Countability and Separability in Tritopological Spaces (δ^*-Countability and δ^*-Separability).

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Abstract
In this paper, we will continue the basic specifications of Tritopological spaces begun by author. Furthermore, we introduce the new concepts and definitions of Countability and Separability in Tritopological spaces namely δ^*-Countability and δ^*-Separability. The properties for them are established. Also we introduce some definitions, theorems, examples and obtained some results.
Keywords: δ^*-First Countable space, δ^*-second Countable space, δ^*-separable space, Tritopological property, δ^*-dense, δ^*-local base, δ^*-base, δ^*-homeomorphism, δ^*-adherent point, δ^*-open set, Tritopological space.

1. Introduction

Throughout this paper we adopt the terminology and notations of (Asmhan, 2004), (Asmhan et al., 2011), (Repovšet al., 2010) and (Bourbaki, 1966). We consider X and Y as a finite sets and the following conventions : (X,T), (X,T,Ω), (X,T,Ω,ρ) will always denoted to Topological space, Bitopological space and Tritopological space respectively .

In (O.Njastad., 1965) α-open set was defined as: Let (X,T) be a topological space , and let A be asubset of X, then A is said to be a α-open set iff $A \subseteq T-\mathrm{int}(T-\mathrm{cl}(T-\mathrm{int}(A)))$, and the family of all α-open sets is denoted by $\alpha.O(X)$. The complement of α-open set is called α-closed set. We suggested the reader is referred to (Mashour et al., 1973), (Mrsevic & Reilly, 1996) and (Maheshwari & Thakur, 1985) for the detail definitions and notations.

(Kelly, 1963) has been introduced the notion of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. There are several works dedicated to the investigation of bitopologies, i.e. pairs of topologies on the same set.

In (Jaleel, 2003) the definition of δ-open set in bitopological space was introduced on the basis of α-open set in topological space. which defined as: Let (X,T,Ω) be a Bitopological space, and let A be a subset of X, then A is said to be a δ-open set iff $A \subseteq T-\mathrm{int}(\Omega-\mathrm{cl}(T-\mathrm{int}(A)))$, and the family of all δ-open sets is denoted by $\delta.O(X)$. The complement of δ-open set is called δ-closed set, and notice that $\delta.O(X)$ does not represent a topology (not always represent a topology).

We suggested the reader is referred to (Jabbar & Nasir, 2010), (Datta, 1977) and (Arya & Nour, 1988) for the detail definitions and notations to bitopological spaces.

In (Kovar, 2000) tritopological spaces was first initiated by modify the concept of θ-regularity for spaces with 2 and 3 topologies. Also (Luay A.,2003) has been introduced the notion of tritopological spaces in which was the main topic in his paper and dealt with in detail and clear. Where they define it as a spaces equipped with three (arbitrary) topologies, i.e. triple of topologies on the same set. And in (Asmhan, 2004) the definition of δ^*-open set in tritopological space was introduced on the basis of δ -open set in bitopological space (Jaleel, 2003) and α-open set in topological space (O.Njastad., 1965). Which defined as: Let (X,T,Ω,ρ) be a tritopological space, a subset A of X is said to be δ^*-open set iff $A \subseteq T-\mathrm{int}(\Omega-\mathrm{cl}(\rho-\mathrm{int}(A)))$, and the family of all δ^*-open sets is denoted by $\delta^*.O(X)$. The complement of δ^*-open set is called a δ^*-closed set, and we notice that $\delta^*.O(X)$ does not represent a topology (not always represent a topology).

(Asmhan, 2011) defined the connectedness in tritopological space. And (Asmhan et al., 2011) defined the base in tritopological space. Also (Asmhan, 2009) and (Asmhan & Hanan, 2007) introduced a relationships among separation axioms, and a relationships among some types of continuous and open functions in topological, bitopological and tritopological spaces.

In the following we will mention some basic definitions and notations in tritopological space from (Asmhan, 2004) and (Asmhan et al., 2011), which we need in this work.

(X,T,Ω,ρ) is called discrete tritopological space with respect to δ^*-open if $\delta^*.O(X)$ contains all subsets on X. And (X,T,Ω,ρ) is called indiscrete topological space with respect to δ^*-open if $\delta^*.O(X)=\{X,\emptyset\}$.

Let (X,T,Ω,ρ) be a tritopological space, and let $x \in X$, A a subset N of X is said to be a δ^*-nhd of a point x iff there exists a δ^*-open set U such that $x \in U \subset N$. The set of all δ^*-nhds of a point x is denoted by $\delta^*-N(x)$.

Let (X,T,Ω,ρ) be a tritopological space. A nonempty collection δ^*- $\beta(x)$ of δ^*-neighbourhoods of x is called a δ^*-base for the δ^*-neighbourhood system of x (δ^*-Local Base) iff for every δ^*-neighbourhood N of x there is a $B \in \delta^*$- $\beta(x)$ such that $B \subset N$ .We then also say that δ^*-$\beta(x)$ is a δ^*-Local Base at x or a fundamental system of a δ^*-neighbourhoods of x. If δ^*-$\beta(x)$ is a δ^*-Local Base at x, then the members of δ^*-$\beta(x)$ are called basic δ^*-neighbourhoods of x.

If $\delta^*.O(X)$ represent a topology then every point x in X has a δ^*-local base in tritopology, and If $\delta^*.O(X)$ does not represent a topology then not every point x in X has a δ^*-local base in tritopology .

A collection δ^*- β of a subsets of X is said to form a δ^*-base for the tritopology (T, Ω, ρ) iff :

I- $\delta^*\text{-}\beta \subset \delta^*.O(X)$.

II- for each point $x \in X$ and each δ^*-neighbourhood N of x there exists some $B \in \delta^*\text{-}\beta$ such that $x \in B \subset N$.

If $\delta^*.O(X)$ represent a topology then the tritopology (T, Ω, ρ) has a δ^*-base. And If $\delta^*.O(X)$ does not represent a topology then the tritopology (T, Ω, ρ) has not a δ^*-base.

The function f: $(X, T, \Omega, \rho) \to (Y, T', \Omega', \rho')$ is said to be a δ^*-continuous at $x \in X$ iff for every δ^*-open set V in Y containing f(x) there exists δ^*-open set U in X containing x such that $f(U) \subset V$. We say f is δ^*-continuous on X iff f is δ^*-continuous at each $x \in X$.

Let (X, T, Ω, ρ) and (Y, T', Ω', ρ') are two tritopological spaces and f: $(X, T, \Omega, \rho) \to (Y, T', \Omega', \rho')$ be a function, then f is δ^*-homeomorphism if and only if :

I- f is bijective (one to one , on to).

II- f and f^{-1} are δ^*-continuous.

Let (X, T, Ω, ρ) be a tritopological space, and $A \subset X$, the intersection of all δ^*-closed sets containing A is called δ^*-closure of A and denoted by $\delta^* - cl(A)$; i.e. $\delta^* - cl(A) = \cap \{F: F \text{ is } \delta^* - \text{closed}, A \subset F\}$.

Let (X, T, Ω, ρ) be tritopological space and let Y be asubset of X. The relative tritopological space for Y is denoted by $(Y, T_y, \Omega_y, \rho_y)$; such that :

$T_y = \{ G \cap Y : G \in T \}$, $\Omega_y = \{ G \cap Y : G \in \Omega \}$ and $\rho_y = \{ G \cap Y : G \in \rho \}$

then $(Y, T_y, \Omega_y, \rho_y)$ is called the subspace of tritopological space (X, T, Ω, ρ). And the relative tritopological space for Y with respect to δ^*-open sets is the collection $\delta^*.O(X)_y$ given by ; $.\delta^*O(X)_y = \{ G \cap Y : G \in \delta^*.O(X)\}$.

This paper contain 4 sections. In Section 2, we introduce a new definition of two types of δ^*-countability in tritopological space (δ^*-first countable space and δ^*-second countable space), and we gives some theorems, exampels and remarks. In Section 3, we define the separability in tritopological spaces (δ^*-separability), also gives some theorems, examples and remarks. In Section 4, we obtain the main and important results of δ^*-countability and δ^*-separability in tritopological space.

This paper deals with two novel subjects in Tritopological space which named (δ^*-Countability and δ^*-Separability) will explain as fellows.

2. Countability in Tritopological Spaces (δ^*-Countability).

δ^*-countability axioms is the name used to refer to a set of properties of a tritopological space which have to do with the existence of countable sets, or countable families of δ^*-open sets, satisfying certain conditions. They are not axioms in the strict sense of the word, but they are usually named as such because one may think of them as additional basic properties that one can ask from a tritopological space.

Countability in tritopological space have two types (δ^*-first countable space and δ^*-second countable space) will define as fellows.

2.1 δ^*-First Countable Space.

2.1.1 Definition:

A Tritopological space (X, T, Ω, ρ) is said to satisfies first axiom of δ^*-countability if each point of X possesses a countable δ^*-local base such a tritopological space is said to be a δ^*-first countable space.

2.1.2 Remark:

If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-first Countable space. And If $\delta^*.O(X)$ does not represent a topology then a tritopological space (X, T, Ω, ρ) can not be a δ^*-first countable space.

Remark above is result from remarks (If $\delta^*.O(X)$ represent a topology then every point x in X has a δ^*-local base in tritopology) And (If $\delta^*.O(X)$ does not represent a topology then not every point x in X has a δ^*-local base in tritopology) (Asmhan et al., 2011). And the following examples show that.

2.1.3 Example:

Consider the discrete tritopological space (X, T, Ω, ρ). Since in a discrete tritopological space, every subset of X is δ^*-open (i.e. $\delta^*.O(X)$ contains all subsets on X (Asmhan, 2004)). In particular the singleton {x}, $x \in X$ is δ^*-open and so is a δ^*-nhd of x. Also every δ^*-nhd N (i.e. δ^*-open set containing x in this case) of x must be a super set of {x}, hence the collection {{x}} consisting of the single δ^*-nhd {x} of X constitutes a δ^*-local base at

x . But a collection {{x}} consisting of a single member is countable. Hence there exists a δ^*-countable base at each point of X. Hence the discrete tritopological space (X, T, Ω, ρ) is δ^*-first Countable space.
(i.e. If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-first countable space).

2.1.4 Example:

Let $X = \{a, b, c, d\}$, $T = \{X, \emptyset, \{c, d\}\}$
, $\Omega = \{X, \emptyset, \{a, b, c\}, \{a\}\}$
and $\rho = \{X, \emptyset, \{d\}, \{c, d\}, \{a, d\}\}$

$(X, T), (X, \Omega), (X, \rho)$ are three topological spaces, then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^*.O(X) = \{ X, , \emptyset, \{a, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$$

Clearly that $\delta^*.O(X)$ does not represent a topology. Then the δ^*-Local Base at each of the points a , b , c , d is given by :

δ^*-β(a) ={{a,d}} , δ^*-β(c) ={{c,d}}

But there is no δ^*-β(b) because the intersection of {a,b,d} and {b,c,d} is {b,d} is not δ^*-open set (not exist) , i.e. there is no set B such that B is a subset of all δ^*-nhds of b .

Again , there is no δ^*-β(d) because the intersection of {a,d} and {c,d} is {d} is not δ^*-open set (not exist), i.e. there is no set D such that D is a subset of all δ^*-nhds of d.

Hence not every point x in X has a δ^*-local base in tritopology, Hence not every point x in X has a δ^*-countable local base in tritopology.

(i.e. If $\delta^*.O(X)$ does not represent a topology then a tritopological space (X, T, Ω, ρ) can not be a δ^*-first countable space)

2.1.5 Definition:

A property of a tritopological space (X, T, Ω, ρ) is said to be hereditary property if every sub space of a tritopological space (X, T, Ω, ρ) has that property.

2.1.6 Definition:

A property of a tritopological space (X, T, Ω, ρ) is said to be tritopological property iff it is preserved under δ^*-homeomorphism.

2.1.7 Theorem:

The property of being a δ^*-first countable space is a hereditary property.

Proof:

Let a tritopological space (X, T, Ω, ρ) be δ^*-first countable space and let $(Y, T_y, \Omega_y, \rho_y)$ be a subspace of a tritopological space (X, T, Ω, ρ). We have to show that $(Y, T_y, \Omega_y, \rho_y)$ is also a δ^*-first countable space. Let y be any arbitrary point of Y, then $y \in X$ (since $Y \subset X$). And since (X, T, Ω, ρ) is δ^*-first countable space, then the collection δ^* - $\beta_y(y) = \{Y \cap B: B \in \delta^* - \beta(y)\}$ forms a δ^*-countable local base at y in $\delta^*.O(X)_y$ for $(Y, T_y, \Omega_y, \rho_y)$.

2.1.8 Theorem:

The property of being a δ^*-first countable space is a tritopological property.

Proof:

Let (Y, T', Ω', ρ') be a δ^*-homeomorphic image of a δ^*-first countable space (X, T, Ω, ρ) under a δ^*-homeomorphism $f: (X, T, \Omega, \rho) \to (Y, T', \Omega', \rho')$. To show that (Y, T', Ω', ρ') is also a δ^*-first countable space. Let $y \in Y$ be arbitrary. Since f is onto, there exists $x \in X$ such that $y = f(x)$. Since X is a δ^*-first countable space there exists a δ^*-countable local base, say $\delta^* - \beta(x)$ at x .

We shall show that the collection $\{f[B]: B \in \delta^* - \beta(x)\}$ is a δ^*-countable local base at y. Surely this collection is countable.

Now let $M \in \delta^*.O(Y)$ be any δ^*-nhd of y. Then there exists a δ^*-open set G in $\delta^*.O(Y)$ such that $f(x) = y \subset G \subset M$, then $x \in f^{-1}[G] \subset f^{-1}[M]$.

Since f is δ^*-continuous function, $f^{-1}[G]$ is a δ^*-open set in $\delta^*.O(X)$ containing x. Since $\delta^* - \beta(x)$ is a δ^*-local base at x, there exists $B \in \delta^* - \beta(x)$such that $x \in B \subset f^{-1}[G] \subset f^{-1}[M]$.

This implies that $y = f(x) \in f[B] \subset M$. This shows that $\{f[B]: B \in \delta^* - \beta(x)\}$ is a δ^*-countable local base at y. Hence (Y, T', Ω', ρ') is δ^*-first countable space.

2.2 δ^*-second countable spaces.

2.2.1 Definition:

Let (X, T, Ω, ρ) be a tritopological space. The space is said to be a δ^*-second countable (or to satisfy the second axiom of δ^*-countability in tritopology) iff there exists a δ^* -countable base for a tritopology.

2.2.2 Remark:

If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-second countable space. And If $\delta^*.O(X)$does not represent a topology then a tritopological space (X, T, Ω, ρ) can not be a δ^*-second countable space.

Remark above is result from the remarks (If $\delta^*.O(X)$ represent a topology then the tritopological space (X, T, Ω, ρ) has a δ^*-base) And (If $\delta^*.O(X)$ does not represent a topology then the tritopological space (X, T, Ω, ρ) has not a δ^*-base) (asmhan et al., 2011). And the following examples show that.

2.2.3 Example:

Consider the discrete tritopological space (X, T, Ω, ρ). Then the collection δ^*-$\beta = \{\{x\}: x \in X\}$ consisting of all singleton subsets of X is a δ^*-base for a tritopology (T, Ω, ρ). For each singleton set is δ^*-open so that δ^*-$\beta \subset \delta^*.O(X)$. Also for each $x \in X$ and each δ^*-nhd N of x, $\{x\} \in \delta^*$-β is such that $x \in \{x\} \subset N$. Then the collection $\{\{x\}\}$ consisting of a single member is countable. Hence there exists a δ^*-countable base for a tritopology (T, Ω, ρ), hence discrete tritopological space (X, T, Ω, ρ) is δ^*-second Countable space.

(i.e. If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-second countable space).

2.2.4 Example:

Let $X = \{a, b, c, d\}$, $T = \{X, \emptyset, \{c, d\}\}$
, $\Omega = \{X, \emptyset, \{a, b, c\}, \{a\}\}$
and $\rho = \{X, \emptyset, \{d\}, \{c, d\}, \{a, d\}\}$

$(X, T), (X, \Omega), (X, \rho)$ are three topological spaces, then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^*.O(X) = \{ X, , \emptyset, \{a, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$$

Clearly that $\delta^*.O(X)$ does not represent a topology .

Then this tritopology has not a δ^*-base because the second condition of the δ^*-base definition in (Asmhan et al., 2011) is not hold. because the intersection of $\{a, b, d\}$ and $\{b, c, d\}$ is $\{b, d\}$ which is not a δ^*-open set (not exist) , i.e. each δ^*-nhds of b contains {b,d} , but {b,d} is not exist in $\delta^*.O(X)$ because $\delta^*.O(X)$ does not represent a topology . .

Again , because the intersection of $\{a, d\}$ and $\{c, d\}$ is $\{d\}$ which is not a δ^*- open set (not exist) , i.e. each δ^*-nhds of d contains {d} , but {d} is not exist in $\delta^*.O(X)$ because $\delta^*.O(X)$ does not represent a topology. Hence there is no a δ^*-base for a tritopology (T, Ω, ρ), hence there is not exist a δ^*-countable base for a tritopology (T, Ω, ρ), hence this tritopological space is not a δ^*-second countable space. (i.e If $\delta^*.O(X)$ does not represent a topology then a tritopological space (X, T, Ω, ρ) can't be a δ^*-second countable space).

2.2.5 Theorem:

Every δ^*-second countable space is δ^*-first countable space.

Proof:

Let (X, T, Ω, ρ) be a δ^*-second countable space and let δ^*-β be a δ^*-countable base for a tritopology (T, Ω, ρ). Let p any arbitrary number of X . If δ^*-β , consists of all those members of δ^*-β which contains p , then it is easy to see that δ^*-β (p) is a δ^*-countable local base at p. It follows that (X, T, Ω, ρ) is δ^*-first countable space.

2.2.6 Remark:

The converse of above theorem is, however, not true. For example every discrete tritopological space is δ^*-first countable space, but it is not δ^*-second countable.

2.2.7 Theorem:

The property of being a δ^*-second countable space is a hereditary property.

Proof:

Let a tritopological space (X, T, Ω, ρ) be δ^*-second countable space and let δ^*-β be a δ^*-countable base for a tritopology (T, Ω, ρ). Let $(Y, T_y, \Omega_y, \rho_y)$ be a subspace of a tritopological space (X, T, Ω, ρ). By using theorem in (Asmhan et al., 2011) (Let $(Y, T_y, \Omega_y, \rho_y)$ be a subspace of a tritopological space (X, T, Ω, ρ) and let δ^*-β be a δ^*-base for a tritopology (T, Ω, ρ). Then δ^*- $\beta_y(y) = \{Y \cap B: B \in \delta^* - \beta(y)\}$ is a δ^*-base for a subspace $(Y, T_y, \Omega_y, \rho_y)$), then δ^*- $\beta_y(y) = \{Y \cap B: B \in \delta^* - \beta(y)\}$ is a δ^*-base for a subspace $(Y, T_y, \Omega_y, \rho_y)$. Also δ^*-$\beta_y(y)$ surely countable. Hence there exists a δ^*-countable base for $(Y, T_y, \Omega_y, \rho_y)$. It follows that every subspace of a tritopological space (X, T, Ω, ρ) is δ^*-second countable space.

2.2.8 Theorem:

The property of being a δ^*-second countable space is a tritopological property.

Proof:

Let (Y, T', Ω', ρ') be a δ^*-homeomorphic image of a δ^*-second countable space (X, T, Ω, ρ) under a δ^*-homeomorphism $f: (X, T, \Omega, \rho) \to (Y, T', \Omega', \rho')$. Let δ^*-β be a δ^*-countable base for a tritopology (T, Ω, ρ). We shall show that the collection $\{f[B]: B \in \delta^* - \beta\}$ is a δ^*-countable base for (T', Ω', ρ'). Surely this collection is countable since δ^*-β is countable. Further since f is δ^*-open, each $f[B]$ is δ^*-open set in $\delta^*.O(Y)$, and let G be any δ^*-open subset of $\delta^*.O(Y)$. Then $f^{-1}[G]$ is a δ^*-open since f is δ^*-continuous and since δ^*-β be a δ^*-countable base for a tritopology (T, Ω, ρ). We have $f^{-1}[G] = \cup\{B: B \in D \subset \delta^*\text{-}\beta\}$, then $f[f^{-1}[G]] = f[\cup\{B: B \in D]$ or $G = \cup\{f[B]: B \in D\}$.

Thus any δ^*-open set in $\delta^*.O(Y)$ is expressible is a union of members of $\{f[B]: B \in \delta^* - \beta\}$. Hence $\{f[B]: B \in$

$\delta^* - \beta$ } is a δ^*-countable base for (T', Ω', ρ'). And therefore Y is δ^*-second countable space.

3. Separability In Tritopological Spaces (δ^*- Separability).

3.1 Definition :

Let (X, T, Ω, ρ) be a tritopological space and let A be subsets of X . Then
A is said to be a δ^*-dense in B iff $B \subset \delta^*$-cl(A).
A is said to be a δ^*-dense in X or everywhere δ^*-dense iff δ^*-cl(A) = X.

3.2 Example:

Let $X = \{a, b, c, d\}$, $T = \{X, \emptyset, \{b, c, d\}\}$
, $\Omega = \{X, \emptyset, \{a\}\}$
and $\rho = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}\}$

$(X, T), (X, \Omega), (X, \rho)$ are three topological spaces, then (X, T, Ω, ρ) is a tritopological space, such that :

$$\delta^*.O(X) = \{ X, , \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c, d\}\}$$

$$\delta^*.C(X) = \{ X, , \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, d\}, \{a, b\}, \{c, d\}, \{a, d\}, \{b, c\}, \{b, c, d\}, \{a, b, d\}, \}$$

Clearly that $\delta^*.O(X)$ does not represent a topology .
Then the sets $\{a, c, d\}$ and $\{a, b, c\}$ are δ^*-dense in X because δ^*-cl($\{a, c, d\}$) = X and δ^* -cl($\{a, b, c\}$) = X .
And $\{a, c, d\}$ is δ^*-dense in $\{a, c\}$ since $\{a, c\} \subset \delta^*$-cl($\{a, c, d\}$) = X

3.3 Definition

A tritopological space (X, T, Ω, ρ) is said to be a δ^*-separable iff X contains a countable δ^*-dense subset in $\delta^*.O(X)$, that is, iff there exists a countable subset say A of X such that δ^* -cl(A) = X.

3.4 Remark

If $\delta^*.O(X)$ represent or does not represent a topology then a tritopological space can be a δ^*-separable space. (as in example (3.2) above (X, T, Ω, ρ) does not represent a topology, but clearly it is represent a δ^*-separable space).

3.5 Remark

In fact any discrete tritopological space (X, T, Ω, ρ), where X is an uncountable set is not δ^*-separable.
And the discrete tritopological space (X, T, Ω, ρ) is δ^*-separable iff X is countable.
Because the only δ^*-dense subset of X is X itself. Hence X contains a countable δ^*-dense subset iff X is countable.

3.6 Theorem

A δ^*-continuous image of a δ^*-separable space is δ^*-separable, so the property of a space being δ^*-separable is a tritopological property.

Proof:

Let (X, T, Ω, ρ) be a δ^*-separable space and let f be a δ^*-continuous mapping of (X, T, Ω, ρ) onto another space (Y, T', Ω', ρ') so that Y is a δ^*-continuous image of X. We want to show that (Y, T', Ω', ρ') is δ^*-separable.

Let A be a countable δ^*-dense subset of X. (Such a set always exists since X is δ^*-separable). Then f[A] is surely countable.

Now since A is δ^*-dense in X, δ^*-cl(A) = X so that Y= f[X] = $f[\delta^* - cl(A)]$ and by the δ^*-continuity of f , $f[\delta^* - cl(A)] \subset \delta^* - cl(f[A])$. Thus $Y \subset \delta^* - cl(f[A])$. But $\delta^* - cl(f[A]) \subset Y$ always. Hence Y= $\delta^* - cl(f[A])$ and so f[A] is δ^*-dense in Y. Thus f[A] is a countable δ^*-dense subset of Y and consequently Y is δ^*-separable. In particular, homeomorphic image of a δ^*-separable space is δ^*-separable space. It follows that δ^*-separability is a tritopological property.

3.7 Definition:

Let A be a subset of a tritopological space X and let $x \in X$. Then x is called a δ^*-Adherent point of A iff every δ^*-nhd of x contains a point of A. The set of δ^*-Adherent points of A is called the δ^*-Adherence of A and shall be denoted by δ^*-Adh(A).

3.8 Theorem:

Every δ^*-second countable space is δ^*-separable.

Proof:

Let X be a δ^*-second countable space. We shall show that it is δ^*-separable. Since the space is δ^*-second countable there exists δ^*-β a countable δ^*-base for a tritopology (T, Ω, ρ). We choose a point b from each member B of δ^*-β . Let D be the set which is evidently countable . We now show that D is δ^*-dense in X. Let x be any arbitrary point of X and let G be any δ^*-open nhd of x. Since δ^*-β is a δ^*-base, there exists at least one B ∈ δ^*-β such that x ∈ B ⊂ G. By definition of D, b ∈ D is such that b ∈ B ⊂ G. Thus G contains a point of D. Hence x is an δ^*-adherent point of D. Since x is arbitrary we have δ^*-cl(D)= X. It follows that D is a countable δ^*-dense in X. Hence X is δ^*-separable.

3.9 Corollary:
Every δ^*-second countable space is hereditarily δ^*-separable.
Proof:
We know that every subspace of a δ^*-second countable space is δ^*-second countable and hence δ^*-separable by the above theorem.

4. Conclusions

- If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-first countable space. And If $\delta^*.O(X)$ does not represent a topology then a tritopological space (X, T, Ω, ρ) can not be a δ^*-first countable space.
- If $\delta^*.O(X)$ represent a topology then a tritopological space (X, T, Ω, ρ) can be a δ^*-second countable space. And If $\delta^*.O(X)$ does not represent a topology then a tritopological space (X, T, Ω, ρ) can not be a δ^*-second countable space.
- If $\delta^*.O(X)$ represent or does not represent a topology then a tritopological space can be a δ^*-separable space.
- δ^*-first countability is hereditary as well as a tritopological property.
- δ^*-second countability implies δ^*-first countability but not conversely.
- δ^*-second countability is hereditary as well as a tritopological property.
- δ^*-seperability is a tritopological property.
- δ^*-second countability implies δ^*-seperability.
- Every δ^*-second countable space is hereditarily δ^*-separable.

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