Transmuted Kumaraswamy Exponentiated Inverse Rayleigh Distribution

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Abstract:

A new family of distributions known as Transmuted Kumaraswamy Exponentiated Inverse Rayleigh distribution TKEIR is suggested and studied. Also f has a shape of well-known sub-models like the Inverse Rayleigh distribution, Kumaraswamy Inverse Rayleigh, the Kumaraswamy Exponentiated Inverse Rayleigh distribution Transmuted Kumaraswamy Inverse Rayleigh distribution, the Exponentiated Inverse Rayleigh distribution and Transmuted Inverse Rayleigh distribution. Some statistical properties of the recent distribution comprise its moments; moment generating function, hazard functions, and order statistic are derived. In general, maximum likelihood estimates of the model parameters are obtained. In practice comparisons of the TKEIR and its sub-models have been made. Results revealed that the new model is the best.

Key words: Rayleigh, sub-models, moments, maximum likelihood, Moment generating function, Hazard rate function

1. Introduction:

As apparent from different studies there are several distribution functions like, the Kumaraswamy probability distribution (1980) used for random processes with double bounded in piratical hydrology. Kumaraswamy distribution is a family of continuous probability distributions. It is double bounded distribution denoted by, $kw(a, b)$ defined on the zero to one interval with cumulative distribution function (cdf) assumed to be:

$$F_{kum}(x) = 1 - (1 - x^a)^b \ , \quad 0 < x < 1 \tag{1}$$

The probability density function (pdf) comparable to (1) is supposed as:

$$f_{kum}(x) = abx^{a-1}(1 - x^a)^{b-1} \ , \quad 0 < x < 1 \tag{2}$$

It is worthy to note that the parameters defining the shape are $a > 0, b > 0$ according to KW whose probability density function has similar basic properties of the Beta distribution (Jones2009 and Cordeiro et al. 2010, 2012) it is synonyms to the Beta distribution on the parameters values: that is unimodal for $a > 1$ *and* $b > 1$ uniantimodal for $a < 1$ *and* $b < 1$; increasing for $a > 1$ *and* $b \leq 1$ decreasing for $a \leq 1$ *and* $b > 1$ and constant for $a = b = 1$ according to investigation by Jones (2009) are advantageous since normalizing constant is simple, formulae are obviously simple for the distribution and quantile functions not containing any specific functions, generates simple expression for random variable, plain L-moments equations for and for order statistics' moments of simpler equations, all benefit from KW distribution,, according to Jones (2009), the simple equations of Beta distribution ease the way for generating moments and a symmetric distribution with one parameter moment estimation and offer various ways for physical processes to generate distribution. Empirical hydrological studies employed this distribution since it appears to be as an exceptional substitute to the Beta distribution, (Koutsoyiannis and Xanthopoulos 1989). However the studies of the new Kumaraswamy class of Nadarajah (2008), Selim and Badr (2016). Cordeiro and de Castro (2009), also, generalized distributions (identified as the *Kw*–G distribution) based on the Kumaraswamy distribution (represented by *Kw* distribution). Nearly all formulae for the probability characteristics of the *Kw*–G distribution are derived by them. the *Kw* distribution has established huge interest in hydrology and related areas (Cordeiro et al. 2010, 2012; Fletcher and Ponnambalam 1996; Ganji et al. (2006), Ponnambalam et al. 2001; Sundar & Subbiah 1989 and Seifi et al. 2000). The *Kw* distribution enabled applications of some problems in hydrology and to different bounded natural phenomena on both sides as documented by all these studies.

The features of the transmuted Inverse Weibull distribution have been examined by Khan et al. (2014) Ashour et al. (2013), and Elbatal et al. (2013). Then studies and discussion by Aryal (2013) focused on certain characteristics of the transmuted Lomax distribution, the transmuted quasi Lindley distribution and the transmuted log-logistic distribution of this family, also, Aryal and Tsokos et al. (2009, 2011) took into consideration the extreme value of the transmuted distributions, the transmuted Gumbel distribution by means of climate data, the transmuted Weibull distribution and the practical analysis of actual data groups. Khan et al. (2013 a, b, c) as well introduced modifications for Weibull transmuted distribution, the transmuted generalized inverse Weibull distribution and the transmuted generalized exponential distribution, in addition to the parametrical models generalization via suitable model transformation. Many different families of lifetime distributions have been intensively studied adding a shape parameter. Recently, Ahmad et al. (2014) explained some mathematical findings of the transmuted Kumaraswamy distribution via the quadratic rank transmutation map according to Shaw et al. (2009). Merovici (2013 a, b, &2014) derived the families of the transmuted Rayleigh distribution, the transmuted generalized Rayleigh distribution and the transmuted Lindley distribution. Lately the transmuted Kumaraswamy distribution and evaluated some mathematical outcomes has been studied by Ahmad et al. (2015).

Haq (2016) employed a four parameter Kumarawamy Exponentiated Inverse Rayleigh distribution (KEIR) to study some of its proprieties such survival and hazard function. The proposed map by Shaw et al. (2009) concerning transmutation of quadratic rank was also employed by Muhammad Khan et .al (2016) to develop the three-parameter transmuted Kumaraswamy distribution (TKw) identified by Concerning this method a random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) fulfills the following expression.

$$\mathrm{F(x)(1+\lambda)G(x)+\lambda G^2(x)} \tag{3}$$

$$\mathrm{f(x)=g(x)[1+\lambda-2\lambda G(x)]} \quad ,|\lambda| \leq 1 \tag{4}$$

G(x) represents the base line distribution cumulative distribution function of and f(x) and g(x) are the resultant probability density functions of F(x) and G(x) respectively. However λ ranges [-1,1], see (Aryal *et. al* 2009). At $\lambda = 0$ generalized distribution decreases to parent distribution. Numerous generalized distributions, Transmuted Power Function, Kumaraswamy Exponentiated Inverse Rayleigh (KEIR) distribution, Transmuted will be generalized.

The Exponentiated Inverse Rayleigh distribution probability density and cumulative distribution function are given below

$$f(x) = \frac{2\alpha\theta}{x^3}\left(e^{-\frac{\alpha\theta}{x^2}}\right) \quad \& \; F(x) = \left(e^{-\frac{\alpha\theta}{x^2}}\right) \tag{5}$$

Kumaraswamy Exponentiated Inverse Rayleigh distribution's probability density and cumulative distribution function of are specified below

$$f(x) = \frac{2ab\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{(b-1)} \quad ; x \geq 0\,, ab\alpha\theta > 0 \tag{6}$$

The cumulative distribution function of KEIR distribution is:

$$F(x) = 1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b} \tag{7}$$

This means an expansion for the probability density function and cumulative distribution function based on KEIR density function. It is worth mentioning that all this work relates to Haq (2016). The binomial expansion of equation (6) and the probability density function yields:,

$$f(x) = \frac{2ab\alpha\theta}{x^3}\sum_{i=1}^{\infty}(-1)^i\binom{b-1}{i}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \tag{8}$$

The final expression of probability density function is obtained by writing the term $\binom{b-1}{i}$ as $\frac{\Gamma(b)}{i!\Gamma(b-1)}$ so is as follows:

$$f(x) = \frac{2ab\alpha\theta}{x^3}\sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \tag{9}$$

In this study, we intend to generalize four-parameter Kumaraswamy Exponentiated Inverse Rayleigh distribution KEIR to provide new distribution with five parameters by adding fifth parameter denoting the transmuted Kumaraswamy Exponentiated Inverse Rayleigh distribution as TKEIR..

The rest of the study is organized as follows. The TKEIR will be defined in section 2. Section 3 devoted to the investigation of the TKEIR properties such as moments, and moment generating function, modes, section 4 is designated to order statistics; estimation of TKEIR parameters using maximum likelihood estimation method are proposed in section 5. Real data analysis and comparison of TKEIR results .with other selected distributions are placed in section 6.

2. The TKEIR distribution

The pdf and the cdf of TKEIR distribution will be introduced through locating the KEIR reference point functions (6) and (7) in Equations (3) and (4), to get the cdf and pdf of the TKEIR distribution as below:

$$F(x)=\left[1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right]\left[(1+\lambda)-\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)\right]$$

$$;\ a,b,\alpha,\theta,\lambda>0, x>0 \qquad (10)$$

and

$$f(x)=\frac{2ab\alpha\theta}{x^3}\sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)}$$

$$\times\left[(1+\lambda)\ -2\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)\right] \qquad (11)$$

The shape parameters are a,b,α and θ; and λ represents diverse forms of the theme distribution of a transmuting parameter. The five combinations of the shape parameters of the pdf and cdf of TKEIR distribution are delivered in Figure 1 and 2, respectively. The shapes in Figure 1 reveal monotonically decreasing or positively skewed pdfs of TKEIR distribution. At $\lambda > 0$ the TKEIR distribution approaches the KEIR distribution.

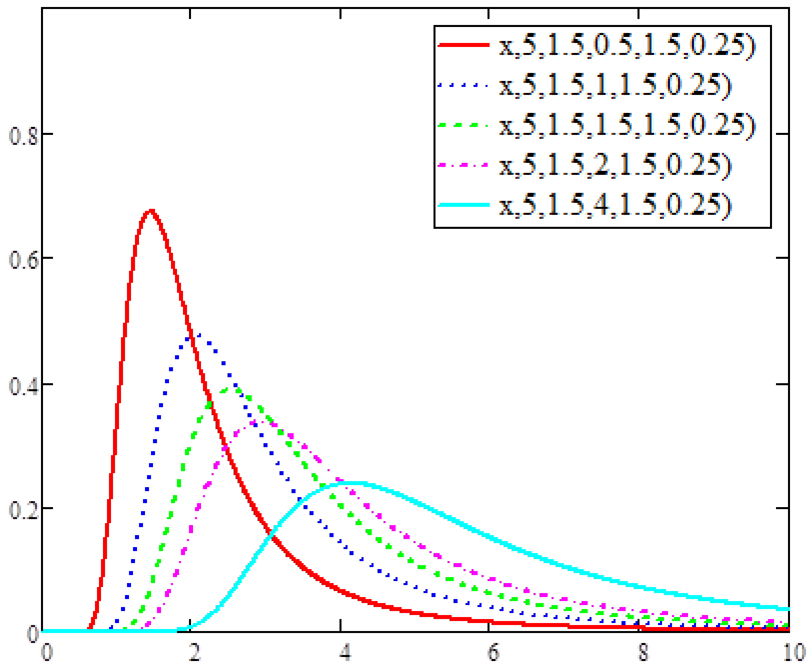

Figure (1): TKEIR's Potential Shapes of the Density Function.

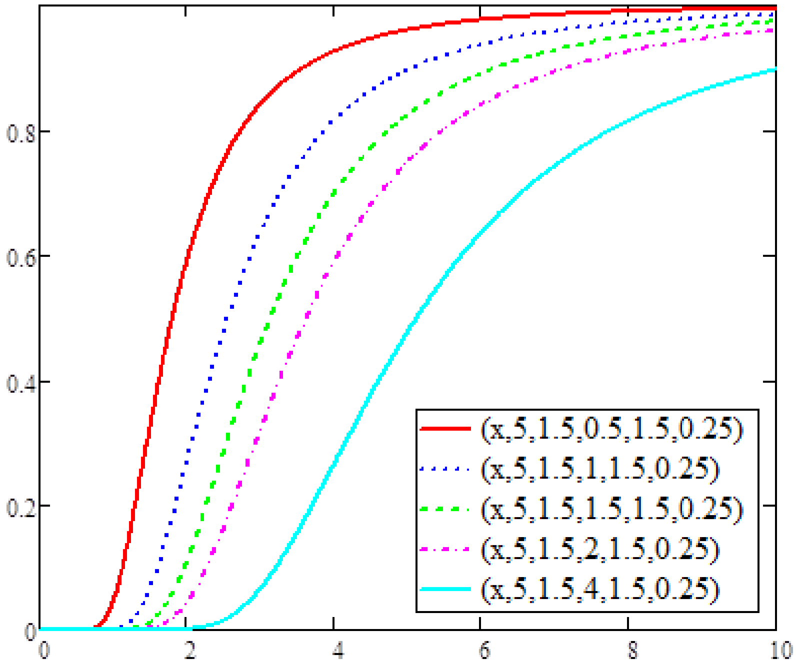

Figure (2): TKEIR's Potential Shapes of the Cumulative Density Function

2.1 Some sub-models of the TKEIR

The TKEIR distribution shows plentiful flexibility since various distributions can be obtained at changing parameter $(a, b, \theta \, \& \alpha, \lambda)$. The theme distribution comprises as distinct cases six well-known probability distributions as exemplified below.

1. At $a = b = \alpha = \lambda = 1$ the Inverse Rayleigh distribution, $IR(\theta, x)$ is obtained.
2. At $\alpha = \lambda = 1$ the Kumaraswamy Inverse Rayleigh distribution, $KIR(a, b, \theta, x)$ is obtained.
3. At $\lambda = 1$ we get the Kumaraswamy Exponentiated Inverse Rayleigh is obtained distribution, $KIER(a, b, \alpha, \theta, x)$ is obtained.
4. At $\alpha = 1$ we get the Transmuted Kumaraswamy Inverse Rayleigh distribution $TKIR(a, b, \alpha, \lambda, x)$ is obtained.
5. At $a = b = \lambda = 1$ we get the Exponentiated Inverse Rayleigh distribution, $EIR(\alpha, \theta, x)$ is obtained
6. At $a = b = \alpha = 1$ we get the Transmuted Inverse Rayleigh distribution, $TIR(\lambda, \theta, x)$ is obtained.

2.2 Survival Function

It deals with the probability of failure of an item before some time t, is defined as S(x)=1-F(x). Survival function of TKEIR is $s_x(a, \alpha, \theta, b, \lambda)$ found by means of the expression of distribution given as follows:

$$s(x) = 1 - \left[1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b \left[(1+\lambda) - \lambda\left(1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)\right]\right] \quad (12)$$

Series expansion the survival function is yield by the following expression:

$$s(x) = 1 - \left[1 - \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-1+1)} \left(e^{-\frac{\theta a\alpha}{x^2}}\right)^i \left[(1+\lambda) - \lambda \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(b+1)}{i!\,\Gamma(b-1+1)} \left(e^{-\frac{\theta a\alpha}{x^2}}\right)^i\right]\right] \quad (13)$$

2.3 Hazard Function

The hazard rate function of the random variable X with probability density function f(x), its survival function S(x) is assumed to be:

$$H(x) = \frac{f(x)}{s(x)}$$

$$H(x) = \frac{\frac{2ab\alpha\theta}{x^3} \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(b)}{i!\,\Gamma(b-1)} \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)} \left[(1+\lambda) - 2\lambda\left(1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)\right]}{1 - \left[1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b \left[(1+\lambda) - \lambda\left(1 - \left[1 - \left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^b\right)\right]\right]} \quad (14)$$

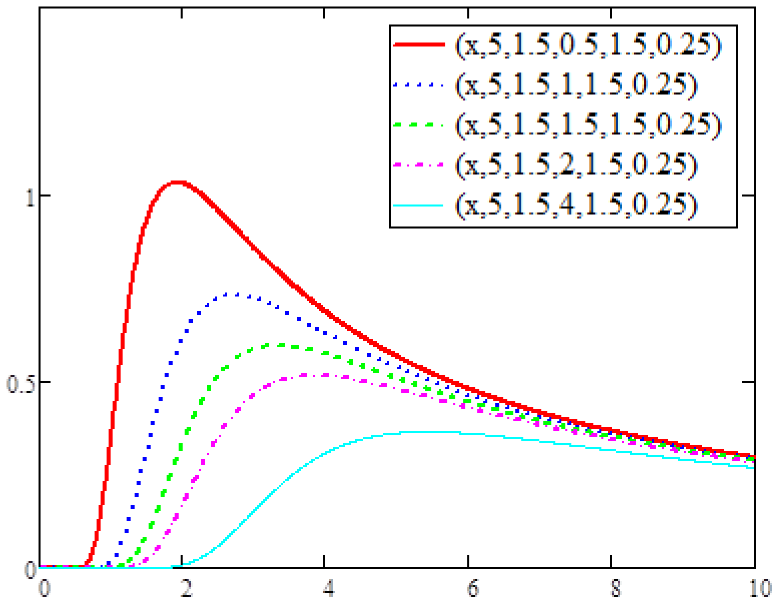

Figure (3): TKEIR' Potential Shapes of the Hazard function.

3 Moments

3.1 The r[th] moments

The TKEIR distribution's r[th] moment declares that μ_r, is set in the following form:

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b})\Gamma\left(1-\frac{r}{2}\right)}{\mathrm{i}!\,\Gamma(\mathrm{b}-1)}\left[\left(\frac{\mathrm{a}\alpha\theta(i+1)^{\frac{r}{2}}(1+\lambda+2\lambda)}{(i+1)} - \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b})\left(\mathrm{a}\alpha\theta(i+2)^{\frac{r}{2}}\right)}{\mathrm{i}!\,\Gamma(\mathrm{b}-\mathrm{i}-1)(i+1)}\right)\right]; r < 2 \quad (15)$$

Proof: The KTEIR's r[th] moment of the distribution as below. It is well known that:

$$\mu_r = \int_0^{\infty} x^r f(x;\ \mathrm{ab}\alpha\theta\lambda)dx$$

$$= 2\mathrm{ab}\alpha\theta \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b})}{\mathrm{i}!\,\Gamma(\mathrm{b}-1)}\left[(1+\lambda)\int_0^{\infty} x^{r-3}\left(\mathrm{e}^{-\frac{\theta}{x^2}}\right)^{a\alpha(\mathrm{i}+1)} dx + 2\lambda \int_0^{\infty} x^{r-3}\left(\mathrm{e}^{-\frac{\theta}{x^2}}\right)^{a\alpha(\mathrm{i}+1)} dx - \int_0^{\infty}\sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b}+1)}{\mathrm{i}!\,\Gamma(\mathrm{b}-1+1)} x^{r-3}\left(\mathrm{e}^{-\frac{\theta}{x^2}}\right)^{a\alpha(\mathrm{i}+1)+a\alpha i} dx\right] \quad (16)$$

By means of the following transformation:

$$y = \frac{a\alpha\theta(\mathrm{i}+1)}{x^2};\quad x = \sqrt{\frac{a\alpha\theta(\mathrm{i}+1)}{y}},\quad dx = -\frac{\left(\frac{a\alpha\theta(\mathrm{i}+1)}{y}\right)^{\frac{3}{2}}}{2a\alpha\theta(\mathrm{i}+1)}dy \quad (17)$$

$$\mu_r = 2\mathrm{ab}\alpha\theta \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b})}{\mathrm{i}!\,\Gamma(\mathrm{b}-1)}\left[(1+\lambda)\int_0^{\infty}\left(\frac{a\alpha\theta(\mathrm{i}+1)}{y}\right)^{\frac{r}{2}-\frac{3}{2}} \mathrm{e}^{-\mathrm{y}} - \frac{\left(\frac{a\alpha\theta(\mathrm{i}+1)}{y}\right)^{\frac{3}{2}}}{2a\alpha\theta(\mathrm{i}+1)}dy - 2\lambda\int_0^{\infty}\left(\frac{a\alpha\theta(\mathrm{i}+1)}{y}\right)^{\frac{r}{2}-\frac{3}{2}} \mathrm{e}^{-\mathrm{y}} \frac{\left(\frac{a\alpha\theta(\mathrm{i}+1)}{y}\right)^{\frac{3}{2}}}{2a\alpha\theta(\mathrm{i}+1)}dy + \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(\mathrm{b}+1)}{\mathrm{i}!\,\Gamma(\mathrm{b}-1+1)}\int_0^{\infty}\left(\frac{a\alpha\theta(\mathrm{i}+2)}{y}\right)^{\frac{r}{2}-\frac{3}{2}} \mathrm{e}^{-\mathrm{y}} \frac{\left(\frac{a\alpha\theta(\mathrm{i}+2)}{y}\right)^{\frac{3}{2}}}{2a\alpha\theta(\mathrm{i}+2)}dy\right] \quad (18)$$

$$= 2ab\alpha\theta\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\Bigg[(1+\lambda)\int_{0}^{\infty}\left(\frac{a\alpha\theta(i+1)}{y}\right)^{\frac{r}{2}}e^{-y}-\frac{1}{2a\alpha\theta(i+1)}dy$$
$$-2\lambda\int_{0}^{\infty}\left(\frac{a\alpha\theta(i+1)}{y}\right)^{\frac{r}{2}}e^{-y}\frac{1}{2a\alpha\theta(i+1)}dy$$
$$+\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b+1)}{i!\,\Gamma(b-1+1)}\int_{0}^{\infty}\left(\frac{a\alpha\theta(i+2)}{y}\right)^{\frac{r}{2}}e^{-y}\frac{1}{2a\alpha\theta(i+2)}dy\Bigg]$$

$$= 2ab\alpha\theta\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\Bigg[-(1+\lambda)\frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+1)}\int_{0}^{\infty}y^{\frac{r}{2}}\,e^{-y}dy$$
$$-2\lambda\frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+1)}\int_{0}^{\infty}y^{\frac{r}{2}}\,e^{-y}dy$$
$$+\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b+1)}{i!\,\Gamma(b-1+1)}\frac{\left(a\alpha\theta(i+2)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+2)}\int_{0}^{\infty}y^{\frac{r}{2}}\,e^{-y}dy\Bigg]$$

$$= 2ab\alpha\theta\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\Bigg[-(1+\lambda)\frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+1)}\Gamma\left(1-\frac{r}{2}\right)$$
$$-2\lambda\frac{\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+1)}\Gamma\left(1-\frac{r}{2}\right)$$
$$+\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b+1)}{i!\,\Gamma(b-1+1)}\frac{\left(a\alpha\theta(i+2)\right)^{\frac{r}{2}}}{2a\alpha\theta(i+2)}\Gamma\left(1-\frac{r}{2}\right)\Bigg]$$

$$\mu_r=\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b+1)\Gamma\left(1-\frac{r}{2}\right)}{i!\,\Gamma(b-1)}\Bigg[\frac{a\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}(1+\lambda+2\lambda)}{(i+1)}$$
$$-\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b)\left(a\alpha\theta(i+2)\right)^{\frac{r}{2}}}{i!\,\Gamma(b-i+1)(i+2)}\Bigg] \qquad (19)$$

3.2 TKEIR's Mean distribution (first moment) is set to be:

$$\mu_1=\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b+1)\Gamma\left(\frac{1}{2}\right)}{i!\,\Gamma(b-1)}\Bigg[\frac{\left(a\alpha\theta(i+1)\right)^{\frac{1}{2}}(1+\lambda+2\lambda)}{(i+1)}$$
$$-\sum_{i=1}^{\infty}(-1)^{i}\frac{\Gamma(b)\left(a\alpha\theta(i+1)\right)^{\frac{1}{2}}}{i!\,\Gamma(b-i+1)(i+2)}\Bigg] \qquad (20)$$

$$E(x) = \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(b+1)\sqrt{\pi}}{i!\,\Gamma(b-1)}\left[\frac{\sqrt{a\alpha\theta(i+1)}(1+\lambda+2\lambda)}{(i+1)} - \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(b)a\sqrt{a\alpha\theta(i+2)}}{i!\,\Gamma(b-i+1)(i+2)}\right] \quad (21)$$

3.3. Moment Generating Function

TKEIR's moment generating function of the is specified in the subsequent form;

$$M_x(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^r}{r!}(-1)^i \frac{\Gamma(b+1)\Gamma\left(1-\frac{r}{2}\right)}{i!\,\Gamma(b-1)}\left[\frac{a\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}(1+\lambda+2\lambda)}{(i+1)} - \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(b)\left(a\alpha\theta(i+2)\right)^{\frac{r}{2}}}{i!\,\Gamma(b-i+1)(i+2)}\right] \quad (22)$$

Proof: the moment generating function computation employs the following relation:

$$M_x(t) = E(e^{tx})\int_0^{\infty} e^{tx} f(x;a,b,\alpha,\theta,\lambda)dx$$

$$= \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\cdots\right)(f(x;a,b,\alpha,\theta,\lambda)dx$$

$$= \int_0^{\infty}\frac{t^r}{r!}x^r(f(x;a,b,\alpha,\theta,\lambda)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(x^r)$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(b+1)\Gamma\left(1-\frac{r}{2}\right)}{i!\,\Gamma(b-1)}\left[\frac{a\left(a\alpha\theta(i+1)\right)^{\frac{r}{2}}(1+\lambda+2\lambda)}{(i+1)} - \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(b)\left(a\alpha\theta(i+2)\right)^{\frac{r}{2}}}{i!\,\Gamma(b-i+1)(i+2)}\right];r<2 \quad (23)$$

4 Order statistics

The life testing and reliability analysis reveal the importance of order statistics The order values of a random sample from TKEIR distribution are symbolized as $X_1, X_2, X_3, \ldots \ldots . X_n$ and $X_{(1)}, X_{(2)}, X_{(3)}, \ldots \ldots . X_{(n)}$ respectively. The following function gives order statistics' probability density function means of:

$$f_{s,n}(x) = \frac{n!}{(s-1)!\,(n-s)!}f(x)[F(x)]^{s-1}[1-F(x)]^{n-s} \quad (24)$$

Derivation of the density of the order statistics that follows the TKEIR distribution is as follows:

$$f_{s,n}(x) = \frac{n!\,2ab\alpha\theta}{(s-1)!\,(n-s)!\,x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b-1}$$

$$\times\left[(1+\lambda)-2\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]$$

$$\times\left[\left[1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right]\left[(1+\lambda)-\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]\right]^{S-1}$$

$$\times\left[1-\left(\left[1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right]\left[(1+\lambda)\quad-\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]\right)\right]^{n-s} \quad (25)$$

The smallest density of the order statistic, is found as:

$$f_{1,n}(x)=\frac{2nab\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b-1}$$

$$\times\left[(1+\lambda)-2\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]$$

$$\times\left[1-\left(\left[1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right]\left[(1+\lambda)-\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]\right)\right]^{n-1} \quad (26)$$

The largest density of the order statistic, is computes as:

$$f_{n,n}(x)=\frac{2nab\alpha\theta}{x^3}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b-1}$$

$$\times\left[(1+\lambda)-2\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]$$

$$\times\left[\left[1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right]\left[(1+\lambda)-\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right]\right]^{n-1} \quad (27)$$

5. Maximum Likelihood Estimation

This section considers the estimation procedure of the ML to the unidentified parameters of TKEIR distribution.

$$f(x)=\frac{2ab\alpha\theta}{x^3}\sum_{i=1}^{\infty}(-1)\frac{\Gamma(b)}{i!\,\Gamma(b-1)}\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha(i+1)}$$

$$\times\left[(1+\lambda)-2\lambda\left(1-\left[1-\left(e^{-\frac{\theta}{x^2}}\right)^{a\alpha}\right]^{b}\right)\right] (28)$$

The sample values of n observations of are written as $X_1, X_2, X_3, \ldots \ldots . X_n$ and the related Log likelihood function of probability density function is set to be:

$$\ln L = n\ln[2] + n\ln[a] + n\ln[b] + n\ln[\alpha] + n\ln[\theta] - 3\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{a\alpha\theta}{x_i^2} + (b-1)\sum_{i=1}^{n}\ln\left[1-\left(e^{-\frac{\theta a\alpha}{x^2}}\right)\right] + \sum_{i=1}^{n}\ln\left[(1+\lambda) - 2\lambda\left(1-\left[1-\left(e^{-\frac{a\alpha\theta}{x^2}}\right)\right]^{b}\right)\right] \quad (29)$$

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\frac{\theta\alpha}{x_i^2} + \sum_{i=1}^{n}\frac{\theta\alpha(b-1)e^{-\frac{a\theta\alpha}{x_i^2}}}{x_i^2\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]} + \sum_{i=1}^{n}\frac{2\lambda b\theta\alpha e^{-\frac{a\theta\alpha}{x_i^2}}\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b-1}}{x_i^2\left[1+\lambda-2\lambda\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]\right]} \quad (30)$$

$$\frac{\partial \ln L}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n}\frac{\theta\alpha}{x_i^2} + \sum_{i=1}^{n}\ln\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right] + \sum_{i=1}^{n}\frac{2\lambda\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\ln\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]}{x_i^2\left[1+\lambda-2\lambda\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]\right]} \quad (31)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{\theta a}{x_i^2} + \sum_{i=1}^{n}\frac{\theta a(b-1)e^{-\frac{a\theta\alpha}{x_i^2}}}{x_i^2\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]} + \sum_{i=1}^{n}\frac{2\lambda ab\theta e^{-\frac{a\theta\alpha}{x_i^2}}\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b-1}}{x_i^2\left[1+\lambda-2\lambda\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]\right]} \quad (32)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\frac{a\alpha}{x_i^2} + \sum_{i=1}^{n}\frac{a\alpha(b-1)e^{-\frac{a\theta\alpha}{x_i^2}}}{x_i^2\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]}$$

$$+\sum_{i=1}^{n}\frac{2\lambda ab\alpha e^{-\frac{a\theta\alpha}{x_i^2}}\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b-1}}{x_i^2\left[1+\lambda-2\lambda\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]\right]} \quad (33)$$

$$\frac{\partial \ln L}{\partial \lambda}=-\sum_{i=1}^{n}\frac{2\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]}{\left[1+\lambda-2\lambda\left[1-\left[1-e^{-\frac{a\theta\alpha}{x_i^2}}\right]^{b}\right]\right]} \quad (34)$$

Obtaining the maximum likelihood estimators $(\hat{a},\hat{b},\hat{\theta},\hat{\alpha},\hat{\lambda})$ for the parameters $(a,b,\theta,\alpha,\lambda)$ will be met by solving the non-linear system equations from (30) to (34) employing numerical analysis of Newton–Raphson method to solve this system.

6. Application

Empirical results of 63 observations originally reported by Bader and Priest (2009) by means of the proposed distribution on real data set of strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows presented in table (1) to give maximum likelihood estimated in the table (2):

Table (1): GPA measured strength for single carbon fibers

1.901	2.396	2.525	2.659	2.937	3.145	3.294	3.501	3.886
2.132	2.397	2.532	2.675	2.937	3.220	3.332	3.537	3.971
2.203	2.445	2.575	2.738	2.977	3.223	3.346	3.554	4.024
2.228	2.454	2.614	2.740	2.996	3.235	3.377	3.562	4.027
2.257	2.474	2.616	2.856	3.030	3.243	3.408	3.628	4.225
2.350	2.518	2.618	2.917	3.125	3.264	3.435	3.852	4.395
2.361	2.522	2.624	2.928	3.139	3.272	3.493	3.871	5.020

The fitted values of the TKEIR model have been compared with 3 models. The of the other fitted pdf of the models are:

Table (2): Real data set ML Estimates

Model	**MLE estimates**				
	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
TKEIR	5.653	8.119	1.257	3.771	0.721
EIR			11.3438	0.73505	
IR			8.3383		
TIR			5.814		-0.46

Table (3) ML Estimates and Information Criteria

	-2ℓ	AIC	AICC	BIC	HQIC
TKEIR	113.248	123.248	124.3	133.964	127.462
EIR	180.677	188.576	188.877	194.963	189.363
IR	182.677	190.677	190.742	194.819	192.519
TIR	193.059	203.059	204.112	213.775	207.274

Models comparisons entailed the consideration of various criteria such as maximized likelihood -2ℓ, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov Smirnov test (KS). Minimum values rule of AIC, BIC, CAIC and HQIC is taken into consideration for selecting the best model to fit. These statistics are given by $AIC = -2\hat{\ell} + 2K, BIC = -2\hat{\ell} + KLog(n), \ , CAIC = -2\hat{\ell} + {}^{2Kn}/_{(n-k-1)}$ where n is sample size, ℓ is log-likelihood and k is number of parameters. Results show that our model satisfied the minimum rule, hence it is the best one.

7. Conclusion

The study aimed to derive five parameter Transmuted Kumaraswamy Exponentiated Inverse Rayleigh TKEIR distributions as a new distribution serves as a modification of TKEIR distribution. The new distribution benefited from the addition of a new parameter leading to increased distribution flexibility. Curves of density and hazard rate have been plotted for selected parameters' values of, in addition to the derivation of moment generating function, entropy, the ordered statistics largest and smallest densities of and the maximum likelihood equations. The usefulness of the model is demonstrated in an applied sample data set by using maximum likelihood method. The derived model proved to be the best compared to other fitted models. The proposed model is hoped attract wider application.

REFERENCES

1. ASHOUR, S. K., ELTEHIWY, M. A., (2013). Transmuted Lomax Distribution, American Journal of Applied Mathematics and Statistics, 1(6), pp. 121−127.
2. AHMAD, A., AHMAD, S. P., AHMED, A., (2015). Characterization and estimation of transmuted Kumaraswamy distribution, Mathematical Theory and Modeling, Vol. 5, No. 9, 168−174.
3. AHMAD, A., S. AHMAD, and A. AHMAD,(2014). Transmuted Inverse Rayleigh Distribution: A Generalization of the Inverse Rayleigh Distribution. Mathematical Theory and Modeling, 4(7): p. 90-98.
4. ARYAL, G. R., TSOKOS, C. P., (2011). Transmuted Weibull distribution: A Generalization of the Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, Vol. 4, No. 2, 89−102.
5. ARYAL, G. R., TSOKOS. C. P., (2009). On the transmuted extreme value distribution with applications. Nonlinear Analysis: Theory, Methods and applications, Vol. 71, 1401−1407.

6. ARYAL, G. R., (2013). Transmuted Log-Logistic Distribution, J. Stat. Appl. Pro. 2, No. 1, 11−20.
7. BALAKRISHNAN, A. N., NAGARAJA, H. N., (1992). A first course in order statistics. New York: Wiley-Interscience.
8. Bader, M. and A. Priest, Statistical aspects of fibre and bundle strength in hybrid composites. Progress in science and engineering of composites, 1982: p. 1129-1136.
9. CORDEIRO, G. M., ORTEGA, E. M. M., NADARAJAH, S., (2010). The Kumaraswamy Weibull distribution with application to failure data. J Frankl Inst 347, 1399–1429.
10. CORDEIRO, G. M., NADARAJAH, S., ORTEGA, E. M. M., (2012). The Kumaraswamy Gumbel distribution, Stat Methods Appl 21: 139–168.
11. CORDEIRO G. M., GOMES, A. E., QUEIROZ DA-SILVA, C., ORTEGA, E. M. M., (2013). The beta exponentiated Weibull distribution, Journal of Statistical Computation and Simulation. 83, 1, 114–138.
12. CORDEIRO, G. M., DE CASTRO M., (2009). A new family of generalized distributions, Journal of Statistical Computation & Simulation, Vol. 00, No. 00, August, 1–17.
13. DUMONCEAUX, R., ANTLE, C. E., (1973). Discrimination between the log-normal and the Weibull distributions. Technometrics 15 (4), 923–926.
14. ELBATHAL, I., ELGARHY, M., (2013). Transmuted Quasi Lindley Distributions: A generalization of the Quasi Lindley Distribution. International Journal of Pure and Applied Sciences and Technology , 18(2), pp. 59–69.
15. FLETCHER, S. C., PONNAMBALAM, K., (1996). Estimation of reservoir yield and storage distribution using moments analysis. J Hydrol 182, 259–275.
16. GANJI, A., PONNAMBALAM, K., KHALILI, D., (2006). Grain yield reliability analysis with crop water demand uncertainty. Stoch Environ Res Risk Assess 20, 259–277.
17. HAQ, M. A.u., (2016). Kumaraswamy Exponentiated Inverse Rayleigh Distribution. Mathematical Theory and Modeling ISSN 2224-5804 (Paper) Vol.6, No.3, ISSN 2225-0522 (Online) www.iiste.org.
18. JONES, M. C., (2009). Kumaraswamy's distribution: a beta-type distribution with some tractability advantages. Stat Methodol 6, 70–91.
19. KENNEY, J. F., KEEPING, E. S., (1962). Mathematics of Statistics. Princeton, NJ.
20. KUMARASWAMY, P., (1980). Generalized probability density-function for double-bounded random-processes. J Hydrol 46, 79–88.
21. KHAN, M. S., KING, R., (2013a). Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution, European Journal of Pure And Applied Mathematics, Vol. 6, No. 1, 66–88.
22. KHAN, M. S., KING, R., (2013b). Transmuted generalized Inverse Weibull distribution, Journal of Applied Statistical Sciences, Vol. 20, No. 3, 15–32.
23. KHAN, M. S., KING, R., HUDSON, I., (2013c). Transmuted generalized Exponential distribution, 57th Annual Meeting of the Australian Mathematical Society, Australia.
24. KHAN, M. S. KING, R. HUDSON., I. L. (2016) .Transmuted Kumaraswamy distribution. STATISTICS IN TRANSITION new series.

25. KHAN, M., SHUAIB, KING, R., HUDSON, I., (2014). Characterizations of the transmuted Inverse Weibull distribution, ANZIAM J., Vol. 55 (EMAC2013 at the Queensland University of Technology from 1st – 4th December 2013), C197–C217.
26. KOUTSOYIANNIS, D, XANTHOPOULOS, T., (1989). On the parametric approach to unit hydrograph identification. Water Resour Manag 3, 107–128.
27. MOORS, J. A., (1998). A quantile alternative for kurtosis. Journal of the Royal Statistical Society, D, 37, 25–32.
28. MEROVCI, F., (2013a). Transmuted Rayleigh distribution. Austrian Journal of Statistics, Vol. 42, No. 1, 21–31.
29. MEROVCI, F., (2013b). Transmuted Lindley distribution. Int. J. Open Problems Compt. Math, 6, 63–72.
30. MEROVCI, F., (2014). Transmuted generalized Rayleigh distribution, J. Stat. Appl. Pro. 3, No. 1, 9–20.
31. NADARAJAH, S., (2008). On the distribution of Kumaraswamy. J Hydrol 348, 568–569.
32. PRINCIPE, J. C., (2009). http://www.cnel.ufl.edu/courses/EEL6814/renyis_entropy. pdf.
33. PONNAMBALAM, K, SEIFI, A, VLACH, J., (2001). Probabilistic design of systems with general distributions of parameters. Int J Circuit Theory Appl 29, 527–536.
34. RÉNYI, A., (1961). On measures of entropy and information. University of California Press, Berkeley, California, 547–561.
35. SUNDAR, V., SUBBIAH, K., (1989). Application of double bounded probability density-function for analysis of ocean waves. Ocean Eng 16, 193–200.
36. SEIFI, A., PONNAMBALAM, K., VLACH, J., (2000). Maximization of manufacturing yield of systems with arbitrary distributions of component values. Ann Oper Res 99, 373–383.
37. SELIM, M. A., BADR A. M., (2016). The Kumaraswamy Generalized Power Weibull Distribution. Mathematical Theory and Modeling 6 : 110-124.
38. SHAW, W. T., BUCKLEY, I. R. C., (2009). The alchemy of probability distributions: beyond Gram–Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map. Technical report, http://arxiv.org/abs/0901.0434.