

Common Fixed Point Theorem for Weakly Compatible Maps in Intuitionistic Fuzzy Metric Spaces

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Abstract
In this paper, we prove some common fixed point theorem for weakly compatible maps in intuitionistic fuzzy metric space for two, four and six self mapping.

1. Introduction

The introduction of the concept of fuzzy sets by Zadeh [1] in1965.Many authors have introduced the concept of fuzzy metric in different ways. Atanassov [2] introduced and studied the concept of intuitionistic fuzzy sets. In 1997 Coker [3] introduced the concept of intuitionistic fuzzy topological spaces .Park [20] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm. Alaca et al. [4] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space. Turkoglu et al. [34] introduced the cocept of compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric space and gave some relations between the concepts of compatible maps and compatible maps of types $(\alpha)\ and\ (\beta)$. Gregory et al. [12], Saadati et al [28],Singalotti et al [27], Sharma and Deshpande [30], Ciric etal [9], Jesic [13], Kutukcu [16] and many others studied the concept of intuitionistic fuzzy metric space and its applications. Sharma and Deshpandey [30] proved common fixed point theorems for finite number of mappings without continuity and compatibility on intuitionistic fuzzy metric spaces. R.P. Pant [23] has initiated work using the concept of R-weakly commuting mappings in 1994.

2. Preliminaries

We begin by briefly recalling some definitions and notions from fixed point theory literature that we will use in the sequel.

Definition 2.1 [Schweizer st. al.1960] - A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norms if $*$ satisfying conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a\ for\ all\ a \in [0,1]$;
(iv) $a * b \le c * d$ wherever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Example of t-norm are $a * b = \min\{a, b\}$ and $a * b = a.b.$

Definition 2.2[Schweizer st. al.1960] A binary operation $\lozenge : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norms if $\lozenge$ satisfying conditions:

(i) $\lozenge$ is commutative and associative;
(ii) $\lozenge$ is continuous;
(iii) $a \lozenge 0 = a$ for all $a \in [0,1]$;
(iv) $a \lozenge b \le c \lozenge d$ wherever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Example of t-norm are $a \lozenge b = \max\{a, b\}$ and $a \lozenge b = \min\{1, a + b\}$.

Definition 2.3 [Alaca, C. et. Al. 2006] A 5-tuple $(X, M, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm, $\lozenge$ is a continuous t -norm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and $t > 0$;
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;
(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) For all $x, y \in X, M(x, y, .): [0, \infty) \to [0,1]$ is left continuous ;
(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$; if and only if $x = y$.
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x, y, t) \lozenge N(y, z, s) \ge N(x, z, t + s)$for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X, N(x, y, .): [0, \infty) \to [0,1]$ is right continuous;
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ *for all* $x, y \in X$;

(M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ *and* $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.4 [Alaca, C. et. Al. 2006]. An intuitionistic fuzzy metric spaces with continuous t-norm $*$ and Continuous t -conorm $\lozenge$ defined by $a * a \geq a, a \in [0,1]$ and $(1 - a) \lozenge (1 - a) \leq (1 - a)$ for all $a \in [0,1]$, Then for all $x, y \in X, M(x, y, *)$ is non-decreasing and, $N(x, y, \lozenge)$ is non-increasing.

Remark 2.5[Park, 2004]. Let (X, d) be a metric space .Define t-norm $a * b = \min\{a, b\}$ and t-conorm$a \lozenge b = \max\{a, b\}$ and for all , $x, y \in X$ and $t > 0$

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space induced by the metric. It is obvious that $N(x, y, t) = 1 - M(x, y, t)$.

Alaca, Turkoglu and Yildiz [Alaca, C. et. Al. 2006] introduced the following notions:

Definition 2.6. Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. Then

(i) a sequence $\{x_n\}$ is said to be Cauchy sequence if, for all $t > 0$ *and* $p > 0$

$$\lim_{n\to\infty} M(x_{n+p,} x_n, t) = 1, N(x_{n+p,} x_n, t) = 0$$

(ii) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if ,for all $t > 0$.

$$\lim_{n\to\infty} M(x_{n,} x, t) = 1, N(x_{n,} x, t) = 0$$

Since $*$ and $\lozenge$ are continuous, the limit is uniquely determined from (v) and (xi) of respectively.

Definition 2.7. An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.8. A pair of self-mappings (f, g) , of an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be compatible if $\lim_{n\to\infty} M(fg\,x_n, gf x_n, t) = 1$ and $\lim_{n\to\infty} N(fg\,x_n, gf x_n, t) = 0$ for every $t > 0$,whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* $z \in X$.

Definition 2.9. A pair of self-mappings (f, g) , of an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be non compatible if $\lim_{n\to\infty} M(fg\,x_n, gf x_n, t) \neq 1$ or nonexistence and $\lim_{n\to\infty} N(fg\,x_n, gf x_n, t) \neq 0$ or nonexistence for every $t > 0$,
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

In 1998, Jungck and Rhoades [Jungck et. al. 1998] introduced the concept of weakly compatible maps as follows:

Definition 2.10. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 2.11 [Alaca, C. et. Al. 2006]. Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space then $f, g : X \to X$ are said to be weakly compatible if they commute at coincidence points.

Lemma 2.12[Alaca, C. et. Al. 2006]. Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X if there exist a number $k \in (0,1)$ such that,

i) $M(y_{n+2,} y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$
ii) $N(y_{n+2,} y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$
for all $t > 0$ and $n = 1,2,3 \ldots \ldots \ldots$then $\{y_n\}$ is a cauchy sequence in X.

Lemma 2.13 Let $(X, M, N, *, \lozenge)$ be an IFM-space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$, $M(x, y, kt) \geq M(x, y, t), N(x, y, kt) \leq N(x, y, t)$ then $x = y$.

3. Main Results -

Theorem 3.1 – Let $(X, M, N, *, \lozenge)$ be an intuitionstic fuzzy metric space with continuous t-norm $*$ and continuous t-norm $\lozenge$ defined by $t * t \geq t$ and $(1 - t) \lozenge (1 - t) \leq (1 - t), \forall\, t \in [0,1]$. Let f and g be weakly compatible self mapping in X s.t.

a) $g(X) \subseteq f(X)$
b) $M(gx, gy, kt) \geq \varphi\{M(fx, fy, t) * M(gx, fy, t) * M(fx, gx, t)\}$
$N(gx, gy, kt) \leq \Psi\{N(fx, fy, t) \lozenge N(gx, fy, t) \lozenge N(fx, gx, t)\}$

Where , $0 < k < 1$ and $\emptyset, \Psi: [0,1] \to [0,1]$ is continuous function s.t. $\emptyset(s) > s$ and $\Psi(s) < s$,for each $0 < s < 1$ and $\varphi(1) = 1$, $\Psi(0) = 0$ with $M(x, y, t) > 0$.

c) If one of $g(X)$ *or* $f(X)$ is complete
Then f and g have a unique common fixed point.

Proof – Let $x_0 \in X$ be any arbitrary point. Since $g(X) \subseteq f(X)$,choose $x_1 \in X$
Such that $y_{2n} = fx_{2n+1} = gx_{2n}$.
Then by (b),

$$M(gx_{2n}, gx_{2n+1}, kt) \geq \varphi \begin{Bmatrix} M(fx_{2n}, fx_{2n+1}, t) * M(gx_{2n}, fx_{2n+1}, t) \\ * M(fx_{2n}, gx_{2n}, t) \end{Bmatrix}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi \begin{Bmatrix} M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n}, t) \\ * M(y_{2n-1}, y_{2n}, t) \end{Bmatrix}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{M(y_{2n-1}, y_{2n}, t) * 1 * M(y_{2n-1}, y_{2n}, t)\}$$
$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{M(y_{2n-1}, y_{2n}, t)\} > M(y_{2n-1}, y_{2n}, t) \text{.............(1)}$$

As $\varphi(s) > s$, for each $0 < s < 1$.

and

$$N(gx_{2n}, gx_{2n+1}, kt) \leq \Psi \begin{Bmatrix} N(fx_{2n}, fx_{2n+1}, t) \lozenge N(gx_{2n}, fx_{2n+1}, t) \\ \lozenge N(fx_{2n}, gx_{2n}, t) \end{Bmatrix}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi \begin{Bmatrix} N(y_{2n-1}, y_{2n}, t) \lozenge N(y_{2n}, y_{2n}, t) \\ \lozenge N(y_{2n-1}, y_{2n}, t) \end{Bmatrix}$$

$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{N(y_{2n-1}, y_{2n}, t) \lozenge 0 \lozenge N(y_{2n-1}, y_{2n}, t)\}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{N(y_{2n-1}, y_{2n}, t)\} < N(y_{2n-1}, y_{2n}, t)$$

As $\Psi(s) < s$ for each $0 < s < 1$.

For all n,

$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$ and $N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n-1}, y_{2n}, t)$

$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t)$ and $N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n+1}, y_{2n}, t)$,

Hence by lemma (2.12) , $\{y_{2n}\}$ is a Cauchy sequence in X. by completeness of X, $\{y_{2n}\}$ =$\{fx_{2n}\}$ is convergent, call z ,

Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$

Now suppose $f(X)$ is complete, so there exist a point $p \ in \ X \ such \ that \ fp = z$

Now from (b),

$M(gp, gx_n, kt) \geq \varphi\{M(fp, fx_n, t) * M(gp, fx_n, t) * M(fp, gp, t)\}$

As $n \to \infty$,

$M(gp, z, kt) \geq \varphi\{M(z, z, t) * M(gp, z, t) * M(z, gp, t)\}$

$M(gp, z, kt) \geq \varphi\{1 * M(gp, z, t) * M(z, gp, t)\}$

$M(gp, z, kt) \geq \varphi\{ M(gp, z, t)\} > M(gp, z, t)$ ....(3)

And

$$N(gp, gx_n, kt) \leq \Psi\{N(fp, fx_n, t) \lozenge N(gp, fx_n, t) \lozenge N(fp, gp, t)\}$$

As $n \to \infty$,

$$N(gp, z, kt) \leq \Psi\{N(z, z, t) \lozenge N(gp, z, t) \lozenge N(z, gp, t)\}$$
$$N(gp, z, kt) \leq \Psi\{0 \lozenge N(gp, z, t) \lozenge N(z, gp, t)\}$$
$$N(gp, z, kt) \leq \Psi\{N(gp, z, t)\} < N(gp, z, t) \text{ ....(4)}$$

From (3) and (4),we have

$$gp = z = fp$$

As $f \ and \ g$ are weakly compatible ,therefore $fgp = gfp \, , i.e, fz = gz$

Now , we show that z is a fixed point of $f \ and \ g$. from (b),

$$M(gz, gx_n, kt) \geq \varphi\{M(fz, fx_n, t) * M(gz, fx_n, t) * M(fz, gz, t)\}$$

As $n \to \infty$,

$$M(gz, z, kt) \geq \varphi\{M(gz, z, t) * M(gz, z, t) * M(gz, gz, t)\}$$
$$M(gz, z, kt) \geq \varphi\{M(gz, z, t) * M(gz, z, t) * 1\}$$
$$M(gz, z, kt) \geq \varphi\{M(gz, z, t)\} > M(gz, z, t) \text{ ....(5)}$$

and

$$N(gz, gx_n, kt) \leq \Psi\{N(fz, fx_n, t) \lozenge N(gz, fx_n, t) \lozenge N(fz, gz, t)\}$$

As $n \to \infty$,

$$N(gz, z, kt) \leq \Psi\{N(gz, z, t) \lozenge N(gz, z, t) \lozenge N(gz, gz, t)\}$$
$$N(gz, z, kt) \leq \Psi\{N(gz, z, t) \lozenge N(gz, z, t) \lozenge 0\}$$
$$N(gz, z, kt) \leq \Psi\{N(gz, z, t)\} < N(gz, z, t) \text{ ....}(6)$$

From (5) and (6),

$$gz = z = fz$$

Hence z is common fixed point of f and g.

Uniqueness – Let w be another fixed point of f and g ,then by (b) ,

$$M(gz, gw, kt) \geq \varphi\{M(fz, fw, t) * M(gz, fw, t) * M(fz, gz, t)\}$$

$$M(z,w,kt) \geq \varphi\{M(z,w,t) * M(z,w,t) * M(z,z,t)\}$$
$$M(z,w,kt) \geq \varphi\{M(z,w,t) * M(z,w,t) * 1\}$$
$$M(z,w,kt) \geq \varphi\{M(z,w,t)\} > M(z,w,t) \quad \text{.........................(7)}$$

and

$$N(gz,gw,kt) \leq \Psi\{N(fz,fw,t) \lozenge N(gz,fw,t) \lozenge N(fz,gz,t)\}$$
$$N(z,w,kt) \leq \Psi\{N(z,w,t) \lozenge N(z,w,t) \lozenge N(z,z,t)\}$$
$$N(z,w,kt) \leq \Psi\{N(z,w,t) \lozenge N(z,w,t) \lozenge 0\}$$
$$N(z,w,kt) \leq \Psi\{N(z,w,t)\} < N(z,w,t) \quad \text{.....................(8)}$$

From (7) and (8),

$$z = w$$

Therefore z is unique common fixed point of f and g.

Theorem – 3.2 Let $(X, M, N,*,\lozenge)$ be an intuitionstic fuzzy metric space with continuous t-norm $*$ and continuous t-norm $\lozenge$ defined by $t * t \geq t$ and $(1-t) \lozenge (1-t) \leq (1-t)$,$\forall\, t \in [0,1]$.Let A, B, S and T be self mappings in X s.t.

a) $A(X) \subseteq S(X)$ and $B(X) \subseteq T(X)$.
b) There exist a constant $k \in (0,1)$, $s.t.$
   $M(Ax,By,kt) \geq \varphi\{M(Tx,Sy,t) * M(Tx,Ax,t) * M(Ax,Sy,t)\}$
   $N(Ax,By,kt) \leq \Psi\{N(Tx,Sy,t) \lozenge N(Tx,Ax,t) \lozenge N(Ax,Sy,t)\}$
   $\forall\, x, y \in X$ and $t > 0$,$0 < k < 1$. where $\varphi, \Psi: [0,1] \to [0,1]$ is continuous function s.t. $\varphi(s) > s$ and $\Psi(s) < s$, for each $0 < s < 1$ and $\varphi(1) = 1$,
   $\Psi(0) = 0$ with $M(x,y,t) > 0$.
c) If one of the $A(X), B(X), S(X)$ and $T(X)$ is complete subspace of X,
   Then $\{A,T\}$ and $\{B,S\}$ have a coincidence point
   More over ,if the pair $\{A,T\}$ and $\{B,S\}$ are weakly compatible ,then A, B, S and T have a unique common fixed point.

Proof – Let x_0 be any arbitrary point since $A(X) \subseteq S(X)$, there is a point $x_1 \in X$ $s.t.$ $Ax_0 = Sx_1$. Again since $B(X) \subseteq T(X)$ for this $x_2 \in X$ $s.t.$ $Bx_1 = Tx_2$ and so on. Then we get a sequence $\{y_n\}$ s.t. $y_{2n} = Ax_{2n} = Sx_{2n-1}$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$, $n = 0,1,2 \ldots \ldots$

Putting $x = x_{2n}$, $y = x_{2n+1}$ in (b) we have,

$$M(Ax_{2n}, Bx_{2n+1}, kt) \geq \varphi\{M(Tx_{2n}, Sx_{2n+1}, t) * M(Tx_{2n}, Ax_{2n}, t) * M(Ax_{2n}, Sx_{2n+1}, t)\}$$
$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n}, t)\}$$
$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{M(y_{2n-1}, y_{2n}, t) * 1\}$$
$$\geq \varphi\{M(y_{2n-1}, y_{2n}, t)\} > M(y_{2n-1}, y_{2n}, t)$$

As $\varphi(s) > s$ for each $0 < s < 1$.
and

$$N(Ax_{2n}, Bx_{2n+1}, kt) \leq \Psi\{N(Tx_{2n}, Sx_{2n+1}, t) \lozenge N(Tx_{2n}, Ax_{2n}, t) \lozenge N(Ax_{2n}, Sx_{2n+1}, t)\}$$
$$N(Ax_{2n}, Bx_{2n+1}, kt) \leq \Psi\{N(y_{2n-1}, y_{2n}, t) \lozenge N(y_{2n-1}, y_{2n}, t) \lozenge N(y_{2n}, y_{2n}, t)\}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{N(y_{2n-1}, y_{2n}, t) \lozenge 0\}$$
$$\leq \Psi\{N(y_{2n-1}, y_{2n}, t)\} < N(y_{2n-1}, y_{2n}, t)$$

As $\Psi(s) < s$ for each $0 < s < 1$.
For all n,
$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$and $N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n-1}, y_{2n}, t)$
$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t)$ and $N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n-1}, y_{2n}, t)$
Hence by lemma (2.12) , $\{y_n\}$ is a Cauchy sequence in X.
Now suppose $S(X)$ is a complete subspace of X.note that the sequence $\{y_{2n}\}$ is contained in $S(X)$ and has a limit in $S(X)$ say u. so we get $Sw = u$. we shall use the fact that subsequence $\{y_{2n+1}\}$ also convergence to u . now putting $x = x_{2n}$,$y = w$ in (b)
and taking $n \to \infty$

$$M(Ax_{2n}, Bw, kt) \geq \varphi\{M(Tx_{2n}, Sw, t) * M(Tx_{2n}, Ax_{2n}, t) * M(Ax_{2n}, Sw, t)\}$$
$$M(u, Bw, kt) \geq \varphi\{M(u,u,t) * M(u,u,t) * M(u,u,t)\}$$
$$= \varphi(1) = 1$$

$i.e$ $M(u, Bw, kt) \geq 1$ ....(3)
Also,

$$N(Ax_{2n}, Bw, kt) \leq \Psi\{N(Tx_{2n}, Sw, t) \lozenge N(Tx_{2n}, Ax_{2n}, t) \lozenge N(Ax_{2n}, Sw, t)\}$$
$$N(u, Bw, kt) \leq \Psi\{N(u,u,t) \lozenge N(u,u,t) \lozenge N(u,u,t)\}$$
$$= \Psi(0) = 0$$

i.e,$N(u, Bw, kt) \leq 0$ ....(4)
from (3) and (4),$u = Bw$.

Since $Sw = Bw = u, i.e\ w$ is the coincidence point of $B\ and\ S$.
As $B(X) \subseteq T(X), u = Bw \Rightarrow u \in T(X)$.let $v \in T^{-1}u$ then $Tv = u$
By putting $x = v, y = x_{2n+1}$ in (b),we get,

$$M(Av, Bx_{2n+1}, kt) \geq \varphi\{M(Tv, Sx_{2n+1}, t) * M(Tv, Av, t) * M(Av, Sx_{2n+1}, t)\}$$

As $n \to \infty$

$$M(Av, u, kt) \geq \varphi\{M(u, u, t) * M(u, Av, t) * M(Av, u, t)\}$$

$$M(Av, u, kt) \geq \varphi\{M(u, Av, t) * 1\} = \varphi\{M(u, Av, t)\} > \{M(u, Av, t)\} \quad ....(5)$$

and

$$N(Av, Bx_{2n+1}, kt) \leq \Psi\{N(Tv, Sx_{2n+1}, t) \lozenge N(Tv, Av, t) \lozenge N(Av, Sx_{2n+1}, t)\}$$

As $n \to \infty$

$$N(Av, u, kt) \leq \Psi\{N(u, u, t) \lozenge N(u, Av, t) \lozenge N(Av, u, t)\}$$

$$N(Av, u, kt) \leq \Psi\{N(u, Av, t) \lozenge 0\} = \Psi\{N(u, Av, t)\} < N(u, Av, t) \quad ....(6)$$

From (5) and (6),we get,

$$Av = u$$

Since $Tv = u$,we have $Av = Tv = u$.thus v is the coincidence point of A and T.If one assume $T(X)$ to be complete ,then an analogous argument establish this claim.
The remaining two cases pertain essentially to the previous cases. Indeed if $B(X)$ is complete then $u \in B(X) \subset T(X)$ and if $A(X)$ is complete then $u \in A(X) \subset S(X)$. Thus (c) is completely established.
Since the pair $\{A, T\}\ and\ \{B, S\}$ are weakly compatible ,i.e.
$B(Sw) = S(Bw) \Rightarrow Bu = Su$ and $A(Tv) = T(Av) \Rightarrow Au = Tu$.
Putting $x = u, y = x_{2n+1}$ in (b),we get

$$M(Au, Bx_{2n+1}, kt) \geq \varphi\{M(Tu, Sx_{2n+1}, t) * M(Tu, Au, t) * M(Au, Sx_{2n+1}, t)\}$$

As $n \to \infty$

$$M(Au, u, kt) \geq \varphi\{M(Au, u, t) * M(Au, Au, t) * M(Au, u, t)\}$$

$$M(Au, u, kt) \geq \varphi\{M(Au, u, t)\} > M(Au, u, t) \quad ....(7)$$

and

$$N(Au, Bx_{2n+1}, kt) \leq \Psi\{N(Tu, Sx_{2n+1}, t) \lozenge N(Tu, Au, t) \lozenge N(Au, Sx_{2n+1}, t)\}$$

As $n \to \infty$

$$N(Au, u, kt) \leq \Psi\{N(Au, u, t) \lozenge N(Au, Au, t) \lozenge N(Au, u, t)\}$$

$$N(Au, u, kt) \leq \Psi\{N(Au, u, t)\} < N(Au, u, t) \quad ....(8)$$

From (7) and (8), implies that $Au = u \Rightarrow Au = Tu = u$
Similarly by putting $x = x_{2n}, y = u$ in (b) and as $n \to \infty$
We have $u = Bu = Su.\ thus\ Au = Su = Tu = u.$
i.e u is a common fixed point of A, B, S and T.
Uniqueness- let $w(w \neq u)$ be another common fixed point of A, B, S and T.
Then by putting $x = u\ , y = w\ in\ (b)$

$$M(Au, Bw, kt) \geq \varphi\{M(Tu, Sw, t) * M(Tu, Au, t) * M(Au, Sw, t)\}$$

$$M(u, w, kt) \geq \varphi\{M(u, w, t) * M(u, u, t) * M(u, w, t)\}$$

$$M(u, w, kt) \geq \varphi\{M(u, w, t) * 1\} > M(u, w, t) \quad ....(9)$$

and

$$N(Au, Bw, kt) \leq \Psi\{N(Tu, Sw, t) \lozenge N(Tu, Au, t) \lozenge N(Au, Sw, t)\}$$

$$N(u, w, kt) \leq \Psi\{N(u, w, t) \lozenge N(u, u, t) \lozenge N(u, w, t)\}$$

$$N(u, w, kt) \leq \Psi\{N(u, w, t) \lozenge 0\} < N(u, w, t)\ . \quad ....(10)$$

From (9) and (10)
$u = w$ for all $x, y \in X$ and $t > 0$.
Therefore u is the unique common fixed point of A, B, S and T.
Theorem – 3.3 Let $(X, M, N, *, \lozenge)$ be an intuitionstic fuzzy metric space with continuous t-norm $*$ and continuous t-norm $\lozenge$ defined by $t * t \geq t\ and\ (1 - t) \lozenge (1 - t) \leq (1 - t)$, $\forall\ t \in [0,1]$. Let A, B, S and T be self mappings in X s.t.

a) $P(X) \subseteq ST(X)\ and\ Q(X) \subseteq AB(X)$
b) There exist a constant $k \in (0,1)\ s.t.$

$$M(Px, Qy, kt) \geq \varphi\{\ M(Px, ABx, t) * M(STy, ABx, t) * (Px, STy, t)\}$$

and

$$N(Px, Qy, kt) \leq \Psi\{\ N(Px, ABx, t) \lozenge N(STy, ABx, t) \lozenge N(Px, STy, t)\}$$

$\forall\ x, y\ \in X\ and\ t > 0\ where\ \varphi, \Psi: [0,1] \to [0,1]$ is continuous function s.t.

$\varphi(s) > s$ *and* $\Psi(s) < s$, for each $0 < s < 1$ and $\varphi(1) = 1$,
$\Psi(0) = 0$ with $M(x, y, t) > 0$.

(c) If one of the $P(X), Q(X), ST(X),$ *and* $AB(X)$ is a complete subspace of X then $\{AB, P\}$ *and* $\{Q, ST\}$ have a coincidence point.

(d) $AB = BA, ST = TS, PB = BP$ *and* $QT = TQ$
Moreover if the pair $\{AB, P\}$ *and* $\{Q, ST\}$ are weakly compatible ,then $A, B, S, T, P,$ *and* Q have a unique common fixed point.

Proof – Let $x_0 \in X$ be an arbitrary point. since $P(X) \subseteq ST(X)$, there exist $x_1 \in X$ s.t. $Px_0 = STx_1 = y_0$. again since $Q(X) \subseteq AB(X)$ for this x_1 there is $x_2 \in X$ s.t.
$Qx_1 = ABx_2 = y_1$ and so on. Inductively we get a sequence $\{x_n\}$ and $\{y_n\}$ in X s.t.
$y_{2n} = Px_{2n} = STx_{2n+1}$ *and* $y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$, $n = 0,1,2 \ldots \ldots$
Putting $x = x_{2n}, y = x_{2n+1}$ in (b) we have,

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \varphi\{ M(Px_{2n}, ABx_{2n}, t) * M(STx_{2n+1}, ABx_{2n}, t) * (Px_{2n}, STx_{2n+1}t)\}$$
$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{ M(y_{2n}, y_{2n-1}, t) * M(y_{2n}, y_{2n-1}, t) * M(y_{2n}, y_{2n}, t)\}$$
$$M(y_{2n}, y_{2n+1}, kt) \geq \varphi\{ M(y_{2n}, y_{2n-1}, t) * 1\} > M(y_{2n}, y_{2n-1}, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$N(Px_{2n}, Qx_{2n+1}, kt) \leq \Psi\{ N(Px_{2n}, ABx_{2n}, t) \Diamond N(STx_{2n+1}, ABx_{2n}, t) \Diamond N(Px_{2n}, STx_{2n+1}, t)\}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{ N(y_{2n}, y_{2n-1}, t) \Diamond N(y_{2n}, y_{2n-1}, t) \Diamond N(y_{2n}, y_{2n}, t)\}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{ N(y_{2n}, y_{2n-1}, t) \Diamond N(y_{2n}, y_{2n-1}, t) \Diamond N(y_{2n}, y_{2n}, t)\}$$
$$N(y_{2n}, y_{2n+1}, kt) \leq \Psi\{ N(y_{2n}, y_{2n-1}, t) \Diamond 0\} < N(y_{2n}, y_{2n-1}, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence we have from (1) and (2)
$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n-1}, t)$ and $N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n}, y_{2n-1}, t)$.
Similarly we also have
$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t)$ and $N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n+1}, y_{2n}, t)$.
$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$ and $N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n+1}, y_{2n}, t)$,
Hence by lemma (2.12) ,$\{y_n\}$ is a Cauchy sequence in X.
Now suppose $AB(X)$ is a complete subspace of X .note that the sequence $\{y_{2n+1}\}$ is contained in $AB(X)$ and has a limit in $AB(X)$ say z. so we get $ABw = z$. we shall use the fact that subsequence $\{y_{2n}\}$ also convergence to z . now putting $x = w$,$y = x_{2n+1}$ in (b)
and taking $n \to \infty$, we have

$$M(Pw, Qx_{2n+1}, kt) \geq \varphi\{ M(Pw, ABw, t) * M(STx_{2n+1}, ABw, t) * (Pw, STx_{2n+1}, t)\}$$

As $n \to \infty$

$$M(Pw, z, kt) \geq \varphi\{ M(Pw, z, t) * M(z, z, t) * (Pw, z, t)\}$$
$$M(Pw, z, kt) \geq \varphi\{ M(Pw, z, t) * 1\} > M(Pw, z, t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and

$$N(Pw, Qx_{2n+1}, kt) \leq \Psi\{ N(Pw, ABw, t) \Diamond N(STx_{2n+1}, ABw, t) \Diamond N(Pw, STx_{2n+1}, t)\}$$

As $n \to \infty$

$$N(Pw, z, kt) \leq \Psi\{ N(Pw, z, t) \Diamond N(z, z, t) \Diamond N(Pw, z, t)\}$$
$$N(Pw, z, kt) \leq \Psi\{ N(Pw, z, t) \Diamond 0\} < N(Pw, z, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3) and (4) , $Pw = z$. Since $ABw = z$ thus we have $Pw = z = ABw$ that is w is coincidence point of P and AB. Since $P(X) \subset ST(X), Pw = z$ implies that $z \in ST(X)$.
let $v \in ST^{-1}z$.then $STv = z$.
Putting $x = x_{2n}$ and $y = v$ in (b), we have

$$M(Px_{2n}, Qv, kt) \geq \varphi\{ M(Px_{2n}, ABx_{2n}, t) * M(STv, ABx_{2n}, t) * (Px_{2n}, STv, t)\}$$

As $n \to \infty$

$$M(z, Qv, kt) \geq \varphi\{ M(z, z, t) * M(z, z, t) * (z, z, t)\}$$
$$M(z, Qv, kt) \geq \varphi\{1\} = 1$$
$$M(z, Qv, kt) \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and

$$N(Px_{2n}, Qv, kt) \leq \Psi\{ N(Px_{2n}, ABx_{2n}, t) \Diamond N(STv, ABx_{2n}, t) \Diamond N(Px_{2n}, STv, t)\}$$
$$N(z, Qv, kt) \leq \Psi\{ N(z, z, t) \Diamond N(z, z, t) \Diamond N(z, z, t)\}$$
$$N(z, Qv, kt) \leq \Psi\{0\} = 0$$
$$N(z, Qv, kt) \leq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (5) and (6),we have $z = Qv$.
Again Putting $x = z$ and $y = x_{2n+1}$ in (b) and as $n \to \infty$

$$M(Pz, Qx_{2n+1}, kt) \geq \varphi\{ M(Pz, ABz, t) * M(STx_{2n+1}, ABz, t) * (Pz, STx_{2n+1}, t)\}$$
$$M(Pz, z, kt) \geq \varphi\{ M(Pz, Pz, t) * M(z, Pz, t) * (Pz, z, t)\}$$
$$M(Pz, z, kt) \geq \varphi\{ M(Pz, z, t) * 1\} > M(Pz, z, t) \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and

$$N(Pz, Qx_{2n+1}, kt) \le \Psi\{ N(Pz, ABz, t) \lozenge N(STx_{2n+1}, ABz, t) \lozenge N(Pz, STx_{2n+1}, t)\}$$

$$N(Pz, z, kt) \le \Psi\{ N(Pz, Pz, t) \lozenge N(z, Pz, t) \lozenge N(Pz, z, t)\}$$

$$N(Pz, z, kt) \le \Psi\{ N(Pz, z, t) \lozenge 0\} < N(Pz, z, t) \text{ ....(8)}$$

From (7) and (8), $Pz = z$. So $Pz = ABz = z$.
By putting $x = x_{2n}$, $y = z$ in (b) and taking $n \to \infty$, $Qz = z$. hence $Qz = STz = z$.
Now putting $x = z$, $y = Tz$ in (b) and using (d), we have
$M(z, Tz, kt) \ge 1$ *and* $N(z, Tz, kt) \le 0$.*thus* $z = Tz$.
Since $STz = z$, *therefore* $Sz = z$.
To prove , $Bz = z$, *we put* $x = Bz$, $y = z$ *in (b) and using (d),*
We have $M(z, Bz, kt) \ge 1$ *and* $M(z, Bz, kt) \le 0$.*Thus* $z = Bz$.
Since $ABz = z$ *therefore* $Az = z$.
By combining the above result we have $Az = Bz = Sz = Tz = Pz = Qz = z$.
that is z *is a common fixed point of* A, B, S, T, P *and* Q.
Uniqueness *– Let* $w (w \ne z)$ *be another common fixed point of* A, B, S, T, P *and* Q *then* $Aw = Bw = Sw = Tw = Pw = Qw = w$.
By putting $x = z, y = w$, *we have* $M(z, w, kt) \ge 1$ *and* $N(z, w, kt) \le 0$.
Hence $z = w$ *for all* $x, y \in X$ *and* $t > 0$.
Therefore z *is the unique common fixed point of* A, B, S, T, P *and* Q.

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