,t) = -(\lambda_1+\lambda_2+\eta(x))I_{0,n}(x,t), \; n \geq 1, \quad (2)$$

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu_1(x)+\lambda^-+\alpha+\beta)P^{(1)}_{m,n}(x,t) + \int_0^\infty R^{(1)}_{m,n}(x,y,t)\beta_1(y)dy$$
$$+ \int_0^\infty E^{(1)}_{m,n}(x,y,t)\gamma_1(y)dy + \lambda_1\sum_{i=1}^{m} C_{1_i}P^{(1)}_{m-i,n}(x,t) + \lambda_2 b\sum_{i=1}^{n} C_{2_i}P^{(1)}_{m,n-i}(x,t) + \lambda_2(1-b)P^{(1)}_{m,n}(x,t),$$
$$m, \; n \geq 0, \quad (3)$$

$$\frac{\partial}{\partial t}P^{(2)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(2)}_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu_2(x)+\lambda^-+\alpha+\beta)P^{(2)}_{m,n}(x,t) + \int_0^\infty R^{(2)}_{m,n}(x,y,t)\beta_1(y)dy$$
$$+ \int_0^\infty E^{(2)}_{m,n}(x,y,t)\gamma_2(y)dy + \lambda_1\sum_{i=1}^{m} C_{1_i}P^{(2)}_{m-i,n}(x,t) + \lambda_2 b\sum_{i=1}^{n} C_{2_i}P^{(2)}_{m,n-i}(x,t) + \lambda_2(1-b)P^{(2)}_{m,n}(x,t),$$
$$m, \; n \geq 0, \quad (4)$$

$$\frac{\partial}{\partial t}E^1_{m,n}(x,y,t) + \frac{\partial}{\partial x}E^1_{m,n}(x,y,t) = -(\lambda_1+\lambda_2+\gamma_1(y)+\xi)E^1_{m,n}(x,y,t) + \lambda_1\sum_{i=1}^{m} C_{1_i}E^1_{m-i,n}(x,y,t)$$
$$+ \lambda_2 b\sum_{i=1}^{n} C_{2_i}E^1_{m,n-i}(x,y,t) + \lambda_2(1-b)E^1_{m,n}(x,y,t) + \xi E^1_{m+1,n}(x,y,t), m, \; n \geq 0, \quad (5)$$

$$\frac{\partial}{\partial t}E^2_{m,n}(x,y,t) + \frac{\partial}{\partial x}E^2_{m,n}(x,y,t) = -(\lambda_1+\lambda_2+\gamma_1(y)+\xi)E^2_{m,n}(x,y,t) + \lambda_1\sum_{i=1}^{m} C_{1_i}E^2_{m-i,n}(x,y,t)$$
$$+ \lambda_2 b\sum_{i=1}^{n} C_{2_i}E^2_{m,n-i}(x,y,t) + \lambda_2(1-b)E^2_{m,n}(x,y,t) + \xi E^2_{m+1,n}(x,y,t), m, \; n \geq 0, \quad (6)$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t) + \frac{\partial}{\partial x}V_{m,n}(x,t) = -(\lambda_1+\lambda_2+\gamma(x)+\xi)V_{m,n}(x,t) + \lambda_1\sum_{i=1}^{m} C_{1_i}V_{m-i,n}(x,t)$$
$$+ \lambda_2 b\sum_{i=1}^{n} C_{2_i}V_{m,n-i}(x,t) + \lambda_2(1-b)V_{m,n}(x,t) + \xi V_{m+1,n}(x,t), m, \; n \geq 0, \quad (7)$$

$$\frac{\partial}{\partial t}D_{m,n}(x,t) + \frac{\partial}{\partial x}D_{m,n}(x,t) = -(\lambda_1+\lambda_2+\xi(x)+\xi)D_{m,n}(x,t) + \lambda_1\sum_{i=1}^{m} C_{1_i}D_{m-i,n}(x,t)$$
$$+ \lambda_2 b\sum_{i=1}^{n} C_{2_i}D_{m,n-i}(x,t) + \lambda_2(1-b)D_{m,n}(x,t) + \xi D_{m+1,n}(x,t), m, \; n \geq 0, \quad (8)$$

$$\frac{\partial}{\partial t}R_{m,n}(x,t)+\frac{\partial}{\partial x}R_{m,n}(x,t)=-(\lambda_1+\lambda_2+\gamma(x)+\xi)R_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}C_{1_i}R_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}C_{2_i}R_{m,n-i}(x,t)+\lambda_2(1-b)R_{m,n}(x,t)+\xi R_{m+1,n}(x,t),m,\ n\geq 0, \tag{9}$$
$$\frac{\partial}{\partial t}D^1_{m,n}(x,y,t)+\frac{\partial}{\partial x}D^1_{m,n}(x,y,t)=-(\lambda_1+\lambda_2+\xi_1(y)+\xi)D^1_{m,n}(x,y,t)+\lambda_1\sum_{i=1}^{m}C_{1_i}D_1^{m-i,n}(x,y,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}C_{2_i}D_1^{m,n-i}(x,y,t)+\lambda_2(1-b)D^1_{m,n}(x,y,t)+\xi D^1_{m+1,n}(x,y,t),m,\ n\geq 0, \tag{10}$$
$$\frac{\partial}{\partial t}D^2_{m,n}(x,y,t)+\frac{\partial}{\partial x}D^2_{m,n}(x,y,t)=-(\lambda_1+\lambda_2+\xi_2(y)+\xi)D^2_{m,n}(x,y,t)+\lambda_1\sum_{i=1}^{m}C_{1_i}D^2_{m-i,n}(x,y,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}C_{2_i}D^2_{m,n-i}(x,y,t)+\lambda_2(1-b)D^2_{m,n}(x,y,t)+\xi D^2_{m+1,n}(x,y,t),m,\ n\geq 0, \tag{11}$$
$$\frac{\partial}{\partial t}R^1_{m,n}(x,y,t)+\frac{\partial}{\partial x}R^1_{m,n}(x,y,t)=-(\lambda_1+\lambda_2+\beta_1(y)+\xi)R^1_{m,n}(x,y,t)+\lambda_1\sum_{i=1}^{m}C_{1_i}R^1_{m-i,n}(x,y,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}C_{2_i}R^1_{m,n-i}(x,y,t)+\lambda_2(1-b)R^1_{m,n}(x,y,t)+\xi R^2_{m+1,n}(x,y,t),m,\ n\geq 0, \tag{12}$$
$$\frac{\partial}{\partial t}R^2_{m,n}(x,y,t)+\frac{\partial}{\partial x}R^2_{m,n}(x,y,t)=-(\lambda_1+\lambda_2+\beta_2(y)+\xi)R^2_{m,n}(x,y,t)+\lambda_1\sum_{i=1}^{m}C_{1_i}R^2_{m-i,n}(x,y,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}C_{2_i}R^2_{m,n-i}(x,y,t)+\lambda_2(1-b)R^2_{m,n}(x,y,t)+\xi R^2_{m+1,n}(x,y,t),m,\ n\geq 0. \tag{13}$$

The above set of equations are to be solved under the following boundary conditions at $x=0,\ y=0.$

$$I_{0,n}(0,t)=(1-\theta)\int_0^\infty P^{(1)}_{0,n}(x,t)\mu_1(x)dx+q(1-\theta)\int_0^\infty P^{(2)}_{0,n}(x,t)\mu_2(x)dx$$
$$+p(1-\theta)\int_0^\infty P^{(2)}_{0,n-1}(x,t)\mu_2(x)dx+\int_0^\infty V_{0,n}(x,t)\gamma(x)dx+\int_0^\infty R_{0,n}(x,t)\beta(x)dx,\quad n\geq 1, \tag{14}$$
$$P^1_{m,n}(0,t)=\lambda_1 C_{m+1}I_{0,n}(t)+(1-\theta)\int_0^\infty P^{(1)}_{m+1,n}(x,t)\mu_1(x)dx$$
$$+q(1-\theta)\int_0^\infty P^{(2)}_{m+1,n}(x,t)\mu_2(x)dx+p(1-\theta)\int_0^\infty P^{(2)}_{m+1,n-1}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty V_{m+1,n}(x,t)\gamma(x)dx++\int_0^\infty R_{m+1,n}(x,t)\beta(x)dx,\ m,\ n\geq 0, \tag{15}$$
$$P^2_{0,n}(0,t)=\lambda_2 C_{n+1}I_{0,0}(t)+\int_0^\infty I_{0,n+1}(x,t)\eta(x)dx+\sum_{i=1}^{n}\lambda_2 C_i\int_0^\infty I_{0,n+1-i}(x,t)dx, \tag{16}$$
$$E^1_{m,n}(x,0,t)=\beta P^1_{m,n}(x,t),m,n\geq 0, \tag{17}$$
$$E^2_{m,n}(x,0,t)=\beta P^2_{m,n}(x,t),m,n\geq 0, \tag{18}$$
$$V_{m,n}(0,t)=(\theta)\big[\int_0^\infty P^{(1)}_{m,n}(x,t)\mu_1(x)dx+q\int_0^\infty P^{(2)}_{m,n}(x,t)\mu_2(x)dx$$
$$+p\int_0^\infty P^{(2)}_{m,n-1}(x,t)\mu_2(x)dx\big],\ m,\ n\geq 0, \tag{19}$$
$$D_{m,n}(0,t)=\lambda^-[\int_0^\infty P^1_{m,n}(x,t)dx+\int_0^\infty P^2_{m,n}(x,t)dx],m,n\geq 0, \tag{20}$$
$$R_{m,n}(0,t)=\int_0^\infty D_{m,n}(x,t)\xi(x)dx,m,n\geq 0, \tag{21}$$
$$D^1_{m,n}(x,0,t)=\alpha P^{(1)}_{m,n}(x,t),\ m,\ n\geq 0, \tag{22}$$
$$D^2_{m,n}(x,0,t)=\alpha P^{(2)}_{m,n}(x,t),\ m,\ n\geq 0, \tag{23}$$

$$R^1_{m,n}(x,0,t) = \int_0^\infty D^1_{m,n}(x,y,t)\xi_1(y)dy,\ m,\ n \geq 0, \quad (24)$$
$$R^2_{m,n}(x,0,t) = \int_0^\infty D^2_{m,n}(x,y,t)\xi_2(y)dy,\ m,\ n \geq 0. \quad (25)$$

We assume that initially there are no customers in the system so that the server is idle.

$$I_{0,n}(0) = P^{(1)}_{m,n}(0) = P^{(2)}_{m,n}(0) = E^{(1)}_{m,n}(0) = E^{(2)}_{m,n}(0) = V_{m,n}(0) = D_{m,n}(0) = R_{m,n}(0) = D^{(1)}_{m,n}(0) = D^{(2)}_{m,n}(0)$$
$$= R^{(1)}_{m,n}(0) = R^{(2)}_{m,n}(0) = 0 \text{ and } I_{0,0}(0) = 1. \quad (26)$$

Next, we define the following probability generating functions:

$$I(x,z_2,t) = \sum_{m=1}^{\infty} z_2^n I_n(x,t),\ A(x,z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(x,t), \quad (27)$$

where $A = P^{(i)},\ E_{(i)},\ V,\ D,\ R,\ D_{(i)},\ R_{(i)}$
which are convergent inside the circle given by $|z_1| \leq 1, |z_2| \leq 1$. Taking Laplace transform from equations (1) to (25) and then solve it we get,

$$\overline{I}_0(x,z_2,s) = \overline{I}_0(0,z_2,s)[1-\overline{M}(a,s)]e^{-(a,s)x}, \quad (28)$$
$$\overline{P}^{(1)}(x,z_1,z_2,s) = \overline{P}^{(1)}(0,z_1,z_2,s)[1-\overline{B^1}(\phi_1(z,s))]e^{-\phi_1(z,s)x}, \quad (29)$$
$$\overline{P}^{(2)}(x,z_1,z_2,s) = \overline{P}^{(2)}(0,z_1,z_2,s)[1-\overline{B^2}(\phi_2(z,s))]e^{-\phi_2(z,s)x}, \quad (30)$$
$$\overline{E}_1(x,y,z_1,z_2,s) = \overline{E}_1(x,0,z_1,z_2,s)[1-\overline{E_1}(B(z,s))]e^{-B(z,s)y}, \quad (31)$$
$$\overline{E}_2(x,y,z_1,z_2,s) = \overline{E}_2(x,0,z_1,z_2,s)[1-\overline{E_2}(B(z,s))]e^{-B(z,s)y}, \quad (32)$$
$$\overline{V}(x,z_1,z_2,s) = \overline{V}(0,z_1,z_2,s)[1-\overline{V}(B(z,s))]e^{-B(z,s)x}, \quad (33)$$
$$\overline{D}(x,z_1,z_2,s) = \overline{D}(0,z_1,z_2,s)[1-\overline{D}(B(z,s))]e^{-B(z,s)x}, \quad (34)$$
$$\overline{R}(x,z_1,z_2,s) = \overline{R}(0,z_1,z_2,s)[1-\overline{R}(B(z,s))]e^{-B(z,s)x}, \quad (35)$$
$$\overline{D}_1(x,y,z_1,z_2,s) = \overline{D}_1(x,0,z_1,z_2,s)[1-\overline{D_1}(B(z,s))]e^{-B(z,s)y}, \quad (36)$$
$$\overline{D}_2(x,y,z_1,z_2,s) = \overline{D}_2(x,0,z_1,z_2,s)[1-\overline{D_2}(B(z,s))]e^{-B(z,s)y}, \quad (37)$$
$$\overline{R}_1(x,y,z_1,z_2,s) = \overline{R}_1(x,0,z_1,z_2,s)[1-\overline{R_1}(B(z,s))]e^{-B(z,s)y}, \quad (38)$$
$$\overline{R}_2(x,y,z_1,z_2,s) = \overline{R}_2(x,0,z_1,z_2,s)[1-\overline{R_2}(B(z,s))]e^{-B(z,s)y}, \quad (39)$$

where

$$A(z,s) = s + \lambda_1[1-C_1(z_1)] + \lambda_2 b[1-C_2(z_2)] + \lambda^- + \alpha + \beta,$$
$$B(z,s) = s + \lambda_1[1-C_1(z_1)] + \lambda_2 b[1-C_2(z_2)] + \xi - \frac{\xi}{z_1},$$
$$\phi_1(z_1,z_2,s)) = (A(z,s) - \alpha\overline{D}_1(B(z,s))\overline{R}_1(B(z,s)) - \beta\overline{E}_1(B(z,s))),$$
$$\phi_2(z_1,z_2,s)) = (A(z,s) - \alpha\overline{D}_2(B(z,s))\overline{R}_2(B(z,s)) - \beta\overline{E}_2(B(z,s))),$$
$$\begin{aligned}\overline{I}_0(0,z_2,s) = {} & \overline{P}^{(1)}_0(0,z_2,s)[(1-\theta)\overline{B}_1(\psi_1[z,s]) + \lambda^-\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_1(\psi_1[z,s])}{\psi_1[z,s]}] \\ & + \theta\overline{B}_1(\psi_1[z,s])\overline{V}(C(z,s))] + \overline{P}^{(2)}_0(0,z_2,s)[(p+q)(1-\theta)\overline{B}_2(\psi_2[z,s]) \\ & + \lambda^-\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_2(\psi_2[z,s])}{\psi_2[z,s]}] + (p+q)\theta\overline{B}_2(\psi_2[z,s])\overline{V}(C(z,s))] \\ & - [(a,s)\overline{I}_{0,0}(s) + 1],\end{aligned} \quad (40)$$

$$\{z_1 - \overline{B}_1(\phi_1[z,s])[1-\theta+\theta\overline{V}(B(z,s))] - \lambda^-\overline{D}(B(z,s))\overline{R}(B(z,s))[\frac{1-\overline{B}_1(\phi_1[z,s])}{\phi_1[z,s]}]\}\overline{P}^{(1)}(0,z_1,z_2,s)$$
$$= \lambda_1 C_1(z_1)[\frac{1-\overline{M}(a,s)}{(a,s)}]\overline{I}_0(0,z_2,s) - \overline{P}^{(1)}_0(0,z_2,s)\{\overline{B}_1(\psi_1[z,s])[1-\theta+\theta\overline{V}(C(z,s))]$$

$$+\lambda^{-}\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_1(\psi_1[z,s])}{\psi_1[z,s]}]\}+\overline{P}_0^{(2)}(0,z_2,s)\{\overline{B}_2(\phi_2[z,s])(p+q)[1-\theta+\theta\overline{V}(B(z,s))]$$
$$+\lambda^{-}\overline{D}(B(z,s))\overline{R}(B(z,s))[\frac{1-\overline{B}_2(\phi_2[z,s])}{\phi_2[z,s]}]-\overline{B}_2(\psi_2[z,s])(p+q)[1-\theta+\theta\overline{V}(C(z,s))]$$
$$-\lambda^{-}\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_2(\psi_2[z,s])}{\psi_2[z,s]}]\}, \quad (41)$$

$$z_2\overline{P}_0^{(2)}(0,z_2,s)=\overline{I}_0(0,z_2,s)\{\overline{M}(a,s)+\lambda_2C_2(z_2)[\frac{1-\overline{M}(a,s)}{(a,s)}]\}+\lambda_2C(z_2)\overline{I}_{0,0}(s). \quad (42)$$

We have to solve (40), (41) and (42).
Letting $z_1=g(z_2)$ in (41) we get,

$$\overline{P}_0^{(1)}(0,z_2,s)\{\overline{B}_1(\psi_1[z,s])[1-\theta+\theta\overline{V}(C(z,s))]+\lambda^{-}\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_1(\psi_1[z,s])}{\psi_1[z,s]}]\}$$
$$=\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]\overline{I}_0(0,z_2,s)+\overline{P}_0^{(2)}(0,z_2,s)\{\overline{B}_2(\phi_3[z,s])(p+q)[1-\theta+\theta\overline{V}(b(z,s))]$$
$$+\lambda^{-}\overline{D}(b(z,s))\overline{R}(b(z,s))[\frac{1-\overline{B}_2(\phi_3[z,s])}{\phi_3[z,s]}]-\overline{B}_2(\psi_2[z,s])(p+q)[1-\theta+\theta\overline{V}(C(z,s))]$$
$$-\lambda^{-}\overline{D}(C(z,s))\overline{R}(C(z,s))[\frac{1-\overline{B}_2(\psi_2[z,s])}{\psi_2[z,s]}]\}, \quad (43)$$

where
$\phi_3(g(z_2),z_2,s))=(A_2(z,s)-\alpha\overline{D}_2(b(z,s))\overline{R}_2(b(z,s))-\beta\overline{E}_2(b(z,s)))$,
$A_2(z,s)=s+\lambda_1[1-C[g(z_2)]]+\lambda_2b[1-C(z_2)]+\lambda^{-}+\alpha+\beta$,
$b(z,s)=s+\lambda_1[1-C[g(z_2)]]+\lambda_2b[1-C(z_2)]+\xi-\frac{\xi}{g(z_2)}$,
substitute (43) into (40), we get

$$\overline{I}_0(0,z_2,s)=\frac{1-(a,s)\overline{I}_{0,0}(s)}{1-\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]}$$
$$+\frac{\overline{P}_0^{(2)}(0,z_2,s)\{\overline{B}_2(\phi_3[z,s])(p+q)[1-\theta+\theta\overline{V}(b(z,s))]+\lambda^{-}\overline{D}(b(z,s))\overline{R}(b(z,s))[\frac{1-\overline{B}_2(\phi_3[z,s])}{\phi_3[z,s]}]\}}{1-\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]}, \quad (44)$$

substitute (44) into (42), we get

$$\overline{P}_0^{(2)}(0,z_2,s)=\frac{\left\{1-[s+\lambda_1(1-C[g(z_2)])+\lambda_2b(1-C(z_2))]\overline{I}_{0,0}(s)\}\{\{1-\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]\}+\{\overline{M}(a,s)+\lambda_2C(z_2)[\frac{1-\overline{M}(a,s)}{(a,s)}]\}\right\}}{\left\{z_2-z_2\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]\}-\{\overline{B}_2(\phi_3[z,s])(p+q)[1-\theta+\theta\overline{V}(b(z,s))]+\lambda^{-}\overline{D}(b(z,s))\overline{R}(b(z,s))[\frac{1-\overline{B}_2(\phi_3[z,s])}{\phi_3[z,s]}]\}\{\overline{M}(a,s)+\lambda_2C(z_2)[\frac{1-\overline{M}(a,s)}{(a,s)}]\}\right\}}, \quad (45)$$

now use (45) into (44), we get,

$$\overline{I}_0(0,z_2,s)=\frac{\left\{\{1-\overline{I}_{0,0}(s)[s+\lambda_1(1-C[g(z_2)])+\lambda_2b(1-C(z_2))]\}\{z_2+\overline{B}_2(\phi_3[z,s])(p+q)\times[1-\theta+\theta\overline{V}(b(z,s))]+\lambda^{-}\overline{D}(b(z,s))\overline{R}(b(z,s))[\frac{1-\overline{B}_2(\phi_3[z,s])}{\phi_3[z,s]}]\}\right\}}{\left\{z_2-z_2\lambda_1C[g(z_2)][\frac{1-\overline{M}(a,s)}{(a,s)}]\}-\{\overline{B}_2(\phi_3[z,s])(p+q)[1-\theta+\theta\overline{V}(b(z,s))]+\lambda^{-}\overline{D}(b(z,s))\overline{R}(b(z,s))[\frac{1-\overline{B}_2(\phi_3[z,s])}{\phi_3[z,s]}]\}\{\overline{M}(a,s)+\lambda_2C(z_2)[\frac{1-\overline{M}(a,s)}{(a,s)}]\}\right\}}, \quad (46)$$

substitute (43), (45) and (46) into (41), we get,

$$\overline{P}^{(1)}(0,z_1,z_2,s)=\frac{\left\{\overline{I_0}(0,z_2,s)[\frac{1-\overline{M}(a,s)}{a,s}]\{1-\{\lambda_1(1-C(z_1))+\lambda_2 b(1-C(z_2))\}\}+\overline{p_0^2}(0,z_2,s)(p+q)\times\{1-[1-\theta+\theta\overline{V}(B(z,s))]\}[1-\overline{B_2}(\phi_2(z,s))]+\lambda^-\{1-\overline{D}(B(z,s))\overline{R}(B(z,s))\}\times[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}]\right\}}{\{z_1-\overline{B}_1(\phi_1[z,s])[1-\theta+\theta\overline{V}(B(z,s))]-\lambda^-\overline{D}(B(z,s))\overline{R}(B(z,s))[\frac{1-\overline{B}_1(\phi_1[z,s])}{\phi_1[z,s]}]\}}, \tag{47}$$

$$\overline{V}(0,z_1,z_2,s)=\theta\big[P^{(1)}(0,z_1,z_2,s)\overline{B}_1(\phi_1(z_1,z_2,s))+(p+q)P_0^{(2)}(0,z_2,s)\overline{B}_2(\phi_2(z_1,z_2,s))\big], \tag{48}$$

$$\overline{D}(0,z_1,z_2,s)=\lambda^-[P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]+P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}]], \tag{49}$$

$$\overline{R}(0,z_1,z_2,s)=\lambda^-[P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]+P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}]]\times\overline{D}(B(z,s)), \tag{50}$$

$$\overline{E}_1(0,z_1,z_2,s)=\beta P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}], \tag{51}$$

$$\overline{E}_2(0,z_1,z_2,s)=\beta P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}], \tag{52}$$

$$\overline{D}_1(0,z_1,z_2,s)=\alpha P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}], \tag{53}$$

$$\overline{D}_2(0,z_1,z_2,s)=\alpha P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}], \tag{54}$$

$$\overline{R}_1(0,z_1,z_2,s)=\alpha P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]\overline{D}_1(B(z,s)), \tag{55}$$

$$\overline{R}_2(0,z_1,z_2,s)=\alpha P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}]\overline{D}_2(B(z,s)). \tag{56}$$

Theorem

The inequality $P^1(1,1)+P^2(1,1)=\rho<1$ is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state, queue size and orbit size distributions are given by

$$\overline{I}_0(z_2,s)=\overline{I}_0(0,z_2,s)[\frac{1-\overline{M}(a,s)}{(a,s)}], \tag{57}$$

$$\overline{P}^{(1)}(z_1,z_2,s)=\overline{P}^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}], \tag{58}$$

$$\overline{P}^{(2)}(z_1,z_2,s)=\overline{P}^{(2)}(0,z_1,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}], \tag{59}$$

$$\overline{V}(z_1,z_2,s)=\theta\big[P^{(1)}(0,z_1,z_2,s)\overline{B}_1(\phi_1(z_1,z_2,s))+(p+q)P_0^{(2)}(0,z_2,s)\overline{B}_2(\phi_2(z_1,z_2,s))\big]\times[\frac{1-\overline{V}(B(z,s))}{B(z,s)}], \tag{60}$$

$$\overline{D}(z_1,z_2,s)=\lambda^-[P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]+P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}]]\times[\frac{1-\overline{D}(B(z,s))}{B(z,s)}], \tag{61}$$

$$\overline{R}(z_1,z_2,s)=\lambda^-[P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]+P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}]] \times\overline{D}(B(z,s))[\frac{1-\overline{R}(B(z,s)}{B(z,s)}], \quad (62)$$

$$\overline{E}_1(z_1,z_2,s)=\beta P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}][\frac{1-\overline{E^1}(B(z,s))}{B(z,s)}], \quad (63)$$

$$\overline{E}_2(z_1,z_2,s)=\beta P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}][\frac{1-\overline{E_2}(B(z,s))}{(B(z,s))}], \quad (64)$$

$$\overline{D}_1(z_1,z_2,s)=\alpha P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}][\frac{1-\overline{D_1}(B(z,s))}{(B(z,s))}], \quad (65)$$

$$\overline{D}_2(z_1,z_2,s)=\alpha P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}][\frac{1-\overline{D_2}(B(z,s))}{B(z,s)}], \quad (66)$$

$$\overline{R}_1(z_1,z_2,s)=\alpha P^{(1)}(0,z_1,z_2,s)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2,s))}{\phi_1(z_1,z_2,s)}]\overline{D}_1(B(z,s))[\frac{1-\overline{R_1}(B(z,s))}{(B(z,s))}], \quad (67)$$

$$\overline{R}_2(z_1,z_2,s)=\alpha P_0^{(2)}(0,z_2,s)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2,s))}{\phi_2(z_1,z_2,s)}]\overline{D}_2(B(z,s))[\frac{1-\overline{R_2}(B(z,s))}{(B(z,s))}]. \quad (68)$$

5 Steady state Analysis: Limiting Behaviour

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s)=\lim_{t\to\infty} f(t)$$

to the equations (57) to (68). In order to determine $I_{(0,0)}$, we use the normalizing condition

$$I_{(0,0)}+I_{(0)}(1)+P^{(1)}(1,1)+P^{(2)}(1,1)+E_1(1,1)+E_1(1,1)+D(1,1)+R(1,1) +V(1,1)+D_1(1,1)+D_2(1,1)+R_1(1,1)+R_2(1,1)=1. \quad (69)$$

The steady state probability for priority and low-priority customers with retrial queueing system, negative arrival, balking, reneging, feedback and emergency and Bernoulli vacation for an unreliable server are

$$I_0(z_2)=I_0(0,z_2)[\frac{1-\overline{M}(a)}{(a)}], \quad (70)$$

$$P^{(1)}(z_1,z_2)=P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}], \quad (71)$$

$$P^{(2)}(z_1,z_2)=P^{(2)}(0,z_1,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}], \quad (72)$$

$$V(z_1,z_2)=\theta\big[P^{(1)}(0,z_1,z_2)\overline{B}_1(\phi_1(z_1,z_2))+(p+q)P_0^{(2)}(0,z_2)\overline{B}_2(\phi_2(z_1,z_2))] \times[\frac{1-\overline{V}(B(z))}{B(z)}], \quad (73)$$

$$D(z_1,z_2)=\lambda^-[P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}]+P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}]] \times[\frac{1-\overline{D}(B(z))}{B(z)}], \quad (74)$$

$$R(z_1,z_2) = \lambda^{-}[P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}] + P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}]] \times \overline{D}(B(z))[\frac{1-\overline{R}(B(z)}{B(z)}], \quad (75)$$

$$E_1(z_1,z_2) = \beta P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}][\frac{1-\overline{E^1}(B(z))}{B(z)}], \quad (76)$$

$$E_2(z_1,z_2) = \beta P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}][\frac{1-\overline{E^2}(B(z))}{(B(z))}], \quad (77)$$

$$D_1(z_1,z_2) = \alpha P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}][\frac{1-\overline{D_1}(B(z))}{(B(z))}], \quad (78)$$

$$D_2(z_1,z_2) = \alpha P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}][\frac{1-\overline{D_2}(B(z))}{B(z)}], \quad (79)$$

$$R_1(z_1,z_2) = \alpha P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(z_1,z_2))}{\phi_1(z_1,z_2)}]\overline{D}_1(B(z))[\frac{1-\overline{R_1}(B(z))}{(B(z))}], \quad (80)$$

$$R_2(z_1,z_2) = \alpha P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(\phi_2(z_1,z_2))}{\phi_2(z_1,z_2)}]\overline{D}_2(B(z))[\frac{1-\overline{R_2}(B(z))}{(B(z))}], \quad (81)$$

where

$$I_0(0,z_2) = \frac{\{\{-I_{0,0}[\lambda_1(1-C[g(z_2)]) + \lambda_2 b(1-C(z_2))]\}\{z_2 + \overline{B}_2(\phi_3[z])(p+q)[1-\theta+\theta\overline{V}(b(z))] + \lambda^{-}\overline{D}(b(z))\overline{R}(b(z))[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}]\}\}}{\left\{z_2 - z_2\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}]\} - \{\overline{B}_2(\phi_3[z])(p+q)[1-\theta+\theta\overline{V}(b(z))] + \lambda^{-}\overline{D}(b(z))\overline{R}(b(z))[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}\}\{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}]\}\right\}}, \quad (82)$$

$$P^{(1)}(0,z_1,z_2) = \frac{\left\{-I_0(0,z_2)[\frac{1-\overline{M}(a)}{a}]\{\lambda_1(1-C(z_1)) + \lambda_2 b(1-C(z_2))\} + p_0^2(0,z_2)(p+q) \times \{1-[1-\theta+\theta\overline{V}(B(z))]\}[1-\overline{B_2}(\phi_2(z))] + \lambda^{-}\{1-\overline{D}(B(z))\overline{R}(B(z))\} \times [\frac{1-\overline{B_2}(\phi_2(z))}{\phi_2(z)}]\right\}}{\{z_1 - \overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(B(z))] - \lambda^{-}\overline{D}(B(z))\overline{R}(B(z))[\frac{1-\overline{B}_1(\phi_1[z])}{\phi_1[z]}]\}}, \quad (83)$$

$$P_0^{(2)}(0,z_2) = \frac{\left\{-[\lambda_1(1-C[g(z_2)]) + \lambda_2 b(1-C(z_2))]I_{0,0}\}\{\{1-\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}]\} + \{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}]\}\right\}}{\left\{z_2 - z_2\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}]\} - \{\overline{B}_2(\phi_3[z])(p+q)[1-\theta+\theta\overline{V}(b(z))] + \lambda^{-}\overline{D}(b(z))\overline{R}(b(z))[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}\}\{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}]\}\right\}}. \quad (84)$$

Let $W_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations (70) to (81), we obtain

$$W_q(z_1,z_2) = I_0(z_2) + P^{(1)}(z_1,z_2) + P^{(2)}(z_1,z_2) + V(z_1,z_2) + E_1(z_1,z_2) + E_2(z_1,z_2) + D(z_1,z_2) + R(z_1,z_2) + D_1(z_1,z_2) + D_2(z_1,z_2) + R_1(z_1,z_2) + R_2(z_1,z_2), \quad (85)$$

$$W_q(z_1,z_2) = \frac{Nr(z_1,z_2)}{\phi_1(z_1,z_2)\phi_2(z_1,z_2)B(z_1,z_2)T_1(z_1,z_2)T(z_2)}, \quad (86)$$

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.12, 2016

where

$$Nr(z_1,z_2) = \phi_1(z_1,z_2)\phi_2(z_1,z_2)B(z_1,z_2)T_1(z_1,z_2)N_1(z_2)[\frac{1-\overline{M}(a)}{a}] + N_2(z_1,z_2)\phi_2(z_1,z_2)M_1(z_1,z_2)$$
$$+ N_3(z_2)\phi_1(z_1,z_2)T_1(z_1,z_2)M_2(z_1,z_2),$$
$$N_1(z_2) = \{-I_{0,0}[\lambda_1(1-C[g(z_2)]) + \lambda_2 b(1-C(z_2))]\}\{z_2 + \overline{B}_2(\phi_3[z])(p+q)[1-\theta+\theta\overline{V}(b(z))]$$
$$+ \lambda^-\overline{D}(b(z))\overline{R}(b(z))[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}]\},$$
$$N_2(z_1,z_2) = -N_1(z_2)[\frac{1-\overline{M}(a)}{a}]\{\lambda_1(1-C(z_1)) + \lambda_2 b(1-C(z_2))\} + N_3(z_2)(p+q)$$
$$\{1-[1-\theta+\theta\overline{V}(B(z))]\}[1-\overline{B_2}(\phi_2(z))] + \lambda^-\{1-\overline{D}(B(z))\overline{R}(B(z))\}[\frac{1-\overline{B_2}(\phi_2(z))}{\phi_2(z)}],$$
$$N_3(z_2) = -[\lambda_1(1-C[g(z_2)]) + \lambda_2 b(1-C(z_2))]I_{0,0}\{\{1-\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}]\}$$
$$+ \{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}],$$
$$M_1(z_1,z_2) = \{(1-\overline{B}_1(\phi_1(z_1,z_2)))[B(z_1,z_2) + \beta(1-\overline{E^1}(B(z_1,z_2))) + \lambda^-(1-\overline{D}(B(z_1,z_2))) + \lambda^-$$
$$\overline{D}(B(z_1,z_2))(1-\overline{R}(B(z_1,z_2))) + \alpha(1-\overline{D}_1(B(z_1,z_2))) + \alpha\overline{D}_1(B(z_1,z_2))(1-\overline{R}_1(B(z_1,z_2)))]$$
$$+ \theta\phi_1(z_1,z_2)\overline{B}_1(\phi_1(z_1,z_2))(1-\overline{V}(B(z_1,z_2)))\},$$
$$M_2(z_1,z_2) = \{(1-\overline{B}_2(\phi_2(z_1,z_2)))[B(z_1,z_2) + \beta(1-\overline{E}_2(B(z_1,z_2))) + \lambda^-(1-\overline{D}(B(z_1,z_2))) + \lambda^-$$
$$\overline{D}(B(z_1,z_2))(1-\overline{R}(B(z_1,z_2))) + \alpha(1-\overline{D}_2(B(z_1,z_2))) + \alpha\overline{D}_2(B(z_1,z_2))(1-\overline{R}_2(B(z_1,z_2)))]$$
$$+ \theta\phi_2(z_1,z_2)(p+q)\overline{B}_2(\phi_2(z_1,z_2))(1-\overline{V}(B(z_1,z_2)))\},$$
$$T(z_2) = z_2 - z_2\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}] - \{\overline{B}_2(\phi_3[z])(p+q)[1-\theta+\theta\overline{V}(b(z))] + \lambda^-\overline{D}(b(z))\overline{R}(b(z))$$
$$[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}]\}\{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}]\},$$
$$T_1(z_1,z_2) = z_1 - \overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(B(z))] - \lambda^-\overline{D}(B(z))\overline{R}(B(z))[\frac{1-\overline{B}_1(\phi_1[z])}{\phi_1[z]}].$$

Now using the normalizing condition (69) we get

$$I_{0,0} = \frac{\{6(\lambda^-)(\xi-\lambda_1 C'[1] - b\lambda_2 C'[1])\,T'[1]\,(T_1)'[1]\}}{Dr} \tag{87}$$

where

$$Dr = \{6\left(\lambda^-\right)(\xi-\lambda_1 E(I) - b\lambda_2 E(I))\,T'[1]\,(T_1)'[1]\}$$
$$+ \frac{6(1-\overline{M}(a))\,(\lambda^-)\,(\xi-\lambda_1 E(I) - b\lambda_2 E(I))\,(N_1)'[1]\,(T_1)'[1]}{a}$$
$$+ 6\,(m_2)'[1]\,(N_3)'[1]\,(T_1)'[1] + 3\,(m_1)'[1]\,(N_2)''[1]$$
$$T'(1,1) = 1 - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 - [\frac{1-\overline{M}(a)}{(a)}]\lambda_2 E(I) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 E(I)E(I_1) + \frac{1}{\lambda^-}(\overline{M}(a)$$
$$+ [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(-\lambda^-(-E(D) - E(R) + (E(D) + E(R) - E(V)\theta)$$
$$B_2[\lambda^-])(\xi - b\lambda_2 C'[1] - \lambda_1 E(I)E(I_1)) - (-1 + \overline{B}_2[\lambda^-])(\phi_3)'[1])$$
$$T_1'(1,1) = \{\frac{1}{\lambda^-}((-1+\overline{B}_1[\lambda^-])(\phi_1)'[1] + \lambda^-(1 - E(D)\xi - E(R)\xi + (E(D)+E(R))\lambda_1 E(I)$$
$$+ bE(D)\lambda_2 C'[1] + bE(R)\lambda_2 E(I) + (E(D) + E(R) + E(V)\theta)\overline{B}_1[\lambda^-](\xi - \lambda_1 E(I) - b\lambda_2 E(I))$$
$$- 2(\overline{B}_1)'[\lambda^-](\phi_1)'[1]))\}$$
$$N_1'(1,1) = 2(1 + \overline{B}_2[\lambda^-])E(I)(b\lambda_2 + \lambda_1 E(I_1))$$
$$\phi_1'(1,1) = [(\xi - \lambda_1 E(I) - b\lambda_2 E(I))\{\alpha[E(D_1) + E(R_1)] + \beta E(E_1)\} - \lambda_1 E(I) - b\lambda_2 E(I)]$$
$$\phi_2'(1,1) = [(\xi - \lambda_1 E(I) - b\lambda_2 E(I))\{\alpha[E(D_2) + E(R_2)] + \beta E(E_2)\} - \lambda_1 E(I) - b\lambda_2 E(I)]$$
$$\phi_3'(1,1) = (\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1))\{\alpha[E(D_1) + E(R_1)] + \beta E(E_1)\} - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1)$$

$$N_3'(1,1) = (1 + \overline{M}(a) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 + [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)E(I)(b\lambda_2 + \lambda_1 E(I_1))$$

$$N_2''(1,1) = \{2[\frac{1-\overline{M}(a)}{(a)}](\lambda_1 + b\lambda_2)E(I)N_1'[1] + 2(E(D) + E(R) + E(V)\theta - E(V)\theta\overline{B}_1[\lambda^-] - (E(D) + E(R))\overline{B}_2[\lambda^-])(\xi - \lambda_1 E(I) - b\lambda_2 E(I))N_3'[1]\}$$

$$M_1'(1,1) = \{((-1 - \beta E(E_1) - (E(D) + E(R))\lambda^- - \alpha[E(D_1) + E(R_1))(-1 + \overline{B}_1[\lambda^-]) + E(V)\theta\lambda^-\overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I) - b\lambda_2 E(I))\}$$

$$M_2'(1,1) = \{((-1 - \beta E(E_2) - (E(D) + E(R))\lambda^- - \alpha[E(D_2) + E(R_2)])(-1 + \overline{B}_2[\lambda^-]) + E(V)\theta\lambda^-\overline{B}_2[\lambda^-])(\xi - \lambda_1 E(I) - b\lambda_2 E(I))\}$$

Equation (87) gives the probability that the server is idle. Substituting equation (87) in equation (86), we have completely and explicitly determined $Wq(z_1, z_2)$, the probability generating function of the queue size.

6 Stochastic Decomposition

In this section we study the stochastic decomposition property of the system size distribution of our model. Stochastic decomposition for retrial models has also been found in Yang et al.[11] and Yang and Templeton[12]. The existence of the stochastic decomposition property for our model can be demonstrated easily by showing that

$$\Phi(z) = \Psi(z)\Pi(z). \tag{88}$$

where $\Psi(z)$ is the PGF of the queue size distribution of an batch arrival retrial queueing system with priority service, negative arrival, balking, reneging, feedback and emergency and Bernoulli vacation schedule for an unreliable server, which can be obtained for formula (86) by putting $\overline{M}(a) = 1$. Thus we have

$$\Psi(z) = \frac{nr_1(z_1, z_2)}{\phi_1(z_1,z_2)\phi_2(z_1,z_2)B(z_1,z_2)T_1(z_1,z_2)T(z_2)},$$

where

$$nr_1 = N_2(z_1,z_2)\phi_2(z_1,z_2)M_1(z_1,z_2) + N_3(z_2)\phi_1(z_1,z_2)T_1(z_1,z_2)M_2(z_1,z_2).$$

$$N_2(z_1,z_2) = N_3(z_2(p+q)\{1 - [1 - \theta + \theta\overline{V}(B(z))]\}[1 - \overline{B_2}(\phi_2(z))] + \lambda^-\{1 - \overline{D}(B(z))\overline{R}(B(z))\} [\frac{1-\overline{B_2}(\phi_2(z))}{\phi_2(z)}],$$

$$N_3(z_2) = -2[\lambda_1(1 - C[g(z_2)]) + \lambda_2 b(1 - C(z_2))]I_{0,0},$$

$\Pi(z)$ is the PGF of the conditional distribution of the number of customer in the orbit given that the system is idle= $\frac{I_{0,0}+I_0(z_2)}{I_{0,0}+I_0(1)}$ which is equal to

$$\Pi(z) = \frac{nr_2}{dr}$$

where

$$nr_2 = T'(1)\{z_2 - z_2\lambda_1 C[g(z_2)][\frac{1-\overline{M}(a)}{(a)}] - z_2[\frac{1-\overline{M}(a)}{(a)}][\lambda_1(1 - C[g(z_2)]) + \lambda_2 b(1 - C(z_2))] - \{\overline{B}_2(\phi_3[z])(p+q)[1 - \theta + \theta\overline{V}(b(z))] + \lambda^-\overline{D}(b(z))\overline{R}(b(z))[\frac{1-\overline{B}_2(\phi_3[z])}{\phi_3[z]}]\} \times \{\{\overline{M}(a) + \lambda_2 C(z_2)[\frac{1-\overline{M}(a)}{(a)}]\} + [\frac{1-\overline{M}(a)}{(a)}][\lambda_1(1 - C[g(z_2)]) + \lambda_2 b(1 - C(z_2))]\}\}\}$$

$$dr = \{T(z_2)\{1 - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1\} + \{\lambda_1 E(I)E(I_1) + \lambda_2 bE(I)\}[\frac{1-\overline{M}(a)}{(a)}] - (\overline{M}(a) + [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(-\lambda^-(-E(D) - E(R) + (E(D) + E(R) - E(V)\theta)\overline{B}_2[\lambda^-]) \times (\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1)) - (-1 + \overline{B}_2[\lambda^-])(\phi_3)'[1])\}$$

7 Performance Measures

Theorem: If the system is in steady state conditions, then we have

1. The system is free with probability $I_{0,0}$ which is given by (87).
2. The system is occupied with probability

$$I_0(1)+P^1(1,1)+P^2(1,1)$$
$$=\frac{\lambda^- T_1'(1,1)N_1'(1)[\frac{1-\overline{M}(a)}{a}]+N_2''(1,1)(1-\overline{B}_1(\lambda^-))+N_3'(1,1)T_1'(1,1)(1-\overline{B}_2(\lambda^-))}{T_1'(1,1)\lambda^-)T'(1)}$$

3. The server is idle with probability

$$I_{0,0}+I_0(1)=1-\frac{N_2''(1,1)M_1'(1,1)+2T_1'(1,1)M_2'(1,1)N_3'(1)}{2B'T'(1)T_1'(1,1)}.$$

4. The server is busy with probability

$$P^1(1,1)+P^2(1,1)=\frac{N_2''(1,1)(1-\overline{B}_1(\lambda^-))+N_3'(1,1)T_1'(1,1)(1-\overline{B}_2(\lambda^-))}{T_1'(1,1)\lambda^-)T'(1)}.$$

5. The server is on emergency vacation in priority service with probability

$$E_1(1,1)=\frac{\beta N_2''(1,1)(1-\overline{B}_1(\lambda^-))E(E_1)}{2T_1'(1,1)\lambda^-)T'(1)}.$$

6. The server is on emergency vacation in non-priority service with probability

$$E_2(1,1)=\frac{\beta N_3'(1)(1-\overline{B}_2(\lambda^-))E(E_2)}{\lambda^-)T'(1)}.$$

7. The server is waiting for repair with probability of priority and non priority services

$$D(1,1)=\frac{E(D)\{N_2''(1,1)(1-\overline{B}_1(\lambda^-))+2N_3'(1)(1-\overline{B}_2(\lambda^-))T_1'(1,1)\}}{2T_1'(1,1)T'(1)}.$$

8. The server is under repair with probability of priority and non priority services

$$R(1,1)=\frac{E(R)\{N_2''(1,1)(1-\overline{B}_1(\lambda^-))+2N_3'(1)(1-\overline{B}_2(\lambda^-))T_1'(1,1)\}}{2T_1'(1,1)T'(1)}.$$

9. The server is on vacation for both priority and non priority services with probability

$$V(1,1)=\frac{\theta E(V)\{N_2''(1,1)\overline{B}_1(\lambda^-)+2(p+q)N_3'(1)\overline{B}_2(\lambda^-)T_1'(1,1)\}}{2T_1'(1,1)T'(1)}.$$

10. The server is waiting for repair with probability of priority service

$$D_1(1,1)=\frac{\alpha N_2''(1,1)(1-\overline{B}_1(\lambda^-))E(D_1)}{2\lambda^- T_1'(1,1)T'(1)}.$$

11. The server is waiting for repair with probability of non-priority service

$$D_2(1,1)=\frac{\alpha N_3'(1)(1-\overline{B}_2(\lambda^-))E(D_2)}{\lambda^- T'(1)}.$$

12. The server is under repair with probability of priority service

$$R_1(1,1)=\frac{\alpha N_2''(1,1)(1-\overline{B}_1(\lambda^-))E(R_1)}{2\lambda^- T_1'(1,1)T'(1)}.$$

13. The server is under repair with probability of non-priority service

$$R_2(1,1)=\frac{\alpha N_3'(1)(1-\overline{B}_2(\lambda^-))E(R_2)}{\lambda^- T'(1)}.$$

Proof

Note that

$$I_0(1)+P^1(1,1)+P^2(1,1)=\lim_{\substack{z_1\to 1\\ z_2\to 1}}[I_0(z_2)+P^{(1)}(z_1,z_2)+P^{(2)}(z_1,z_2)],$$

$$I_{0,0}+I_0(1)=I_{0,0}+\lim_{z_2\to 1}I_0(z_2), P^{(1)}(1,1)+P^{(2)}(1,1)=\lim_{\substack{z_1\to 1\\ z_2\to 1}}[P^{(1)}(z_1,z_2)+P^{(2)}(z_1,z_2)],$$

$E_1(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} E_1(z_1,z_2), E_2(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} E_2(z_1,z_2), D(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} D(z_1,z_2),$

$R(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} R(z_1,z_2), V(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} V(z_1,z_2), D_1(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} D_1(z_1,z_2),$

$D_2(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} D_2(z_1,z_2), R_1(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} R_1(z_1,z_2), R_2(1,1) = \lim_{\substack{z_1\to 1\\ z_2\to 1}} R_1(z_1,z_2),$

by direct calculation we get the above formulae.

Theorem:

The availability of the server and failure of the server under the steady state conditions are given by

$$A_v = 1 - \frac{N_2''(1,1)M_1'(1,1) + 2T_1'(1,1)M_2'(1,1)N_3'(1)}{2B'T'(1)T_1'(1,1)}$$

the server failure by the arrival of negative customer

$$M_{f_1} = \frac{\lambda^-\{N_2''(1,1)(1-\overline{B}_1(\lambda^-)) + N_3'(1,1)T_1'(1,1)(1-\overline{B}_2(\lambda^-))\}}{T_1'(1,1)\lambda^-)T'(1)}$$

and the server failure by the random breakdown

$$M_{f_2} = \frac{\alpha\{N_2''(1,1)(1-\overline{B}_1(\lambda^-)) + N_3'(1,1)T_1'(1,1)(1-\overline{B}_2(\lambda^-))\}}{T_1'(1,1)\lambda^-)T'(1)}$$

Proof: By considering the following equations we get the results

$$A_v = I_{0,0} + \lim_{\substack{z_1\to 1\\ z_2\to 1}}[P^{(1)}(z_1,z_2) + P^{(2)}(z_1,z_2)]$$

and

$$M_{f_2} = \lambda^-[P^{(1)}(1,1) + P^{(2)}(1,1)],$$
$$M_{f_2} = \alpha[P^{(1)}(1,1) + P^{(2)}(1,1)].$$

8 The Average Queue Length:

The mean number of customers in priority queue and in the orbit under the steady state condition is given by

$$L_{q_1} = \frac{d}{dz_1}W_{q_1}(z_1,1)|_{z_1=1} \tag{89}$$

$$L_{q_2} = \frac{d}{dz_2}W_{q_2}(1,z_2)|_{z_2=1}, \tag{90}$$

respectively, then

$$L_{q_1} = \frac{Dr''(1)Nr'''(1) - Dr'''(1)Nr''(1)}{3Dr''(1)^2}, \tag{91}$$

$$L_{q_2} = \frac{dr'''(1)nr''''(1) - nr'''(1)dr''''(1)}{4dr'''(1)^2}, \tag{92}$$

where

$$Nr''(1) = \frac{1}{a}2\lambda^-((1-\overline{M}(a))N_1\lambda^-(\xi - \lambda_1 C'[1])(T_1)'[1] + a((m_1)'[1](N_2)'[1] + N_3(m_2)'[1](T_1)'[1]))$$

$$\begin{aligned}
Nr'''(1) = {} & \frac{6(1-\overline{M}(a))\lambda N_1(\xi - \lambda_1 E(I))(T_1)'[1](\phi_1)'[1]}{a} + 6N_3(m_2)'[1](T_1)'[1](\phi_1)'[1] \\
& + 6(m_1)'[1](N_2)'[1](\phi_2)'[1] + \frac{6(1-\overline{M}(a))\lambda N_1(\xi - \lambda_1 E(I))(T_1)'[1](\phi_2)'[1]}{a} \\
& + \frac{3(1-\overline{M}(a))\lambda^2 N_1(T_1)'[1](-2\xi - \lambda_1 E(I[I-1]))}{a} + 3\lambda(N_2)'[1](m_1)''[1] + 3\lambda N_3(T_1)'[1](m_2)''[1] \\
& + 3\lambda(m_1)'[1](N_2)''[1] + \frac{3(1-\overline{M}(a))\lambda^2 N_1(\xi - \lambda_1 E(I))(T_1)''[1]}{a} + 3\lambda N_3(m_2)'[1](T_1)''[1]
\end{aligned}$$

$$Dr''(1) = 2T(\lambda^-)^2(\xi - \lambda_1 E(I))(T_1)'[1]$$

$$Dr'''(1) = 6T\lambda^-(\xi - \lambda_1 E(I))(T_1)'[1](\phi_1)'[1] + 6T\lambda^-(\xi - \lambda_1 E(I))(T_1)'[1](\phi_2)'[1] + 3T(\lambda^-)^2(T_1)'[1](-2\xi - \lambda_1 E(I[I-1])) + 3T(\lambda^-)^2(\xi - \lambda_1 E(I))(T_1)''[1]$$

$$T(1,1) = 1 - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 - [\frac{1-\overline{M}(a)}{(a)}]\lambda_2 E(I) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 E(I)E(I_1) + \frac{1}{\lambda^-}(\overline{M}(a) + [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(-\lambda^- - E(D) - E(R) + (E(D) + E(R) - E(V)\theta)\overline{B}_2[\lambda^-])(\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1)) - (-1 + \overline{B}_2[\lambda^-])(\phi_3)'[1])$$

$$N_1(1) = I_{0,0}(1 + \overline{B}_2[\lambda^-])E(I)(b\lambda_2 + \lambda_1 E(I_1))$$

$$N_3(1) = (1 + \overline{M}(a) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 + [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)I_{0,0}E(I)(b\lambda_2 + \lambda_1 E(I_1))$$

$$N_2'(1) = N_1[\frac{1-\overline{M}(a)}{(a)}]\lambda_1 E(I) + N_3(V\theta(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I)) + (1 - \overline{B}_2[\lambda^-])(E(D)(\xi - \lambda_1 E(I)) + E(R)(\xi - \lambda_1 E(I))))$$

$$N_2''(1) = N_1[\frac{1-\overline{M}(a)}{(a)}]\lambda_1 E(I[I-1]) + N_3(-E(V^2)\theta(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))^2 - 2E(V)\theta(\xi - \lambda_1 E(I)) (\overline{B}_1)'[\lambda^-](\phi_1)'[1] - \frac{2(1 - \overline{B}_2[\lambda^-])(E(D)(\xi - \lambda_1 E(I)) + E(R)(\xi - \lambda_1 E(I)))(\phi_2)'[1]}{\lambda^-} - 2(E(D)(\xi - \lambda_1 E(I)) + E(R)(\xi - \lambda_1 E(I)))(\overline{B}_2)'[\lambda^-](\phi_2)'[1] + E(V)\theta(1 - \overline{B}_1[\lambda^-]) (-2\xi - \lambda_1 E(I[I-1])) + (1 - \overline{B}_2[\lambda^-])(-E(D^2)(\xi - \lambda_1 E(I))^2 - 2E(D)E(R)(\xi - \lambda_1 E(I))^2 - E(R^2)(\xi - \lambda_1 E(I))^2 + E(D)(-2\xi - \lambda_1 E(I[I-1])) + E(R)(-2\xi - \lambda_1 E(I[I-1]))))$$

$$T_1'(1) = \frac{1}{\lambda^-}((-1 + \overline{B}_1[\lambda^-])(\phi_1)'[1] + \lambda^-(1 - E(D)\xi - E(R)\xi + (E(D) + E(R))\lambda_1 E(I) + (E(D) + E(R) + E(V)\theta)\overline{B}_1[\lambda^-](\xi - \lambda_1 E(I)) - 2(\overline{B}_1)'[\lambda^-](\phi_1)'[1]))$$

$$T_1''(1) = E(D)^2(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))^2 + 2E(D)E(R)(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))^2 + E(R^2)(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))^2 - E(V^2)\theta\overline{B}_1[\lambda^-](\xi - \lambda_1 E(I))^2 + \frac{2E(D)(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))(\phi_1)'[1]}{\lambda^-} + \frac{2E(R)(1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))(\phi_1)'[1]}{\lambda^-} + 2E(D)(\xi - \lambda_1 E(I))(\overline{B}_1)'[\lambda^-](\phi_1)'[1] + 2E(R)(\xi - \lambda_1 E(I))(\overline{B}_1)'[\lambda^-](\phi_1)'[1] + 2E(V)\theta(\xi - \lambda_1 E(I))(\overline{B}_1)'[\lambda^-](\phi_1)'[1] + \frac{2(1 - \overline{B}_1[\lambda^-])(\phi_1)'[1]^2}{(\lambda^-)^2} + \frac{2(\overline{B}_1)'[\lambda^-](\phi_1)'[1]^2}{\lambda^-} - E(D)(1 - \overline{B}_1[\lambda^-])(-2\xi - \lambda_1 E(I[I-1])) - E(R)(1 - \overline{B}_1[\lambda^-])(-2\xi - \lambda_1 E(I[I-1])) + E(V)\theta\overline{B}_1[\lambda^-](-2\xi - \lambda_1 E(I[I-1])) - 2(\phi_1)'[1]^2(\overline{B}_1)''[\lambda^-] - \frac{(1 - \overline{B}_1[\lambda^-])(\phi_1)''[1]}{\lambda^-} - 2(\overline{B}_1)'[\lambda^-](\phi_1)''[1]$$

$$M_1'(1,1) = ((-1 - \beta E(E_1) - (E(D) + E(R))\lambda^- - \alpha[E(D_1) + E(R_1)]) (-1 + \overline{B}_1[\lambda^-]) + E(V)\theta\lambda^-\overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))$$

$$M_1''(1,1) = (1 + \beta E(E_1) + \lambda^-[E(D) + E(R)] + \alpha[E(D_1) + E(R_1)])[-(2\xi + \lambda_1 E(I[I-1])) (1 - \overline{B}_1[\lambda^-]) - 2(\xi - \lambda_1 E(I[I-1]))(\overline{B}_1)'[\lambda^-](\phi_1)'[1]] - (1 - \overline{B}_1[\lambda^-])(\xi - \lambda_1 E(I))^2 (\beta E(E_1^2) + \lambda^-[E(D^2) + E(R^2) + 2E(R)E(D)] + \alpha[E(D_1^2) + E(R_1^2) + 2E(R_1)E(D_1)]) + 2E(V)\theta(\xi - \lambda_1 E(I))[(\phi_1)'[1]\overline{B}_1[\lambda^-] + \lambda^-(\overline{B}_1)'[\lambda^-]] - (\xi - \lambda_1 E(I))^2 E(V^2)\theta\lambda^-\overline{B}_1[\lambda^-] - (2\xi + \lambda_1 E(I[I-1]))V\theta\lambda^- B_1[\lambda^-]$$

$$M_2'(1,1) = ((-1 - \beta E(E_2) - (E(D) + E(R))\lambda^- - \alpha[E(D_2) + E(R_2)])(-1 + \overline{B}_2[\lambda^-]) + E(V)\theta\lambda^-\overline{B}_2[\lambda^-])(\xi - \lambda_1 E(I))$$

$$M_2''(1,1) = (1 + \beta E(E_2) + \lambda^-[E(D) + E(R)] + \alpha[E(D_2) + E(R_2)])[-(2\xi + \lambda_1 E(I[I-1])) (1 - \overline{B}_2[\lambda^-]) - 2(\xi - \lambda_1 E(I))(\overline{B}_2)'[\lambda^-](\phi_2)'[1]] - (1 - \overline{B}_2[\lambda^-])(\xi - \lambda_1 E(I))^2$$

$$
\begin{aligned}
&\times(\beta E(E_2^2)+\lambda^-[E(D^2)+E(R^2)+2E(R)E(D)]+\alpha[E(D_2^2)+E(R_2^2)+2E(R_2)E(D_2)]) \\
&+2E(V)\theta(\xi-\lambda_1E(I))[(\phi_2)'[1]\overline{B}_2[\lambda^-]+\lambda^-(\overline{B}_2)'[\lambda^-]]-(\xi-\lambda_1E(I))^2E(V^2)\theta\lambda^-\overline{B}_2[\lambda^-] \\
&-(2\xi+\lambda_1E(I[I-1]))E(V)\theta\lambda^-\overline{B}_2[\lambda^-]
\end{aligned}
$$

$$
\begin{aligned}
nr''' &= -\frac{6b(1-\overline{M}(a))\lambda_2(\lambda^-)^2E(I)(N_1)'[1](T_1)'[1]}{a}+6\lambda^-(m_2)'[1](N_3)'[1](T_1)'[1] \\
&+3\lambda^-(m_1)'[1](N_2)''[1] \\
nr'''' &= -\frac{24b(1-\overline{M}(a))\lambda_2\lambda^-E(I)(N_1)'[1](T_1)'[1](\phi_1)'[1]}{a}+24(m_2)'[1](N_3)'[1](T_1)'[1](\phi_1)'[1] \\
&-\frac{24b(1-\overline{M}(a))\lambda_2\lambda^-E(I)(N_1)'[1](T_1)'[1](\phi_2)'[1]}{a} \\
&-\frac{12b(1-\overline{M}(a))\lambda_2(\lambda^-)^2(N_1)'[1](T_1)'[1]E(I[I-1])}{a}+12\lambda^-(N_3)'[1](T_1)'[1](m_2)''[1] \\
&-\frac{12b(1-\overline{M}(a))\lambda_2(\lambda^-)^2C'[1](T_1)'[1](N_1)''[1]}{a}+12(m_1)'[1](\phi_2)'[1](N_2)''[1] \\
&+6\lambda^-(m_1)''[1](N_2)''[1]+12\lambda^-(m_2)'[1](T_1)'[1](N_3)''[1] \\
&-\frac{12b(1-\overline{M}(a))\lambda_2(\lambda^-)^2E(I)(N_1)'[1](T_1)''[1]}{a}+12\lambda^-(m_2)'[1](N_3)'[1](T_1)''[1] \\
&+4\lambda^-(m_1)'[1](N_2)^{(3)}[1] \\
dr''' &= -6b\lambda_2(\lambda^-)^2E(I)T'[1](T_1)'[1] \\
dr'''' &= -24b\lambda_2\lambda^-E(I)T'[1](T_1)'[1](\phi_1)'[1]-24b\lambda_2\lambda^-E(I)T'[1](T_1)'[1](\phi_2)'[1] \\
&-12b\lambda_2(\lambda^-)^2T'[1](T_1)'[1]E(I[I-1])-12b\lambda_2(\lambda^-)^2E(I)(T_1)'[1]T''[1] \\
&-12b\lambda_2(\lambda^-)^2E(I)T'[1](T_1)''[1] \\
\phi_3'(1,1) &= (\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))\{\alpha[E(D_2)+E(R_2)]+\beta E(E_2)\}-b\lambda_2E(I)-\lambda_1E(I)E(I_1) \\
\phi_1'(1) &= -b\lambda_2E(I)-b\lambda_2E(I)(\alpha[E(D_1)+E(R_1)]+\beta E(E_1)) \\
\phi_1''(1) &= -b\lambda_2(b\lambda_2E(I)^2(\alpha[E(D_1^2)+E(R_1^2)+2E(D_1)E(R_1)]+\beta E(E_1^2)+(1+\beta E(E_1) \\
&+\alpha[E(D_1)+E(R_1)])E(I[I-1])) \\
\phi_2'(1) &= -b\lambda_2E(I)-b\lambda_2E(I)(\alpha[E(D_2)+E(R_2)]+\beta E(E_2)) \\
t'(1) &= 1-[\frac{1-\overline{M}(a)}{(a)}]\lambda_1-[\frac{1-\overline{M}(a)}{(a)}]\lambda_2E(I)-[\frac{1-\overline{M}(a)}{(a)}]\lambda_1E(I)E(I_1)+\frac{1}{\lambda^-}(\overline{M}(a) \\
&+[\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(-\lambda^-(-E(D)-E(R)+(ED)+E(R)-E(V)\theta)\overline{B}_2[\lambda^-])(\xi-b\lambda_2E(I) \\
&-\lambda_1E(I)E(I_1))-(-1+\overline{B}_2[\lambda^-])(\phi_3)'[1]) \\
t''(1) &= -2[\frac{1-\overline{M}(a)}{(a)}]\lambda_1E(I)E(I_1)+\frac{1}{\lambda^-}2[\frac{1-\overline{M}(a)}{(a)}]\lambda_2E(I)(-\lambda^-(-E(D)-E(R)+(E(D)+E(R) \\
&-E(V)\theta)\overline{B}_2[\lambda^-])(\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))-(-1+\overline{B}_2[\lambda^-])(\phi_3)'[1])-[\frac{1-\overline{M}(a)}{(a)}] \\
&\lambda_2E(I[I-1])-[\frac{1-\overline{M}(a)}{(a)}]\lambda_1E(I_1)^2E(I[I-1])-[\frac{1-\overline{M}(a)}{(a)}]\lambda_1E(I)E(I_1[I_1-1])-(\overline{M}(a) \\
&+[\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(-E(D^2)(-1+\overline{B}_2[\lambda^-])(-\xi+b\lambda_2E(I)+\lambda_1E(I)E(I_1))^2-2E(D)E(R) \\
&(-1+\overline{B}_2[\lambda^-])(-\xi+b\lambda_2E(I)+\lambda_1E(I)E(I_1))^2-E(R^2)(-1+\overline{B}_2[\lambda^-])(-\xi+b\lambda_2E(I) \\
&+\lambda_1E(I)E(I_1))^2+E(V^2)\theta\overline{B}_2[\lambda^-](-\xi+b\lambda_2E(I)+\lambda_1E(I)E(I_1))^2 \\
&-\frac{2E(D)(-1+\overline{B}_2[\lambda^-])(\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))(\phi_3)'[1]}{\lambda^-} \\
&-\frac{2E(R)(-1+\overline{B}_2[\lambda^-])(\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))(\phi_3)'[1]}{\lambda^-} \\
&+2E(D)(\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3t)'[1]+2E(R)(\xi-b\lambda_2E(I) \\
&-\lambda_1E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3)'[1]-2E(V)\theta(\xi-b\lambda_2E(I)-\lambda_1E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3)'[1]
\end{aligned}
$$

$$- \frac{2(-1+\overline{B}_2[\lambda^-])(\phi_3)'[1]^2}{(\lambda^-)^2} + \frac{2(\overline{B}_2)'[\lambda^-](\phi_3)'[1]^2}{\lambda^-} - E(D)(-1+\overline{B}_2[\lambda^-])(2\xi + b\lambda_2 E(I[I-1])$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1]) + E(I)E(I_1[I_1-1]))) - E(R)(-1+\overline{B}_2[\lambda^-])(2\xi + b\lambda_2 E(I[I-1])$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1]) + E(I)E(I_1[I_1-1]))) + E(V)\theta\overline{B}_2[\lambda^-](2\xi + b\lambda_2 E(I[I-1])$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1]) + E(I)E(I_1[I_1-1]))) + \frac{(-1+\overline{B}_2[\lambda^-])(\phi_3)''[1]}{\lambda^-})$$

$$n_1'(1) = 2I_{0,0}E(I)(b\lambda_2 + \lambda_1 E(I_1))$$

$$n_1''(1) = I_{0,0}(-\frac{1}{\lambda^-}2(b\lambda_2 E(I) + \lambda_1 E(I)E(I_1))(-\lambda^-(1 - E(D)\xi - E(R)\xi + b(E(D)+E(R))\lambda_2 E(I)$$
$$+ E(D)\lambda_1 E(I)E(I_1) + E(R)\lambda_1 E(I)E(I_1) + (E(D)+E(R)-E(V)\theta)\overline{B}_2[\lambda^-](\xi - b\lambda_2 E(I)$$
$$- \lambda_1 E(I)E(I_1))) - (-1+\overline{B}_2[\lambda^-])(\phi_3)'[1]) + 2(b\lambda_2 E(I[I-1]) + \lambda_1(E(I_1)^2 E(I[I-1])$$
$$+ E(I)E(I_1[I_1-1]))) - 2E(D)E(R)(-1+\overline{B}_2[\lambda^-])(-\xi + b\lambda_2 E(I) + \lambda_1 E(I)E(I_1))^2$$
$$- E(R^2)(-1+\overline{B}_2[\lambda^-])(-\xi + b\lambda_2 E(I) + \lambda_1 E(I)E(I_1))^2 + E(V^2)\theta\overline{B}_2[\lambda^-](-\xi + b\lambda_2 E(I)$$
$$+ \lambda_1 E(I)E(I_1))^2 - \frac{2E(D)(-1+\overline{B}_2[\lambda^-])(\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1))(\phi_3)'[1]}{\lambda^-}$$
$$- \frac{2E(R)(-1+\overline{B}_2[\lambda^-])(\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1))(\phi_3)'[1]}{\lambda^-}$$
$$+ 2E(D)(\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3)'[1] + 2E(R)(\xi - b\lambda_2 E(I)$$
$$- \lambda_1 E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3)'[1] - 2E(V)\theta(\xi - b\lambda_2 E(I) - \lambda_1 E(I)E(I_1))(\overline{B}_2)'[\lambda^-](\phi_3)'[1]$$
$$- \frac{2(-1+\overline{B}_2[\lambda^-])(\phi_3)'[1]^2}{(\lambda^-)^2} + \frac{2(\overline{B}_2)'[\lambda^-](\phi_3)'[1]^2}{\lambda^-} - E(D)(-1+\overline{B}_2[\lambda^-])(2\xi + b\lambda_2 E(I[I-1])$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1]) + E(I)E(I_1[I_1-1]))) - E(R)(-1+\overline{B}_2[\lambda^-])(2\xi + b\lambda_2 E(I[I-1]$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1] + E(I)E(I_1[I_1-1])) + E(V)\theta\overline{B}_2[\lambda^-](2\xi + b\lambda_2 E(I[I-1]$$
$$+ \lambda_1(E(I_1)^2 E(I[I-1] + E(I)E(I_1[I_1-1])) + \frac{(-1+B_2[\lambda^-])(\phi_3)''[1]}{\lambda^-}))$$

$$n_3'(1) = (1 + \overline{M}(a) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1 + [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)I_{0,0}E(I)(b\lambda_2 + \lambda_1 E(I_1))$$

$$n_3''(1) = I_{0,0}(2[\frac{1-\overline{M}(a)}{(a)}]E(I)^2(\lambda_2 - \lambda_1 E(I_1))(b\lambda_2 + \lambda_1 E(I_1)) + (1 + \overline{M}(a) - [\frac{1-\overline{M}(a)}{(a)}]\lambda_1$$
$$+ [\frac{1-\overline{M}(a)}{(a)}]\lambda_2)(b\lambda_2 E(I[I-1] + \lambda_1(E(I_1)^2 E(I[I-1] + E(I)E(I_1[I_1-1])))$$

$$t_1'(1) = bE(D)\lambda_2(1-\overline{B}_1[\lambda^-])E(I) + bE(R)\lambda_2(1-\overline{B}_1[\lambda^-])E(I) - bE(V)\theta\lambda_2\overline{B}_1[\lambda^-]E(I)$$
$$- \frac{(1-\overline{B}_1[\lambda^-])(\phi_1)'[1]}{\lambda^-} - 2(\overline{B}_1)'[\lambda^-](\phi_1)'[1]$$

$$t_1'(1) = bE(D)\lambda_2(1-\overline{B}_1[\lambda^-])E(I) + bE(R)\lambda_2(1-\overline{B}_1[\lambda^-])E(I) - bE(V)\theta\lambda_2\overline{B}_1[\lambda^-]E(I)$$
$$- \frac{(1-\overline{B}_1[\lambda^-])(\phi_1)'[1]}{\lambda^-} - 2(\overline{B}_1)'[\lambda^-](\phi_1)'[1]$$

$$t_1''(1) = -\frac{1}{(\lambda^-)^2}(b^2\lambda_2^2(\lambda^-)^2(-(E(D)+E(R))^2 + (E(D^2) + 2E(D)E(R) + E(R^2) + E(V^2)\theta)$$
$$\overline{B}_1[\lambda^-])E(I)^2 - 2(\phi_1)'[1]^2 - 2\lambda^-(\overline{B}_1)'[\lambda^-](\phi_1)'[1]^2 + b\lambda_2\lambda^-(2(E(D)+E(R))E(I)(\phi_1)'[1]$$
$$+ \lambda^-(2(E(D)+E(R)+E(V)\theta)E(I)(\overline{B}_1)'[\lambda^-](\phi_1)'[1] - (E(D)+E(R))E(I[I-1])$$
$$+ \overline{B}_1[\lambda^-](-2(E(D)+E(R))E(I)(\phi_1)'[1] + (E(D)+E(R)+E(V)\theta)\lambda^- E(I[I-1]))$$
$$+ 2(\lambda^-)^2(\phi_1'[1])^2(\overline{B}_1)''[\lambda^-] + \lambda^-(\phi_1)''[1] + 2(\lambda^-)^2(\overline{B}_1)'[\lambda^-](\phi_1)''[1] + \overline{B}_1[\lambda^-](2(\phi_1)'[1]^2$$
$$- \lambda^-(\phi_1)''[1]))$$

$$n_2''(1) = 2b[\frac{1-\overline{M}(a)}{(a)}]\lambda_2 E(I)n_1'[1] + 2(-bE(V)\theta\lambda_2(1-\overline{B}_1[\lambda^-])E(I) + (1-\overline{B}_2[\lambda^-])(-bE(D)\lambda_2 E(I)$$
$$- bE(R)\lambda_2 E(I)))n_3'[1]$$

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.12, 2016

$$
\begin{aligned}
n_2'''(1) = & \frac{1}{\lambda^-}3b\lambda_2(b\lambda_2\lambda^-(-1-E(D^2)-2E(D)E(R)-E(V^2)\theta+E(V^2)\theta\overline{B}_1[\lambda^-]+(1+E(D^2)\\
& +2E(D)E(R))\overline{B}_2[\lambda^-])C'[1]^2n_3'[1]\\
& -2(E(D)+E(R))(-1+\overline{B}_2[\lambda^-])E(I)n_3'[1](\phi_2)'[1]+\lambda^-(2E(V)\theta E(I)n_3'[1](\overline{B}_1)'[\lambda^-](\phi_1)'[1]\\
& +2(E(D)+E(R))E(I)n_3'[1](\overline{B}_2)'[\lambda^-](\phi_2)'[1]+[\frac{1-\overline{M}(a)}{(a)}]n_1'[1]E(I[I-1])-E(D)n_3'[1]\\
& E(I[I-1])-E(R)n_3'[1]E(I[I-1])-E(V)\theta n_3'[1]E(I[I-1])+E(V)\theta\overline{B}_1[\lambda^-]n_3'[1]E(I[I-1])\\
& +E(D)\overline{B}_2[\lambda^-]n_3'[1]E(I[I-1]+E(R)\overline{B}_2[\lambda^-]n_3'[1]E(I[I-1])+[\frac{1-\overline{M}(a)}{(a)}]E(I)n_1''[1]\\
& -E(D)E(I)n_3''[1]-E(R)E(I)n_3''[1]-E(V)\theta E(I)n_3''[1]+E(V)\theta\overline{B}_1[\lambda^-]E(I)n_3''[1]\\
& +E(D)\overline{B}_2[\lambda^-]E(I)n_3''[1]+E(R)\overline{B}_2[\lambda^-]E(I)n_3''[1]))\\
m_1'(1,1) = & -b\lambda_2((-1-\beta E(E_1)-(E(D)+E(R))\lambda^--\alpha[E(D_1)+E(R_1)])(-1+\overline{B}_1[\lambda^-])\\
& +E(V)\theta\lambda^-\overline{B}_1[\lambda^-])E(I)\\
m_2'(1,1) = & -b\lambda_2((-1-\beta E(E_2)-(E(D)+E(R))\lambda^--\alpha[E(D_2)+E(R_2)])(-1+\overline{B}_2[\lambda^-])\\
& +E(V)\theta\lambda^-\overline{B}_2[\lambda^-])E(I)\\
m_1''(1,1) = & (1+\beta E(E_1)+\lambda^-[E(D)+E(R)]+\alpha[E(D_1)+E(R_1)])[-(\lambda_2bE(I[I-1]))(1-\overline{B}_1[\lambda^-])\\
& +2(\lambda_2bE(I))(\overline{B}_1)'[\lambda^-](\phi_1)'[1]]-(1-\overline{B}_1[\lambda^-])(\lambda_2bE(I))^2(\beta E(E_1^2)\\
& +\lambda^-[E(D^2)+E(R^2)+2E(R)E(D)]+\alpha[E(D_1^2)+E(R_1^2)+2E(R_1)E(D_1)])\\
& -2E(V)\theta(\lambda_2bC'[1])[(\phi_1)'[1]\overline{B}_1[\lambda^-]+\lambda^-(\overline{B}_1)'[\lambda^-]]-(\lambda_2bE(I))^2V^2\theta\lambda^-\overline{B}_1[\lambda^-]\\
& -(\lambda_2bE(I[I-1])E(V)\theta\lambda^-\overline{B}_1[\lambda^-]\\
m_2''(1,1) = & (1+\beta E(E_2)+\lambda^-[E(D)+E(R)]+\alpha[E(D_2)+E(R_2)])[-(\lambda_2bE(I[I-1])(1-\overline{B}_2[\lambda^-])\\
& +2(\lambda_2bE(I))(\overline{B}_1)'[\lambda^-](\phi_1)'[1]]-(1-\overline{B}_1[\lambda^-])(\lambda_2bE(I))^2(\beta E(E_1^2)\\
& +\lambda^-[E(D^2)+E(R^2)+2E(R)E(D)]+\alpha[E(D_2^2)+E(R_2^2)+2E(R_2)E(D_2)])\\
& -2E(V)\theta(\lambda_2bE(I))[(\phi_2)'[1]\overline{B}_2[\lambda^-]+\lambda^-(\overline{B}_2)'[\lambda^-]]-(\lambda_2bE(I))^2E(V^2)\theta\lambda^-\overline{B}_2[\lambda^-]\\
& -(\lambda_2bE(I[I-1]))E(V)\theta\lambda^-\overline{B}_2[\lambda^-]
\end{aligned}
$$

9 The Average Waiting Time in the Queue:

Average waiting time of a customer in the priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}. \tag{93}$$

Average waiting time of a customer in the orbit is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}, \tag{94}$$

where L_{q_1} and L_{q_2} have been found in equations (91) and (92).

10 Particular Cases

Case: I

If there are no priority arrivals, no balking and reneging, no emergency vacation and Bernoulli vacations, no random breakdown, delayed repair. ie., $\lambda_1 = 0, m = 0, \overline{B}_1(.) = 0, b = 1, \xi = 0, \beta = 0,$

$\theta = 0, p = 0, \alpha = 0$ and we let $P_2(z_2)P(z), \lambda_2 = \lambda$

$$
\begin{aligned}
I(z) &= \frac{I_0\{C(z)\overline{B}(\phi(z))\phi(z) + \lambda^- C(z)(1-\overline{B}(\phi(z)))\overline{D}(B(z))\overline{R}(B(z)) - z\phi(z)\}\{1-\overline{M}(\lambda)\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda^-(1-\overline{B}(\phi(z)))\overline{D}(B(z))\overline{R}(B(z))\}\{C(z) + \overline{M}(\lambda)(1-C(z))\}}, \\
P(z) &= \frac{I_0\{\lambda(C(z)-1)(1-\overline{B}(\phi(z)))\overline{M}(\lambda)\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda^-(1-\overline{B}(\phi(z)))\overline{D}(B(z))\overline{R}(B(z))\}\{C(z) + \overline{M}(\lambda)(1-C(z))\}}, \\
D(z) &= \frac{-I_0\{\lambda^-(1-\overline{B}(\phi(z)))\overline{M}(\lambda)(1-\overline{D}(B(z)))\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda^-(1-\overline{B}(\phi(z)))\overline{D}(B(z))\overline{R}(B(z))\}\{C(z) + \overline{M}(\lambda)(1-C(z))\}}, \\
R(z) &= \frac{-I_0\{\lambda^-\overline{D}(B(z))(1-\overline{B}(\phi(z)))\overline{M}(\lambda)(1-\overline{R}(B(z)))\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda^-(1-\overline{B}(\phi(z)))\overline{D}(B(z))\overline{R}(B(z))\}\{C(z) + \overline{M}(\lambda)(1-C(z))\}}.
\end{aligned}
$$

This result is coincide with Kirupa. K and Udaya Chandrika. K[10]

Case: II

If there are no priority arrivals, no negative arrivals, no balking and reneging, no emergency vacation and no feedback. ie.,$\lambda_1 = 0, m = 0, \overline{B}_1(.) = 0, b = 1, \xi = 0, p = 0, \beta = 0, \lambda^- = 0$ and we let $P_2(z_2) = P(z), \lambda_2 = \lambda$

$$
\begin{aligned}
I(z) &= \frac{I_0\{C(z)\overline{B}(\phi(z))\phi(z)[1-\theta+\theta\overline{V}(B(z))] - z\}\{1-\overline{M}(\lambda)\}}{z - \{\overline{B}(\phi(z))[1-\theta+\theta\overline{V}(B(z))]\}\{C(z) + \overline{M}(\lambda)(1-C(z))\}}, \\
P(z) &= \frac{I_0\lambda(1-C(z))(1-\overline{B}(\phi(z)))\overline{M}(\lambda)}{\phi(z)\{\{\overline{B}(\phi(z))[1-\theta+\theta\overline{V}(B(z))]\}\{C(z) + \overline{M}(\lambda)(1-C(z))\} - z\}}, \\
V(z) &= \frac{I_0\theta\overline{B}(\phi(z))\overline{M}(\lambda)(1-\overline{V}(B(z)))}{\{\overline{B}(\phi(z))[1-\theta+\theta\overline{V}(B(z))]\}\{C(z) + \overline{M}(\lambda)(1-C(z))\} - z}, \\
D(z) &= \frac{I_0\alpha(1-\overline{B}(\phi(z)))\overline{M}(\lambda)(1-\overline{D}(B(z)))}{\phi(z)\{\{\overline{B}(\phi(z))[1-\theta+\theta\overline{V}(B(z))]\}\{C(z) + \overline{M}(\lambda)(1-C(z))\} - z\}}, \\
R(z) &= \frac{I_0\alpha\overline{D}(B(z))(1-\overline{B}(\phi(z)))\overline{M}(\lambda)(1-\overline{R}(B(z)))}{\phi(z)\{\{\overline{B}(\phi(z))[1-\theta+\theta\overline{V}(B(z))]\}\{C(z) + \overline{M}(\lambda)(1-C(z))\} - z\}}.
\end{aligned}
$$

This result is coincide with Gautam Choudhury and Jau-Chaun Ke.[3]

11 Numerical Analysis

The above queueing model is analysed numerically with the following assumptions. We consider the service time, emergency vacation and bernoulli vacation time, repair time and delay time are to be exponentially distributed.
We choose the following values: $\lambda^- = 1$, $\lambda_2 = 2$, $\alpha = 0.8$, $\beta = 0.6$, $\beta_e = 0.8$, $\nu = 6$, $\gamma = 0.3$, $\gamma_1 = 0.5$, $\gamma_2 = 1$, $\beta_1 = 0.3$, $\beta_2 = 0.3$, $\xi = 4$, $\mu = 4$, $\xi_d = 3$, $\xi_1 = 0.5$, $\xi_2 = 3$, $b = 0.8, E(I) = 1$ and $E[I(I-1)] = 0$, we choose that λ_1 takes the values $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ and 0.8 while θ takes the values $0.25, 0.50$ and 0.75. All the values were chosen arbitrarily in order that the stability condition is satisfied. The tables gives the computed values of the proportion of idle time and the performence measures. It is clear from the tables that increasing the values of λ_1 or θ increases the average queue lengths for both priority and nonpriority queues, while the server idle time decreases. All the trends shown by this tables and the graphs are as expected.
Results are presented for the values of λ_1 and θ in the following tables with their corresponding graphical representations.

Table 1: Effect of λ_1 on various queue characteristics

λ_1	$I_{0,0}$	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.6519	1.0482	10.2394	10.4822	5.1197
0.2	0.6398	1.1208	10.5730	5.6039	5.2865
0.3	0.6265	1.1975	10.9252	3.9915	5.4626
0.4	0.6120	1.2786	11.2966	3.1965	5.6483
0.5	0.5962	1.3647	11.6871	2.7294	5.8436
0.6	0.5788	1.4562	12.0963	2.4270	6.0481
0.7	0.5596	1.5537	12.5224	2.2196	6.2612
0.8	0.5385	1.6578	12.9621	2.0722	6.4811

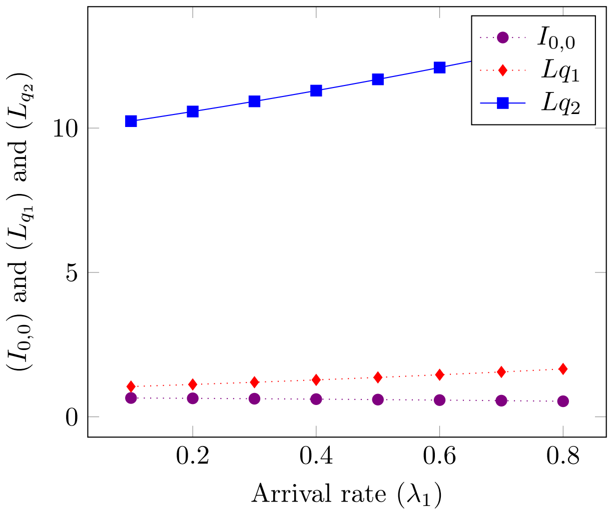

Figure 1: Average queue lengths of priority and non-priority customers verses arrival rate λ_1

Table 2: Effect of λ_1 on various queue characteristics

λ_1	$I_{0,0}$	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.5938	1.2728	8.6291	12.7283	4.3145
0.2	0.5813	1.3244	8.8346	6.6219	4.4173
0.3	0.5677	1.3783	9.0485	4.5945	4.5243
0.4	0.5529	1.4349	9.2700	3.5872	4.6350
0.5	0.5368	1.4941	9.4973	2.9882	4.7487
0.6	0.5192	1.5561	9.7281	2.5936	4.8641
0.7	0.5000	1.6212	9.9583	2.3160	4.9792
0.8	0.4788	1.6894	10.1819	2.1118	5.0910

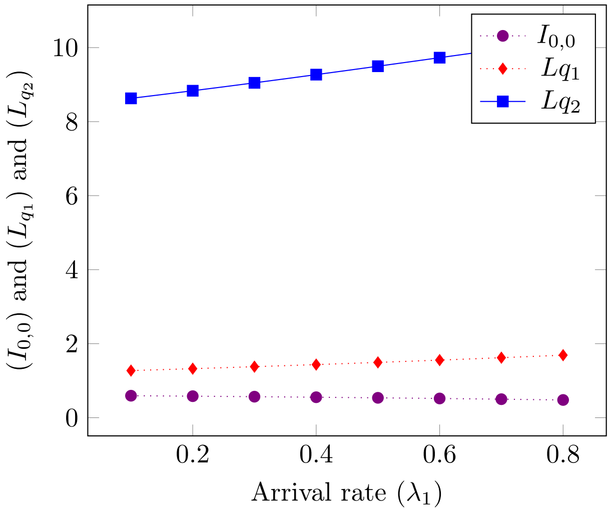

Figure 2: Average queue lengths of priority and non-priority customers verses arrival rate λ_1

Table 3: Effect of λ_1 on various queue characteristics

λ_1	$I_{0,0}$	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.5348	1.4515	8.9243	14.5146	4.4622
0.2	0.5222	1.4825	9.0228	7.4126	4.5114
0.3	0.5087	1.5146	9.1196	5.0487	4.5598
0.4	0.4940	1.5477	9.2122	3.8694	4.6061
0.5	0.4781	1.5818	9.2968	3.1637	4.6484
0.6	0.4608	1.6168	9.3682	2.6947	4.6841
0.7	0.4420	1.6526	9.4191	2.3608	4.7096
0.8	0.4214	1.6889	9.4392	2.1111	4.7196

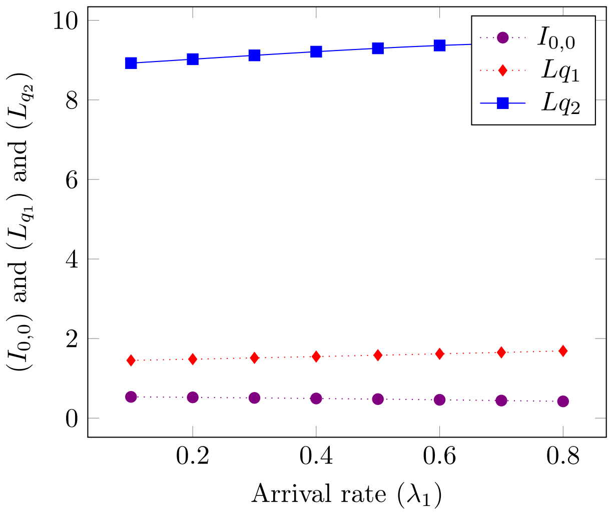

Figure 3: Average queue lengths of priority and non-priority customers verses arrival rate λ_1

References

[1] Ayyappan. G and S.Shyamala (2014) Transient behavior of $M^{[X]}/G/1$ retrial queueing model with non persistence customers, random breakdown, delaying repair and Bernoulli vacation, *Mathematical Theory and Modeling*, Vol.4 No.3

[2] Atencia. I,Moreno. P (2005) A single-server retrial queue with general retrial times and Bernoulli schedule, *Applied Mathematics and Computation,* 162, $855 - 880$.

[3] Gautam Choudhury, Jau-Chuan Ke (2012) A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, *Applied Mathematical Modelling,* $36, 255 - 269$.

[4] Gomez Corral. A (1999) Stochastic analysis of a single server retrial queue with general retrial times, *Nav. Logis.*, vol.46. pp.$561 - 581$.

[5] Jain. M and Charu Bhargava (2008) Bulk arrival retrial queue with unreliable server and priority subscribers, *International Journal of Operations Research,* Vol. 5, No. $4, 242 - 259$.

[6] Jau-Chuan Ke, Fu-Min Chang (2009) Modified vacation policy for $M/G/1$ retrial queue with balking and feedback, *Computers and Industrial Engineering,* $57, 433 - 443$.

[7] Jinbiao Wu, Zhaotong Lian (2013) A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule, *Computers and Industrial Engineering,* $64, 84 - 93$.

[8] Khalaf. R.F, Madan.K.C, Lukas.C.A. (2011). An $M_{[X]}/G/1$ queue with Bernoulli schedule general vacation times, general extended vacation, random breakdown, general delay times for repairs to start and general repair times, *Journal of Mathematics Research,* Vol.3, No.4.

[9] Kirupa. K, Udaya Chandrika. K (2010) Single-server retrial queueing system with two different vacation policies, *Int. J. Contemp.Math.Sciences,*Vol. 5, No. $32, 1591 - 1598$.

[10] Kirupa. K, Udaya Chandrika. K (2014) Batch arrival retrial G-queue and an unreliable server delayed repair,*International Journal of Innovative Research in Science, Engineering and Technology,*Vol. 3.

[11] Yang. T, Templeton.J.G.C.(1987) A survey on retrial queue,*Queueing Systems.* Vol. $2, 201 - 233$.

[12] Yang. T, Posner.M.J.M, Templeton.J.G.C, Li.H (1994) An approximation method for the $M/G/1$ retrial queue with general retrial times, *Eur.J.Oper.Res.* Vol. $76, 552 - 562$.