# Some Bounds For The Spectral Radius Of Hadamard Product & Kronecker Product Of Matrices.

Alaa Mohammad Saeed Abu Alruz

Financial & Banking Sciences Department, Al-Balqaa Applied University, P.O.Box 107, Amman 11118 – Jordan

**Abstract**

The main aim of this study was to discuss some bounds for the Spectral Radius of the Hadamard Product of matrices.

This study presents several spectral radius inequalities for sums, product ( hadamard product), and comutators of matrices, and it exposes to some properties of the hadamard product and the relationship between hadamard product and kronecker product for spectral radius of matrix.

Applications of these results are also given.

**Keywords**: Spectral Radius, Hadamard product, Kronecker Product

## 1. Introduction

The spectral radius function is one of the most important functions of matrices. It is closely related to matrix norms and the numerical radius.

Many functionals in matrix analysis are submultiplicative with respect to ordinary matrix multiplication, but the spectral radius is not. However, for nonnegative or positive semidefinite matrices A,B ∈ Mn, the spectral radius is submultiplicative with respect to the Hadamard (entry-wise) product:

$r(A°B) \leq r(A)r(B)$. ((see Theorem 3.3.1).

This result, among other interesting properties of the spectral radius and the hadamard product, can be found in the famous book of Halmos (1982) and that of Horn and Johnson (1985). Equalities and inequalities for the spectral radii of hadamard product and kronecker product of matrices have been given by Zhang (1999), Cheng, G-H., (2005) Barra and Boumazgour (2001). Spectral radius inequalities for sums, products, and commutators of matrices have been recently given by Kittaneh (2005). These inequalities are based on a spectral radius inequality for partitioned matrices due to Hou and Du (1995).

The material of this research has been arranged by sections, spread out in three parts. The arrangement of the subject matter is given in such a way to give a brief survey of results related to the spectral radius.

 

Firstly, the study introduces some preliminary results in matrix theory that will be very useful in this research. These include some elements of the spectral theory, positive definite matrices, nonnegative and positive matrices, and matrix norm.

Secondly, the study deals with matrix norms and introduces the concept of spectral radius. Special emphasis is given to properties of the spectral radius, and presents several inequalities for the spectral radii of sums, products, and commutators of matrices.

Finally, the study introduces some basic definitions and properties of Hadamard product and Kronecker product of matrices. Also it gives and proves some bounds for the spectral radius of Hadamard product of matrices.

## 2. Previous studies

1) The study of **(**Pattrawut Chansangiam_, Patcharin Hemchote, Praiboon Pantaragphong, 2009) aimed to develop inequalities for Kronecker products and Hadamard products of positive definite matrices. A number of inequalities involving powers, Kronecker powers, and Hadamard powers of linear combination of matrices are presented. In particular, H¨older inequalities and arithmetic mean-geometric mean inequalities for Kronecker products and Hadamard products are obtained as special cases.

2) The study of ( Roger A. Horn and Fuzhen Zhang, 2010) aimed to prove Zhan's conjecture (the spectral radius of the Hadamard product of two square nonnegative matrices is not greater than the spectral radius of their ordinary product), and a related inequality for positive semidefinite matrices, using standard facts about principal sub matrices, Kronecker products, and the spectral radius.

3) The study of (Dongjun Chen and Yun Zhang, 2015) presented some spectral radius inequalities for nonnegative matrices. Used the ideas of Audenaert, and then proved the inequality which may be regarded as a Cauchy--Schwarz inequality for spectral radius of nonnegative matrices

$$r(A^\circ B) \leq [r(A^\circ A)]^{1/2}[r(B^\circ B)]^{1/2}.$$

In addition, new proofs of some related results due to Horn and Zhang, Huang were also given. Finally, it interpolated Huang's inequality by proving

$$r(A_1{}^\circ A_2{}^\circ \dots {}^\circ A_k) \leq [r(A_1A_2 \dots A_k]^{1-\frac{2}{k}}[r((A_1{}^\circ A_1) \dots (A_k{}^\circ A_k)]^{\frac{1}{k}} \leq r(A_1A_2 \dots A_K).$$

On the spectral radius of Hadamard products of nonnegative matrices

4) The study of (Koenraad M.R**., 2010)** aimed to prove an inequality for the spectral radius of products of non-negative matrices conjectured by Zhan. And showed that for all n × n non-negative matrices A and B, $r(A^\circ B) \leq r[(A^\circ A)(B^\circ B)]^{\frac{1}{2}} \leq r(AB)$, in which ◦ represents the Hadamard product.
5) The study of (Maozhong Fang, 2007) aimed to prove an upper bound for the spectral radius of the Hadamard product of nonnegative matrices and a lower bound for the minimum eigenvalue of the Fan product of M-matrices.
6) The study of ( M. Goldberg, G. Zwas, 1974) characterized all *n*x*n* matrices whose spectral radius equals their spectral norm. it showed that for $n \geqslant 3$ the class of these matrices contains the normal matrices as a subclass.
7) The study of (Zejun Huang, 2010) aimed to prove the spectral radius inequality $r(A_1{}^\circ A_2{}^\circ \dots {}^\circ A_k) \leq r(A_1A_2 \dots A_k)$for nonnegative matrices using the ideas of Horn and Zhang. It obtained the inequality $\|A^\circ B\| \leq r(A^TB)$ for nonnegative matrices, which improves Schur's classical inequality $\|A^\circ B\| \leq \|A\|\|B\|$, where $\|.\|$ denotes the spectral norm. It also gave counterexamples to two conjectures about the Hadamard product.

## 3. Fundamentals of Matrix Analysis

### 3.1 Basic Results in Matrix Theory :-

Let $M_{mn}$ denote the space of all $m \times n$ complex matrices and let $M_n$ denote the algebra of all $n \times n$ complex matrices.

 

**Definition 3.1.1**: Let $A \epsilon M_n$ , Then a complex number λ is called an eigenvalue of A, if there exists a nonzero vector $x \in C^n$. Such that $Ax = \lambda x$. Such a vector $x$ is called an eigenvector of A associated with λ.

**Definition 3.1.2**: If $A \in M_n$, then det(λI – A) = 0 is called the characteristic equation of A, where det (.) is the determinant function. The polynomial p(λ ) = det (λI – A) is called the characteristic polynomial of A. the set of all eigenvalues of A is called the spectrum of A, denoted by σ (A).

**Theorem 3.1.3**: If $A \in M_n$, then $\lambda \in \sigma$ (A) is an eigenvalue of A if and only if det(λI – A) = 0

**Definition 3.1.4**: Let A= $[a_{ij}] \in M_n$. Then

(1) the trace of A is given by trA $=\sum_{i=1}^{n} a_{ii}$.
(2) the transpose of A is given by $A^t = [a_{ji}]$and $A^* = [\overline{a_{ji}}]$ is called the adjoint of A.

**Theorem 3.1.5**: For all A, B ∈ $M_n$

1) σ(AB)=σ (BA)
2) If σ(A) = $\{\lambda_1,\ldots,\lambda_n\}$, then det(A) $=\prod_{j=1}^{n} \lambda_j$, and tr(A)= $\sum_{j=1}^{n} \lambda_j$.
3) $\sigma(A^*) = \{\bar{\lambda}: \lambda \in \sigma(A)\}$.

**Theorem 3.1.6**: Let A,B ∈ $M_n$, and let $\alpha \in C$. Then

1) det AB = (det A)(det B)
2) det $(\alpha A) = \alpha^n \det A$.
3) $\sigma(A^k) = (\sigma(A))^k = \{\lambda^k : \lambda \in \sigma(A)\}$, where k is a natural number.
4) $\sigma(A^t)=\sigma(A)$ .
5) for any matrix A with rank at most 1, σ(A)= {trA,0}

**Theorem 3.1.7**: Let A, B Є $M_n$, a Є C. Then

1) tr(A + B)= trA +trB.
2) traA = a trA.
3) trAB = trBA.
4) tr0 = 0 , tr$I_n$ = n , where $I_n$ is the idenitity matrix of order n.

**Theorem3.1.8**: Let A,B Є $M_n$ , $\alpha$ Є C. Then

(1) $(A^*)^* = A$.
(2) $(A+B)^* = A^* + B^*$
(3) $(\alpha A^*) = \bar{\alpha} A^*$.
(4) $(A B)^* = B^* A^*$
(5) $\det(A^*) = \overline{\det(A)}$
(6) $trA^* = \overline{trA}$.
(7) $trA^* A \geq 0$ .
(8) $\sigma(A^*) = \overline{\sigma(A)}$.

**Definition 3.1.9**: A,B Є $M_n$ are called similar if there exists invertible S ∈ $M_n$ such that

$B = S^{-1} AS$ or $A = SBS^{-1}$

**Theorem 3.1.10**: Similar matrices have the same eigenvalues.

**Corollary 3.1.11**: Similar matrices have the same determinant and trace.

**Theorem 3.1.12**:(The spectral mapping theorem), Let A Є $M_n$. Then for every polynomial $f$

$\sigma(f(A)) = f(\sigma(A))$.

**Theorem 3.1.13**: ( Cayely - Hamilton ). Every matrix satisfies its characteristic Polynomial (i.e, if $A \in M_n$ and p is the characteristic polynomial of A , then $p(A) = 0$).

**Remark 3.1.14**: Let $A \in M_n$ and let $k \in C$. Then $\sigma(kA) = k\sigma(A)$.

**Definition 3.1.15**: If $A \in M_n$ , then

(1) A is called Hermitian if $A^* = A$.
(2) A is called skew-Hermitian if $A^* = -A$.
(3) A is called unitary if $A^*A = AA^* = I$.
(4) A is called normal if $A^*A = AA^*$
It is obvious that Hermitian, skew-Hermitian and unitary matrices are normal matrices.

**Remark 3.1.16**:

(1) The sum of two Hermitian matrices is Hermitian.
(2) The product of two Hermitian matrices is Hermitian if and only if these matrices commute.
(3) If $A \in M_n$, then $AA^*$ , $A^*A$ , $A + A^*$ are Hermitian.
(4) If $A \in M_n$ is Hermitian, then every eigenvalue of A is a real number.

**Remark 3.1.17**: Let $A \in M_n$ . Then

(1) If A is unitary, then $|\det A| = 1$.
(2) The product of two unitary matrices is unitary.
(3) A is unitary if and only if $A^{-1} = A^*$ is unitary.
(4) If A is unitary, then every eigenvalue of A has modulus one.

**Definition 3.1.18**: If $x = [x_1,x_2,...,x_n]^t$ ,$y = [y_1,y_2,...,y_n]^t \in C^n$ , then the Euclidean inner product of x and y is given by: $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_1$

Note that, $(x, y) = y^*x$

**Remark 3.1.19**: Let x, y, z $\in C^n, \alpha \in C$ Then

(1) $(y, x) = \overline{(x, y)}$
(2) $(\alpha x, y) = \alpha(x, y)$
(3) $(x + y, z) = (x, z) + (y, z)$
(4) $(x, x) = \sum |x_i|^2 \geq 0$ with equality iff x=0
(5) $(x, \alpha y) = \bar{\alpha}(x, y)$
(6) $(z, x + y) = (z, x) + (z, y)$

**Definition 3.1.20**: A matrix $B \in M_n$ is said to be unitarily equivalent to $A \in M_n$ if there is a unitary matrix $U \in M_n$ such that $B = U^*AU$.

**Theorem 3.1.21**: ( Schur's unitary triangularization theorem ). Let $A \in M_n$ with $\sigma(A) = \{\lambda_1,\lambda_2, \dots ,\lambda_n\}$. Then there is a unitary matrix $U \in M_n$ such that $U^*AU = T$, where $T = [t_{ij}] \in M_n$ is an upper triangular matrix with diagonal entries $t_{ii} = \lambda_i$, for i = 1, 2, … ,n.

**Theorem 3.1.22**: (Spectral theorem for normal matrices). A ∈ $M_n$ is normal if and only if A is unitarily equivalent to a diagonal matrix (that is, A = $UDU^*$, where D is diagonal and U is unitary)

## 3.2 Positive Definite Matrices

**Definition 3.2.1**: A Hermitim matrix A ∈ $M_n$ is said to be positive definite, if (Ax,x)> 0 for all nonzero x ∈ $C^n$ , and it is called positive semidefinite, if (Ax,x) ≥ 0 for all x ∈ $C^n$

**Remark 3.2.2**:
(1) The sum of any two positive definite (semidefinite) matrices of the same size is positive definite (semidefinite).
(2)The product of any two positive definite (semidefinite) matrices is positive definite (semidefinite) if and only if the two matrices commute.
(3) Each eigenvalue of a positive definite (semidefinite) matrix is a positive( nonnegative) real number.
(4) A Hermitian matrix whose eigenvalues are positive (nonnegative) is positive definite (positive semidefinite).
(5) The trace and determinant of a positive definite (semidefinite) matrix are positive (nonnegative) real numbers.

**Theorem 3.2.3**: Let A ∈ $M_n$ be a positive semidefinite (definite) matrix and let k ≥1 be a given integer .Then there exists a unique positive semidefinite (definite) matrix B such that A = $B^k$ ,written as B = $A^{1/k}$.

**Theorem 3.2.4**: A matrix A ∈ $M_n$ is positive semidefinite if and only if A = $BB^*$ for some B ∈ $M_n$ . In the positive definite case B is taken to be invertible.

**Definition 3.2.5**: The eigenvalues of the matrix $\langle A\rangle = (A^*A)^{1/2}$ are called the singular values of A. They are denoted by $s_1$ (A), $s_2$ (A), ... , $s_n$ (A) and they are arranged in nonincreasing order so that $s_1$ (A) ≥ $s_2$ (A) ≥ …≥ $s_n$(A ).

**Theorem 3.2.6**: (Singular value decomposition). If A ∈ $M_n$, then A may be written in the form A = $VDW^*$, where V,W ∈ $M_n$ are unitary, and the matrix D = diag($s_1$(A),$s_2$(A),...,$s_n$(A).

**Theorem 3.2.7**: (Polar decomposition). If A ∈ $M_n$, then there exists a unitary matrix U ∈ $M_n$ such that A=U $\langle A\rangle$

**Remark 3.2.8**: Let A ∈ $M_n$ and let U, V ∈ $M_n$ be unitary, Then
(1)The matrices $A^*A$ and $AA^*$ are unitarily equivalent, and hence, they have the same eigenvalues.
(2) The matrices $\langle UAV\rangle$ and (A) are unitarily equivalent, which implies that $s_j$ (UAV)= $s_j$ (A) for all j = 1,2,...,n.
(3) lf A is normal with eigenvalues λ (A) ordered in such a way that | $\lambda_1$ (A) |≥ … ≥| $\lambda_n$ (A) |, then $s_j$ (A)= | $\lambda_j$(A) | for all j= 1,2,…., n.

**Definition 3.2.9**: A matrix A = $[a_{ij}]$ ∈ $M_n$ is called diagonally dominant if | $a_{ii}$ | > $\sum_{i\neq j}|a_{ij}|$ for i = 1, 2, … , n.

**Definition 3.2.10**: Let A∈ $M_{mn}$. For index sets $\alpha\epsilon\{1, 2, ..., m\}$ and $\beta \in\{1, 2, .., n\}$, we denote the submatrix that lies in the rows of A indexed by $\alpha$ and the columns indexed by ß as A($\alpha$, $\beta$). If m = n and $\beta = \alpha$, the submatrix A($\alpha$, $\alpha$) is called a principal submatrix of A and is abbreviated A($\alpha$).

**Definition 3.2.11**: Let B = [$b_{ij}$ ]∈ $M_n$ and A = [$a_{ij}$ ] ∈ $M_n$ .We write
B ≥ 0 if all $b_{ij}$ ≥ 0
B > 0 if all $b_{ij}$ > 0

 

$A \geq B$ if $A-B \geq 0$
$A > B$ if $A - B > 0$.

If $A \geq 0$, we say that A is a nonnegative matrix, and if $A > 0$, we say that A is a positive matrix. We define $|A| \equiv [\,|a_{ij}|\,]$ .

**Theorem 3.2.12**: Let A, B, C, D $\in M_n$ , Then

(1) $|A| \geq 0$ and $|A| = 0$ if and only if $A = 0$.
(2) $|aA| = |a||A|$, for all complex numbers a.
(3) $|A+B| \leq |A| + |B|$.
(4) If $A \geq 0$, $B \geq 0$, and $a, b \geq 0$, then $aA + bB \geq 0$.
(5) If $A \geq B$ and $C \geq D$, then $A+C \geq B+D$.
(6) If $A \geq B$ and $B \geq C$, then $A \geq C$.

**Theorem 3.2.13**: Let A, B, C, D $\in M_n$, and let $x \in C^n$. Then

(1) $|Ax| \leq |A|\,|x|$.
(2) $|AB| \leq |A||B|$.
(3) $|A^m| \leq |A|^m$ for all m = 1,2,…
(4) If $0 \leq A \leq B$ and $0 \leq C \leq D$, then $0 \leq AC \leq BD$.
(5) If $0 \leq A \leq B$, then $0 \leq A^m \leq B^m$ for all m = 1, 2, …
(6) If $A \geq 0$, then $A^m \geq 0$, and if $A > 0$, then $A^m > 0$ for all m = 1, 2, …
(7) If $A > 0$, $x \geq 0$, and $x \neq 0$, then $Ax > 0$.
(8) If $A \geq 0$, $x > 0$, and $Ax = 0$, then $A = 0$.

### 3.3 Matrix Norm

**Definition 3.3.1:** let $V$ be a vector space over a field F. A Function $\|\,.\,\|: V \to R$ is a vector norm if for all x, y $\in V$

(1) $\|x\| \geq 0, and \|x\| = 0 \; if \; and \; only \; if \; x = 0$ (positivity) .

(2) $\|cx\| = |c|\|x\| \; for \; all \; scalars \; c \in F$ (Homogenity).

(3) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

For a vector $x = (x_1, x_2, \dots, x_n) \in C^n$, we define

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} , \qquad 1 \leq p < \infty$$

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p = \max_{1\leq j\leq n} |x_j|.$$

For each $1 \leq p \leq \infty$, $\|x\|_p$ defines a norm on $C^n$. These are called the p-norms or $\ell_p$ norms. While for $0 < p < 1$ this defines a quasinorm. Instead of the triangle inequality we have.

$$\|x + y\|_p \leq 2^{1/p-1}\left(\|x\|_p + \|y\|_p\right), \qquad 0 < p < 1$$

**Definition 3.3.2:** A function $N: M \to R$ is a matrix norm if for all $A, B \in M_n$ it satisfies the following axioms :

(1) $N(A) \geq 0$ and $N(A) = 0$ if and only $A = 0$ .
(2) $N(aA) = |a|N(A)$ for all complex numbers $a$ .
(3) $N(A + B) \leq N(A) + N(B)$.
(4) $N(AB) \leq N(A)N(B)$.

**Remark 3.3.3:** A vector norm on $M_n$, that is a function that satisfies (1) – (3) and not necessarily (4) , is often called a generalized matrix norm .

**Examples 3.3.4**: Let $= \lfloor a_{ij} \rfloor \in M_n$ . Then

(1) The $\ell_1$norm is defined by $\|A\|_1 = \sum_{i,j=1}^{n} |a_{ij}|$ .

Note that $\|.\|_1$ is a matrix norm because

$$\|AB\|_1 = \sum_{i,j=1}^{n} \left|\sum_{k=1}^{n} a_{ik} b_{kj}\right| \leq \sum_{i,j,k=1}^{n} |a_{ik} b_{kj}|$$

$$\leq \sum_{i,j,k,m=1}^{n} |a_{ik} b_{mj}| = (\sum_{i,k=1}^{n} |a_{ik}|)(\sum_{j,m=1}^{n} |b_{mj}|$$

$$=\|A\|_1 \|B\|_1$$

(2) The $\ell_2$ norm (or the Euclidean norm) is defined by

$$\|A\|_2 = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

Note that $\|.\|_2$ is a matrix norm because

$$\|AB\|_2^2 = \sum_{i,j=1}^{n} \left|\sum_{k=1}^{n} a_{ik} b_{kj}\right|^2$$

$$\leq \sum_{i,j=1}^{n} (\sum_{k=1}^{n} |a_{ik}|^2)(\sum_{m=1}^{n} |b_{mj}|^2)$$

$$= \|A\|_2^2 \|B\|_2^2$$

This inequality is just the Cauchy – Schwarz inequality. When applied to matrices, this norm is sometimes called the forbenius norm, the Schur norm, or the Hilbert – Schmidt norm .

(3) The maximum column sum matrix norm is defined by

$$|||A|||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

(4) The maximum row sum matrix norm is defined by

$$|||A|||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

Note that, the norms $|||.|||_1,\ and\ |||.|||_\infty$ are induced by the $\ell_1 and\ \ell_\infty$ vectors norms, respectively, and hence must be matrix norms.

(5) The spectral norm ( or the usual operator norm )is defined by

$$\|A\| = \max_{\|x\|=1} \|Ax\| .$$

Note that, for any matrix $A \in M_n$,

(a)$\|A\| = \max_{\|x\|=\|y\|=1} |(y, Ax)|$ .

(b) $If\ A \in M_n$ is Hermitian, then, $\|A\| = \max_{\|x\|=1} |(x, Ax)|$.

(c) If $A \in M_n$ is a unitary, then $\|A\| = 1$.

(d) $\|A^k\| \leq \|A^k\|$, for $k = 1,2, \ldots$.

(e) If $|A| \leq |B|$ , then$\|A\|_2 \leq \|B\|_2$ .

(6) The norm $\|A\|_\infty = \max_{1 \leq i,j \leq n} |a_{ij}|$ is a generalized matrix norm.

## 4. Spectral Radius Inequalities

 

### 4.1 Properties of the Spectral Radius

The following properties of the spectral radius can be found in horn and Johnson (1985).

**Definition 4.1.1**: The spectral radius $r(A)$ of a matrix $A \in M_n$ is

$$r(A) = \max\{|\lambda|: \lambda \text{ is an eigenvalue of } A\}.$$

Observe that if λ is any eigenvalue of, then $|\lambda| \leq r(A)$.

**Theorem 4.1.2**: If $N(.)$is any matrix norm and if $A \in M_n$, then $r(A) \leq N(A)$.

**Proof:**

Let $\lambda \in \sigma(A)$ such that $|\lambda| = r(A)$ and let $x \in C^n$ be a nonzero vector such that $Ax = \lambda x$.

If $X = [x: x: \dots : x]$, then $|\lambda| N(X) = N(\lambda X) = N(AX) \leq N(A)N(X)$. Since $N(X) \neq 0$, we have $|\lambda| \leq N(A)$, and so $r(A) \leq N(A)$.

**Corollay 4.1.3**: If $A \in M_{n\,,}$ then $(A) \leq \|A\|$ , and equality holds if $A$ is normal .

**Remark 4.1.4**: If $A, B \in M_n$, then $r(AB) = r(BA)$.

To see this , use the fact $\sigma(AB) = \sigma(BA)$.

**Theorem 4.1.5**: If $A, B \in M_n$ and $AB = BA$, Then

(1) $r(A + B) \leq r(A) + r(B)$.
(2) $r(AB) \leq r(A)r(B)$.

**Proof:**

(1) Since $AB = BA$, by schur's theorem there is a unitary matrix $U \in M_n$ such that

$U^*AU$and $U^*BU$ are both upper triangular,

I.e,
$$T_{1=}U^*AU = \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & \dots\dots & a_{1n} \\ 0 & \lambda_2 & a_{23} & \dots\dots & a_{2n} \\ 0 & 0 & \lambda_3 & \dots\dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots\dots & \lambda_n \end{bmatrix}$$

Where $\lambda_i$, $i = 1,2, \dots, n$ are the eivgenvalues of $A$, and

$$T_2 = U^*BU = \begin{bmatrix} \mu_1 & b_{12} & b_{13} & \dots & b \\ 0 & \mu_2 & b_{23} & \dots & b_{2n} \\ 0 & 0 & \mu_3 & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_n \end{bmatrix}$$

Where $\mu_i, i = 1,2, \dots, n$ are the eivgenvalues of B.

Note that $\sigma(A) = \sigma(T_1) = \{\lambda_i : i = 1, \dots, n\}$, and $\sigma(B) = \sigma(T_2) = \{\mu_I : i = 1, \dots n\}$.

Now, $\sigma(A + B) \subseteq \sigma(A) + \sigma(B)$, and $\sigma(AB) \subseteq \sigma(A)\sigma(B)$.

So, we have $r(A + B) \leq r(A) + r(B)$ and$r(AB) \leq r(A)r(B)$.

**Remark 4.1.6**: If $A, B \in M_n$do not commute, then Theorem 4.1.5 is false. To see this consider the following example.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $r(A) = 0$ and $r(B) = 0$, but $r(A + B) = 1$.

So ,$r(A + B) = 1 > r(A) + r(B) = 0$.

**Lemma 4.1.7**: If $\in M_n$ , then

$$r(A) = \inf\{\|S^{-1}AS\|: S \in M^n \ \ is\ invertible\}.$$

Proof:

If $S \in M_n$ invertible, then

$$r(A) = r(S^{-1}AS) \leq \|S^{-1}AS\| .$$

So,

$$r(A) \leq \inf\{\|S^{-1}AS\|: S \in M^n \ \ is\ invertible\}.$$

By the Schur traingularization theorem , there is a unitary matrix U and an upper triangular matrix T with diagonal entries $\lambda_1, \dots . \lambda_n$ ( the eigenvalues of $A$), such that $U^*AU = T$

Set $D_1 = diag(t, t^2, t^3, \dots t^n)$ for $t > 0$, and comput $D_t T D_t^{-1}$.

$$D_t T D_t^{-1} = \begin{bmatrix} \lambda_1 & t^{-1}d_{12} & t^{-2}d_{13} & \dots & t^{-n+1}d_{1n} \\ 0 & \lambda_2 & t^{-1}d_{23} & \dots & t^{-n+2}d_{2n} \\ 0 & 0 & \lambda_3 & \dots & t^{-n+3}d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Let $S_{1=}UD_t^{-1}$ . Then $S_t^{-1}AS_t = D_t U^* A U D_t^{-1} = D_t T D_t^{-1}$ .

Thus, for $t > 0$ large enough, we can be certain that the sum of all the absolute values of the off-diagonal entries of $S_t^{-1}AS_t$, can be made arbitrary small. It follows that

$$\|S_t^{-1}AS_t - diag(\lambda_1, \dots, \lambda_n)\| \to 0 \text{ as } t \to \infty.$$

Thus

$$\|S_t^{-1}AS_t\| \to \|diag(\lambda_1, \dots, \lambda_n)\| = \max_{1 \leq i \leq n}|\lambda_i| = r(A).$$

Now,

$$r(A) \leq inf\|S^{-1}AS\| \leq \|S_t^{-1}AS_t\|, \text{ for all } t > 0 .$$

Letting $t \to \infty$, we obtain

$$r(A) \leq \inf \|S^{-1}AS\| \leq r(A), \text{ and so}$$

$r(A) = \inf \|S^{-1}AS\|$ .

## 5. The Spectral Rdius of the Hadamard product of matrices

### 5.1 The Hamdamard product

**Definition 5.1.1** :
$If\ A = \lfloor a_{ij} \rfloor, B = \lfloor b_{ij} \rfloor \epsilon\ M_{mn}, then\ the\ hadamard\ product\ of\ A\ and\ B\ is\ the\ matrix\ A^\circ B = \lfloor a_{ij} b_{ij} \rfloor \epsilon\ M_{mn}$ .

**Theorem 5.1.2** : (Zhang , 1999). If A, B $\epsilon$ $M_n$ and A, B are positive semi definite, Then
$A^\circ B\ is\ positive\ semi\ definite$ .

**Proof :**

For every vector $\in C^n$, We have

$$((A^\circ B)x, x) = x^*(A^\circ B)x\ .$$

If X = $diag(x_1, x_2, \dots, x_n)$, and $\bar{X} = diag\ (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$, then

$x^*(A^\circ B)$ x = tr $(\bar{X}AXB^t)$

$$= tr\left(\left(B^{1/2}\right)^t \bar{X}A^{1/2}A^{1/2}X\left(B^{1/2}\right)^t\right) \qquad \text{by (Theorem 3.1.7)}$$

$$= tr\left(\left(A^{1/2}X\left(B^{1/2}\right)^t\right)^*\left(A^{1/2}X\left(B^{1/2}\right)^t\right)\right) \geq 0$$

by (Theorem 3.1.8)

**Some properties of the Hadamard product** :

**Theorem 5.1.3**: (Zhang 1999) . Let A, B and C $\in M_{mn}$. Then

1- $A^\circ B = B^\circ A$ .

2- $K(A^\circ B) = (AK)^\circ B = A^\circ(KB), where\ K\ is\ a\ scalar$ .

3-(A$\pm B)\ {}^\circ C = (A^\circ C) \pm (B^\circ C)$.

4- if A and B are diagonal matrices , then $A^\circ B = AB$ .

5-$(A^\circ B)^t = A^t \circ B^t$.

6-$(A^\circ B)^* = A^* \circ B^*$.

7- If $A \in M_n\ and\ A \geq 0\ and\ if\ \tilde{A}\ is\ any$ principal submatrix of A, then $r(\tilde{A}) \leq r(A)$

8-if $A \geq 0\ and\ B \geq 0, then\ A^\circ B \geq 0$

9- if $I_n \epsilon\ M_n\ is\ the\ identity\ matrix\ , then\ A\ {}^\circ I_n = diag(a_{11}, a_{22}, \dots \dots, a_{nn})$.

10-let A,B and C be positive semidefinite and if A-B is positive semidefinite, then $(A^\circ C) - (B^\circ C)$ is positive semidefinite.

11- If $A = \lfloor a_{ij} \rfloor \epsilon\ M_n\ is\ positive\ semidefinite, then\ \left\lfloor |a_{ij}|^2 \right\rfloor is\ positive\ semidefinite$.

12-If A and B are Hermitian matrices ,then A°B is Hermitian .

13- If $D_1, D_2\ \epsilon\ M_n\ are\ diagonal\ matrices, then\ A^\circ(D_1BD_2) = D_1(A^\circ B)D_2$ .

Proof of (12) :

$let\ D_1 = Diag(\lambda_1, \lambda_2, \dots \dots \lambda_n)\ and\ D_2 = diag\ (\mu_1, \mu_{2,\dots\dots\dots}\mu_n)$ .then

 

$$D_1BD_2 = \begin{bmatrix} \lambda_1\mu_1 b_{11} & \lambda_1\mu_2 b_{12} & \lambda_1\mu_3 b_{13} & \dots & \lambda_1\mu_n b_{1n} \\ \lambda_1\mu_1 b_{21} & \lambda_1\mu_2 b_{22} & \lambda_1\mu_3 b_{23} & \cdots & \lambda_2\mu_n b_{2n} \\ \lambda_1\mu_1 b_{31} & \lambda_1\mu_2 b_{32} & \lambda_1\mu_3 b_{33} & \ddots & \lambda_3\mu_n b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_n\mu_1 b_{n1} & \lambda_n\mu_2 b_{n2} & \lambda_n\mu_3 b_{n3} & \cdots & \lambda_n\mu_n b_{nn} \end{bmatrix}$$

$$A^{\circ}(D_1\, BD_2) = \left[\begin{bmatrix} \lambda_1\mu_1 b_{11} & \lambda_1\mu_2 b_{12} & \dots & \lambda_1\mu_n b_{1n} \\ \lambda_1\mu_1 b_{21} & \lambda_1\mu_2 b_{22} & \cdots & \lambda_2\mu_n b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n\mu_1 b_{n1} & \lambda_n\mu_2 b_{n2} & & \lambda_n\mu_n b_{nn} \end{bmatrix}\right]$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix}$$

$= D_1\,(A^{\circ}B)D_2$ .

**Theorem 5.1.4** : (Zhang , 1999) . Let A,B $\epsilon$ $M_n$ be positive semidefinite . then

$$A^2{}^{\circ}B^2 \geq (A^{\circ}B)^2 \,.$$

Proof :

Let$a_i$ $and$ $b_i$, be i-th columns' of the matrices A and B, respectively. Now, by a Direct computation we have,

$$(AA^*)^{\circ}(BB^*) = (A^{\circ}B)\,(A^* \circ B^*) + \sum_{i \neq j}(a_i{}^{\circ}b_j)(a_i^*\,{}^{\circ}b_j^*) \,.$$

So ,

$$(AA^*)^{\circ}(BB^*) \geq (A^{\circ}B)\,(A^* \circ B^*).$$

Since A and B are positive semidefinite, then

$$A^2{}^{\circ}B^2 \geq (A^{\circ}B)^2.$$

## 5.2 The kronecker Product :

**Definition 5.2.1**:

$B = \lfloor b_{ij} \rfloor \,\epsilon\, M_{pq}$ *is denoted by* $A \otimes B$ The Kronecker Product of, A $= \lfloor a_{ij} \rfloor \,\epsilon\, M_{mn}$ *and*

*and is defined to be the block matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & a_{23}B & \cdots & a_{2n}B \\ a_{31}B & a_{32}B & a_{33}B & \cdots & a_{3n}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & a_{n3}B & \cdots & a_{mn}B \end{bmatrix} \epsilon\, M_{(mp)(nq)}.$$

**Theorem 5.2.2**: (zhang,1999),Let A,B and C $\epsilon$ $M_n$. *then*

(1) (aA)⊗B=A(aB) for all complex numbers a.
(2) $(A \otimes B)^t = A^t \otimes$ $\mathrm{B}^{\mathrm{t}}$.

(3) $(A \otimes B)^* = A^* \otimes B^*$
(4) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
(5) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.
(6) $(A \circ B)$ $ispricipal\ submatrix\ of$ $A \otimes B$
(7) If $A, B \geq 0\ then\ A \otimes B \geq 0$ .

**Lemma 5.2.3:** (Zhang , 1999). Let A, B, C and D $\epsilon M_n. Then$

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

**Proof:**

Let A=$[a_{ih}]$ $and\ C = \lfloor c_{hj} \rfloor. Then$

$A \otimes B = [a_{ih}B]$ and $C \otimes D = \lfloor c_{hj}D \rfloor$,and

$(A \otimes b)(C \otimes D) =$

$$= \begin{bmatrix} (\sum_{h=1}^{n} a_{1h}c_{h1})BD & (\sum_{h=1}^{n} a_{1h}c_{h2})BD & \cdots & (\sum_{h=1}^{n} a_{1h}c_{hn})BD \\ (\sum_{h=1}^{n} a_{2h}c_{h1})BD & (\sum_{h=1}^{n} a_{2h}c_{h2})BD & \cdots & (\sum_{h=1}^{n} a_{2h}c_{hn})BD \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{h=1}^{n} a_{nh}c_{h1})BD & (\sum_{h=1}^{n} a_{nh}c_{h2})BD & \cdots & (\sum_{h=1}^{n} a_{nh}c_{hn})BD \end{bmatrix}$$

$$= \left[\sum_{h=1}^{n} a_{ih}c_{hj}\right] \otimes BD$$

$$= AC \otimes BD .$$

**Theorem 5.2.4** (Zhang , 1999) . Let A$\epsilon M_n\ and\ B\ \epsilon\ M_m\ .then$

$$\sigma(A \otimes B) = \sigma(A)\sigma(B).$$

Proof :

Since $A \in M_n$, Since $A \in M_n\ and\ B \in M_m$, by Schur's unitary triangularization theorem, there are unitary matrices $U \in M_n\ and\ V \in M_m$ such that,

$$U^*\ AU = T_1, and\ V^*AV = T_2\ ,$$

Where $G_1$and $G_2$ are upper triangular with entries diagonal the eigenvalues of A and B, respectively. Then

$$(U \otimes V)^*(A \otimes B)(U \otimes V) = (U^* \otimes V^*)(A \otimes B)(U \otimes V)$$

$$= (U^*AU) \otimes (V^*BV)$$

$$= \mathrm{T}_1 \otimes \mathrm{T}_2.$$

Now, we note that, $T_1 \otimes T_2$ is upper triangular with entries diagonal

$$\{\lambda_1\mu_1, \lambda_1\mu_2, \dots, \lambda_1\mu_m, \lambda_2\mu_1, \lambda_2\mu_2, \dots \lambda_2\mu_m, \dots \lambda_n\mu_1, \lambda_n\mu_2, \dots \lambda_n\mu_m, \}\ ,$$

Where σ(A) = $\{\lambda_1, \lambda_2, \dots \dots . \lambda_n\}$ and σ (B) = $\{\mu_1, \mu_2, \dots \dots \dots , \mu_m\}$.

 

So,

$$\sigma(A \otimes B) = \{\lambda_i \mu_j, \quad i = 1, \dots..n, j = 1, \dots., m\}$$

Then ,

$$\sigma(A \otimes B) = \sigma(A)\sigma(B).$$

**Corollary 5.2.5** : Let A$\epsilon M_n$ $and\ B\epsilon\ M_n\ .then$

$$r(A \otimes B) = r(A)r(B).$$

**5.3 Some Bounds for the Spectral Radius of the Hadamard products of Matrices**

The study begins this section by the following theorem which relates the hadamard product with the spectral radiues .

**Theorem 5.3.1**: (Cheng, etal., 2005) let A,B $\epsilon M_n, and\ A, B \geq 0\ .Then$

$$r(A^\circ B) \leq r(A)r(B).$$

**Proof** : We have

$$r(A \otimes B) = r(A)r(B) \qquad \text{(by corollary 5.2.5)}$$

Since A$\otimes$B$\geq 0$ $and\ , A^\circ B\ is\ a\ principal\ submatrix\ of\ A \otimes B, then$

$$r(A^\circ B) \leq r(A \otimes B) = r(A)r(B) \qquad \text{(by property (7) of theorem 5.1.3)}$$

**Theorem 5.3.2** : (cheng, etal., 2005) let $A \geq 0, B \geq 0$ $be\ nxn\ nonnegative\ matrices.$

If there exists a positive diagonal D such that $DBD^{-1}$is diagonally dominant of its column (or row) entries, then

(1) r(A$^\circ B) \leq r(A) \max_{i=1,2,\dots n} b_{ii}$ ………………(1)

and

(2) r(B)$\leq tr(B)$ … … … … … … … … . (2)

**Proof of inequality (1):**

We have

$$A^\circ(DBD^{-1}) = D(A^\circ B)D^{-1} \qquad \text{( by property (13) of theorem 5.1.3)}$$

and Hence,

$$r\left(A^\circ(DBD^{-1})\right) = r(D(A^\circ B)D^{-1}) = r\left(D^{-1}D(A^\circ B)\right) = r(A^\circ B) \qquad \text{(by remark 4.1.4)}$$

Notice that the diagonal entries of B and $DBD^{-1}$ are the same, so we may assume that B is diagonally dominant of its column, (or row) entries. Then

$$A \circ B \leq A\ diag\ (b_{11}, \dots \dots . b_{nm}) \leq A \max_{i=1,2,\dots.,n} b_{ii}$$

So

$$r(A \circ B) \leq r\big(Adiag(b_{11}, b_{22}, , , , , , , , , , , b_{nm},)\big) \leq r\big(A \max_{i=1,2,,,,n} b_{ii}\big) \text{ ..........(3)}$$

Then ,

$$r(A \circ B) \leq r\ (\mathrm{A}) \max_{i=1,2,,,,n} b_{ii}.$$

If B is diagonally dominant of its row entries, then

$r(A \circ B) = r(A \circ B)^t = r(A^t \circ B^t)$ ( by theorem 3.1.6 and property ( 5) of theorem 5.1.3) .

Now,

$$\mathrm{A^{t\circ}\ B^t} \leq \mathrm{A^t diag\ (b_{11}, , , , , , , b_{nn)}} \leq \mathrm{A^t} \max_{\mathrm{i=1,2,,,,n}} \mathrm{b_{ii}}.$$

Then,

$$r(A \circ B) = r(A^t \circ B^t) \leq r(A^t) \max_{i=1,2,\dots,n} b_{ii} = r(A) \max_{i=1,2,\dots,n} b_{ii}$$

If B is not diagonally dominant, we have $DBD^{-1}$ is diagonally dominant with diagonal $(b_{11}, \dots, b_{nn})$ entries, and by similarly suppose that $DBD^{-1}$ is diagonally dominant of its column, then

$$\mathrm{A}\circ(\mathrm{DBD^{-1}}) \leq \mathrm{Adiag(b_{11}, \dots, b_{nn})} \leq \mathrm{A} \max_{\mathrm{i=12,,,,,n}} \mathrm{b_{ii}}$$

So

$$r\big(A \circ (DBD^{-1})\big) = r(A \circ B) \leq r(A) \max_{i=1,2,\dots,n} b_{ii}$$

and if $\mathrm{DBD^{-1}}$ is diagonally dominant of it is row entries, then

$$r(A \circ B) = r(A \circ DBD^{-1}) = r(A \circ DBD^{-1})^t = r(A^t \circ (DBD^{-1})^t)$$

Now

$$\mathrm{A^{t\circ}\ (DBD^{-1})^t} \leq \mathrm{A^t diag\ (b_{11}, b_{22}, , , , , , , , , , b_{nn})} \leq \mathrm{A^t} \max_{\mathrm{i=1,2,\dots,n}} \mathrm{b_{ii}}$$

So

$$r(A \circ B) = r(A^t \circ (DBD^{-1})^t) \leq r\ (A^t) \max_{i=1,2,\dots,n} b_{ii} = r(A) \max_{i=1,2,\dots,n} b_{ii}.$$

**Proof of inequality (2)**:

Letting $\mathrm{A=J_n}$, where $\mathrm{J_n}$ is the nxn matrix of all ones in the first inquality (3), we have

$$r(J_n \circ B) \leq r\big(J_n diag\ (b_{11}, \dots, b_{nn})\big).$$

Notice that,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{22} & b_{33} & \cdots & b_{nn} \\ b_{11} & b_{22} & b_{33} & \cdots & b_{nn} \\ b_{11} & b_{22} & b_{33} & \cdots & b_{nn} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{11} & b_{22} & b_{33} & \cdots & b_{nn} \end{bmatrix} = Q$$

And σ(Q)={$0, \sum_{i=1}^{n} b_{ii}$}, (by Theorem 3.1.6)

Then,

$$r(Q) = \sum_{i=1}^{n} b_{ii} = trQ = trB \,.$$

So

$$r(J_n \circ B) \ \leq r(Q).$$

Then,

$$r(B) \leq r(Q) = \ trQ = trB \,.$$

**Remark 5.3.3**: It is not true that if $A, B \in M_n, then\ (A°B) \leq r(A) \max_{i-1,2,\ldots,n} b_{ii}$ , as seen from the following example.

Let A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} and\ , A°B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}. then$

$r(A) = 1, r(B) = 3, r(A°B) = 2\ , \max_{i=1,2,\ldots,n} b_{ii} = 1$

So

$$r(A°B) > r(A)\ \max_{i=1,2,\ldots,n} b_{ii}.$$

**Remark 5.3.4** : it is not true that if A$\geq 0\ and\ B \geq 0$ are both diagonally dominant of its (column) row entries, then

$$r(A°B) \leq \left(\max_{i=1,2,\ldots,n} a_{ii}\right)\left(\max_{i=1,2,\ldots,n} b_{ii}\right)$$

Consider the example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1.5 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, and\ A°B = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

With

$$r(A°B) \cong 4.6180 > 4 = \left(\max_{i=1,2,\ldots,n} a_{ii}\right)\left(\max_{i=1,2,\ldots,n} b_{ii}\right)$$

**Corollary 5.3.5**: (Cheng, et al., 2005). Let $B \geq 0\ be\ n \times n$ nonnegative matrix. If there exists a positive diagonal $D$ such that $DBD^{-1}$ is diagonally dominant of its column (or row) entries, then

$$\max\{r(A°B): A \geq 0,\ \ r(A) = 1\} = \max_{i=1,2,\ldots,n} b_{ii}$$

**Proof** :

Letting $A = I_n$, ( where $I_n\ the\ n \times n$ identity matrix) in inequality(1) of theorem 5.3.2 we have

$$r(I_n{}^\circ B) \leq r(I_n) \max_{i=1,2,\ldots,n} b_{ii}$$

Then

$$r(diag(b_{11},\ldots,b_{nn})) \leq \max_{i=1,2,\ldots,n} b_{ii}$$

But

$$r\big(diag(b_{11},\ldots,b_{nn})\big) = \max_{i=1,2,\ldots,n} b_{ii}$$

So,

$$\max\{r(A^\circ B): A \geq 0,\ \ r(A) = 1\} = \ \max_{i=1,2,\ldots,n} b_{ii}.$$

## 6. Conclusion

In this study, the Hadamard product and kronecker product have good properties when it works with diagonal matrices, positive semidefinite matrices, Hermitian matrices, and nonnegative matrix.

These properties and some theories that have been used ( see theorem 5.3.1and others) can help us to understand and prove some inequalities for the spectral radius of the hadamard product and kronecker product.

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