Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.7, No.5, 2017

A Non-standard discrete model for the solution of first order Ordinary Differential Equations using minimally characterized interpolating function

Obayomi A. A. and *Ayinde S.O

Faculty of Science, Department of Mathematics, Ado Ekiti, Ekiti State

*biskay2003@yahoo.com

ABSTRACT

We present a new set of one step finite difference schemes for the numerical solution of First order differential equations using a combination of an interpolation function and a modification of the resulting schemes by replacing step size h with a suitable function of h as required by the second non-standard modeling rule. The resulting schemes have been applied to some initial value problems and the schemes have been found to possess desirable qualitative properties.

KEYWORDS

Nonstandard methods, Hybrid, Interpolation functions, Non-standard modeling rules, Standard Finite difference methods

INTRODUCTION

Differential equations stem from some academic method of presenting models that best describe a physical phenomenon. They are mathematical models which represent some dynamical state or behavior of such physical phenomena. This also means that the solution may or may not exist. It may also exist, but may not be easy or maybe impossible to represent it in a simple explicit or implicit function. Therefore, finding a solution to such model may require creating a “simpler” mathematical model that can be used to simulate or analyze the original system as presented by the differential equation. This is the area where numerical methods have long played a leading role (or has been in the forefront).

A lot of research work has gone into finding numerical approximations to the solutions of differential equations using finite difference methods. Such approximations are usually based on some acceptable rules and desirable qualities. Early numerical analysts are primarily concerned with standard issues like stability, convergence and consistency of the methods. The works of Lambert (1991), Stretter (1973) and Fatunla (1988) are some of the widely referenced to mention a few. Some of these numerical analysts laid the foundation for general acceptance and suitability of finite difference methods for numerical approximations. Most of these techniques are generally referred to as Standard Numerical methods.

Standard finite difference methods have been found to be more valuable in finding solutions at close ranges and around special grid points like equilibrium and bifurcation points. However looking holistically at the nature of the solution curves and behavioral patterns of the schemes, studies have shown that most of this standard algorithms produce solution curves that does not carry along the qualitative properties of the original dynamic equations(Mickens 1994&2010)

One of the aims of nonstandard method is to develop numerical models that correctly represent the behavioral patterns of the dynamic equation under study. Mickens 1994 and Angueluv and Labuma (2003) have laid a standard foundation for modeling using nonstandard methods that can produce stable schemes that carried along the behavioral pattern of a dynamic system whose initial conditions are known.

This work will follow a mix of some standard techniques and the nonstandard method represented in the works of Mickens1994 with special attention to normalization of the denominator functions. This technique is similar to that of Obayomi et al (2015 ,2016)

In this work we assume a solution that can be represented by a polynomial function with a simple exponential component. The function is then taken through some second order differentiation in order to determine the possible values of the adjoining parameters. The difference form is then approximated using Taylors expansion to obtain a standard numerical model . This numerical model is further extended by the renormalization of the standard discretization function using the Nonstandard modeling rules

Derivation of the schemes

Let's assume an Initial Value Problem possess a solution of the form

$$y(x) = a_o + a_1 x + a_3 e^{-\alpha x} \tag{1}$$

$$y'(x) = a_1 - \alpha a_3 e^{-\alpha x} \tag{2}$$

$$y''(x) = \alpha^2 a_3 e^{-\alpha x} \tag{3}$$

From (1) $a_o = y(x) - a_1 x - a_3 e^{-\alpha x}$ (4)

From (2), $a_1 = y'(x) + \alpha a_3 e^{-\alpha x}$ (5)

From (3), $a_3 = \frac{y''(x)}{\alpha^2 e^{-\alpha x}}$ (6)

From (6) and (5) $a_1 = y'^{(x)} + \alpha e^{-\alpha x} \frac{y''(x)}{\alpha^2 e^{-\alpha x}}$

$$a_1 = y'^{(x)} + \frac{y''(x)}{\alpha} \tag{7}$$

Putting (7) and (6) in (4), we obtain

$$a_o = y(x) - \left((y'(x)) + \frac{y''(x)}{\alpha} \right) - \frac{y''(x)}{\alpha^2}$$

$$a_o = y(x) - y'(x) - \frac{y''(x)}{\alpha} - \frac{y''(x)}{\alpha^2}$$

$$\therefore a_o = y(x) - y'(x) - y''(x) \propto^{-2} \quad (8)$$

The Interpolating function must coincide with the theoretical solution at $x = x_n$ and $x = x_{n+1}$ such that

$$y(x_n) = a_o + a_1 x_n + a_3 e^{-\propto x_n}$$

$$y(x_{n+1}) = a_o + a_1 x_{n+1} + a_3 e^{-\propto x_{n+1}}$$

Let $y'(x) = f_n$, $y''(x) = f_n'$

it follows that : $y(x_{n+1}) - y(x_n) = y_{n+1} - y_n$

And $a_0 + a_1 x_{n+1} + a_3^{e-\alpha x_{n+1}} - = a_0 - a_1 x_n - a_3^{e-\alpha x_n} = y_{n+1} - y_n$

$$y_{n+1} = y_n + a_1(x_{n+1} - x_n) + a_3(e^{-\alpha x_{n+1}} - e^{-\alpha x_n}) \quad (9)$$

If our initial value is given at grid point "a" then

$$x_n = a + nh \text{ and } x_{n+1} = a + (n+1)h.$$

$$(x_{n+1} - x_n) = a + (n+1)h - a - nh = h.$$

Then, $y_{n+1} = y_n + a_1 h + a_3\left(e^{-\alpha(a+(n+1)h)} - e^{-\alpha(a+nh)}\right)$ (10)

Putting (6) and (7) in (10), we have:

$$y_{n+1} = y_n + \left(f_n + \frac{f_n'}{\propto}\right)h + \left\{\frac{f_n'}{\alpha^2 \, e^{-\alpha(a+nh)}}\right\}\left(e^{-\alpha(a+(n+1)h)} - e^{-\alpha(a+nh)}\right) \quad (11)$$

Equation (11) is the required Standard Finite Difference Scheme:

This will be renormalized applying rule 2 of the Nonstandard modeling rules

We will obtain two new schemes by replacing h with a dynamic function of h as follows with the condition that $\psi(h) \to h + 0(h^2)$ as $h \to 0$.

$$\psi = \sin(h) , \quad \psi = \frac{(e^{\lambda h} - 1)}{\lambda}, \psi = \sin(\propto h), \psi = h \qquad \propto, \lambda \in \boldsymbol{\mathcal{R}}$$

The Standard scheme developed using (11) will be named NEW h.

The hybrid scheme is obtained by substituting h for $\varphi = \sin(h)$ and $\varphi = \frac{(e^{\lambda h}-1)}{\lambda}$

which will be named NEW SIN, NEW EXP respectively

Qualitative properties of the new scheme

Definition (Henrici,1962)

Any algorithm for solving a differential equation in which the approximation y_{n+1} to the solution at x_{n+1} can be calculated iff $x_n, y_n and\ h$are known is called a one step method. It is a common practice to write the functional dependence y_{n+1} On the quantities $x_n, y_n and\ h$ in the form $y_{n+1}= y_n + \phi(x_n, y_n, h)$
Where $\phi(x_n, y_n, h)\ is\ the$ incremental function

Theorem (Henrici,1962)

Let the incremental function of the scheme defined in the one step scheme above be continuous and jointly as a function of its arguments in the region defined by
$x \in [a, b] and\ y \in (-\infty, \infty), 0 \le h \le h_0$ Where$h_0 > 0$ and let there exists a constant L such that $\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \le L|y_n - y_n^*|$ for all $(x_n, y_n, h) and\ (x_n, y_n^*, h)$ $in\ the\ region\ just\ defined$ then the relation $(x_n, y_n, 0) = (x_n, y_n^*)$ is a necessary condition for the convergence of the new scheme

Definition (Fatunla, 1988)

A numerical scheme with an incremental $\phi(x_n, y_n, h)$ is said to be consistent with the initial value problem $y' = f(x, y), y(x_0) = y_0$ if the incremental function is identically zero at t_0 when $h = 0$,

Theorem (Fatunla, 1988)

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of the differential equation with the initial condition specified a
$y_0 = y(x_0) = \xi$and $p_0 = p(x_0) = \xi^*$ respectively, such that $|\xi - \xi^*| < \varepsilon$ $\varepsilon > 0$
If the two numerical estimates are generated by the integration scheme , we have
$y_{n+1} = y_n + h\phi(x_n, y_n, h)$
$p_{n+1} = p_n + h\phi(x_n, p_n, h)$
The condition that $|y_{n+1} - p_{n+1}| \le K\ |\xi - \xi^*|$ is the necessary and sufficient condition for the stability and convergence of the schemes.

Proof of Convergence

$$y_{n+1} = y_n + \left(f_n + \frac{f_n'}{\propto}\right)h + \left\{\frac{f_n'}{\alpha^2\ e^{-\alpha(a+nh)}}\right\}\left(e^{-\alpha(a+(n+1)h)} - e^{-\alpha(a+nh)}\right)$$

$$y_{n+1} = y_n + \left(f_n + \frac{f_n'}{\propto}\right)h + \left\{\frac{f_n'}{\alpha^2}\right\}\left(e^{-\alpha h} - 1\right)$$

Simplify to obtain

$$y_{n+1} = y_n + \{h\}f_n + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\}f_n'$$

(12)

The incremental function can be written as

$$\phi(x_n, y_n, h) = hf_n + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\}f_n'$$

$$\phi(x_n, y_n, h) = Af_n + Bf_n'$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = A[f(x_n, y_n, h) - f(x_n, y_n^*, h)] + B[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= A[f(x_n, y_n) - f(x_n, y_n^*)] + B[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= A\left[\frac{\partial f(x_n,\bar{y})}{\partial y_n}(y_n - y_n^*)\right] + B\left[\frac{\partial f'(x_n,\bar{y})}{\partial y_n}(y_n - y_n^*)\right] \quad (13)$$

L1 = $\mathrm{SUP}_{(x_n,y_n)\in D} \frac{\partial f(x_n,\bar{y})}{\partial y_n}$ and

L2 = $\mathrm{SUP}_{(x_n,y_n)\in D} \frac{\partial f'(x_n,\bar{y})}{\partial y_n}$

then

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = A[L1(y_n - y_n^*)] + B[L2(y_n - y_n^*)] \quad (14)$$

Let $M = |A.L1 + B.L2|$

$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \leq M|y_n - y_n^*|$ which is the condition for convergence

Consistency of the schemes

$$y_{n+1} = y_n + hf_n + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\}f_n'$$

$$y_{n+1} = y_n + h\left\{f_n + \left[\frac{(e^{-\alpha h}-1)}{h\alpha^2} + \frac{1}{\alpha}\right]f_n'\right\}$$

$$y_{n+1} = y_n + h\,\phi(x_n, y_n, h)$$

When $h = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$

Stability of the schemes

Consider the equation

$$y_{n+1}= y_n+hf_n(x_n,y_n) +\left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\} f_n'(x_n,y_n)$$

$$y_{n+1}= y_n+\{A\}f_n(x_n,y_n) +\{B\}f_n'(x_n,y_n) \quad (15)$$

Let $p_{n+1}= p_n+\{A\}f_n(x_n,P_n) +\{B\}f_n'(x_n,P_n)$

$$y_{n+1} - p_{n+1}= y_n - p_n+ \{A\}[f_n(x_n,y_n) - f_n(x_n,P_n)] +\{B\}[f_n'(x_n,y_n) - f_n'(x_n,P_n)]$$

$$= y_n - p_n+ A[\frac{\partial f(x_n,p_n)}{\partial p_n}(y_n - p_n)] + B[\frac{\partial f'(x_n,p_n)}{\partial p_n}(y_n - p_n)] \quad (16)$$

L1 = $\text{SUP}_{(x_n,y_n)\in D} \frac{\partial f(x_n,p_n)}{\partial p_n}$ and

L2 = $\text{SUP}_{(x_n,y_n)\in D} \frac{\partial f'(x_n,p_n)}{\partial p_n}$

$$y_{n+1} - p_{n+1}= y_n - p_n+ A.L1(y_n - p_n) + B.L2(y_n - p_n)$$

$$|y_{n+1} - p_{n+1}|= |y_n - p_n|+ [A.L1 + B.L2]|(y_n - p_n)| \quad (17)$$

Let N = $|1+ [A.L1 + B.L2]|$

$$|y_{n+1} - p_{n+1}|\leq N\ |y_n - p_n|$$

Let $y_0= y(x_0) = \xi$ and $p_0= p(x_0) = \xi^*$ then

$$|y_{n+1} - p_{n+1}|\leq K\ |\xi-\xi^*| \quad (18)$$

Application to some Initial Value Problems

These Initial value problems are from the books of Shepley Ross and also D.G. Zill

Example 1

$$y' = x^2 + y, y(0) = 1 \quad (19)$$

$$f_n = y'(x_n) = x_n^2 + y_n$$

$$f_n' = y''(x_n) = 2x_n + y_n' = 2x_n + x_n^2 + y_n$$

The standard scheme is

$$y_{n+1} = y_n + \left(x_n^2 + y_n + \frac{2x_n+x_n^2+y_n}{\propto}\right)\text{h}+ \left\{\frac{2x_n+x_n^2+y_n}{\propto^2\ e^{-\propto(a+nh)}}\right\}\left(e^{-\propto(a+(n+1)h)} - e^{-\propto(a+nh)}\right) \quad (20)$$

Set a= 0

$$y_{n+1} = y_n + \left(x_n^2 + y_n + \frac{2x_n+x_n^2+y_n}{\propto}\right)\text{h}+ \left\{\frac{2x_n+x_n^2+y_n}{\propto^2\ e^{-\alpha nh}}\right\}\left(e^{-\propto((n+1)h)} - e^{-\propto nh}\right) \quad (21)$$

The two hybrid schemes of (21) will be obtained by changing h to $\psi = \sin(h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$

The Analytic solution is $y = 2e^{x} - x - 1$

The Nonstandard scheme using rules 2 and 3 will be obtained by replacing the denominator h by ψ and approximating y non-locally

$$y' = x^2 + y \tag{22}$$

$$\frac{(y_{n+1} - y_n)}{\psi} = x_n^2 + cy_{n+1} + dy_n$$

$$y_{n+1} = y_n + \frac{y_n(1+\psi d) + \psi x_n^2}{(1-\psi c)} \tag{23}$$

Example 2

$$y' = 4x - 2y, \quad y(0) = 3 \tag{24}$$

$$f_n = y'_{(x_n)} = 4x_n - 2y_n$$

$$f'_n = y''(x_n) = 4 - 2y'_n = 4 - 8x_n + 4y_n$$

$$y_{n+1} = y_n + hf_n + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\} f'_n$$

$$y_{n+1} = y_n + h(4x_n - 2y_n) + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\}(4 - 8x_n + 4y_n) \tag{25}$$

The two hybrid schemes of (25) will be obtained by changing h to $\psi = \sin(h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$

The Nonstandard scheme using rules 2 and 3 will be obtained by replacing the denominator h by ψ and approximating y non-locally

$$y' = 4x - 2y \tag{26}$$

$$\frac{(y_{n+1} - y_n)}{\psi} = 4x_n + 2cy_{n+1} + 2dy_n$$

$$y_{n+1} = y_n + \frac{y_n(1-2\psi d) + 4x\psi}{(1+2\psi c)} \tag{27}$$

Example 3

$\frac{dy}{dx}(e^{2x}y) = (2x - e^{2x}y^2)$, y(0)=2, $y(x) = \mathrm{e}^{-\mathrm{x}}(2x^2 + 4)^{1/2}$ (28)

$$f_n = y'(x_n) = \frac{2x_n - y_n^2 e^{2x_n}}{y_n e^{2x_n}}$$

$$f_n' = y''(x_n) = [(4x_n - 2)y_n + (y_n^2 e^{2x_n} + 2x_n)f_n]/(y_n^2 e^{2x_n})$$

$$y_{n+1} = y_n + hf_n + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\} f_n'$$

$$y_{n+1} = y_n + h\left(\frac{2x_n - y_n^2 e^{2x_n}}{y_n e^{2x_n}}\right) + \left\{\frac{(e^{-\alpha h}-1)}{\alpha^2} + \frac{h}{\alpha}\right\} \frac{[(4x_n - 2)y_n + (y_n^2 e^{2x_n} + 2x_n)f_n]}{(y_n^2 e^{2x_n})} \quad (29)$$

The two hybrid schemes of (29) will be obtained by changing h to $\psi = \sin(h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$

The Nonstandard scheme using rules 2 and 3 will be obtained by replacing the denominator h by ψ and approximating y^2 non-locally

Replace y^2 *in* (28) *by* $(ay_k y_{k+1} + by_k^2)$, $a + b = 1$ and h by ψ

$$\frac{y_{k+1} - y_k}{\varphi} = \frac{2x - e^{2x}(ay_k y_{k+1} + by_k^2)}{e^{2x} y_k}$$

$$y_{k+1}(e^{2x}y_k + a\varphi y_k e^{2x}) = e^{2x}y_k^2 + 2\varphi x - b\varphi e^{2x} y_k^2$$

$$y_{k+1} = \frac{2\varphi x + e^{2x} y_k^2 (1 - b\varphi)}{e^{2x} y_k (1 + a\varphi)} \quad (30)$$

Numerical experiment

The schemes have been tested using various step sizes and the behavior of the curves were consistent. We present below the 3D graphs for the scheme using step size h=0.01

Example 1 Schemes of $y' = x^2 + y, y(0) = 1$

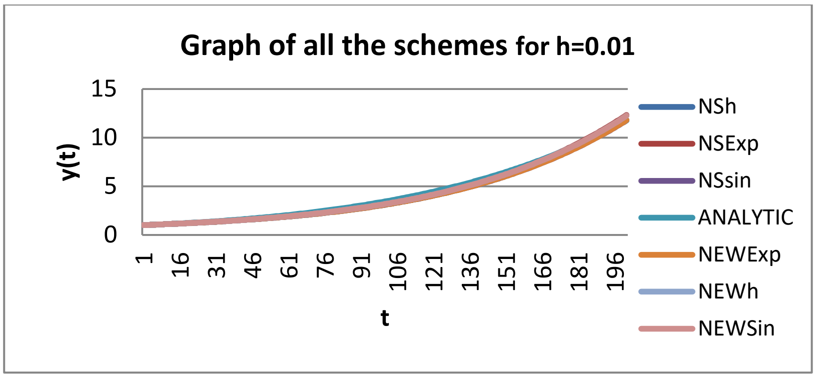

Fig 1: solution curves for the standard, hybrid and Nonstandard schemes of example 1

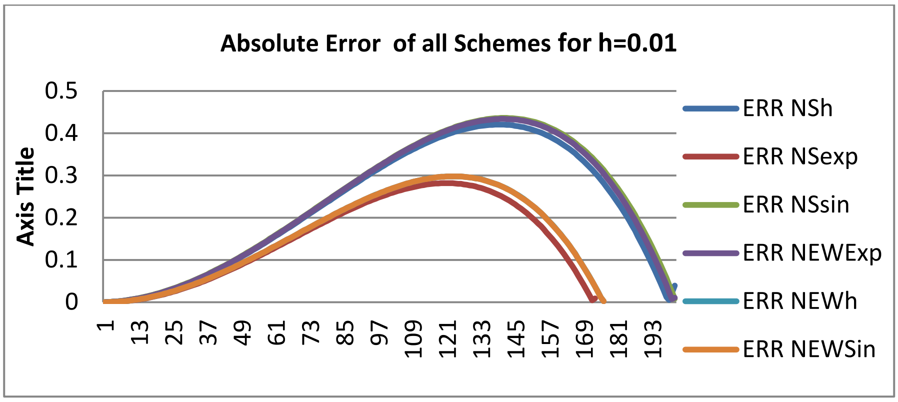

Fig 2: Graph of absolute Error for the standard, hybrid and Nonstandard schemes of example 1

Example 2 Schemes of $y' = 4x - 2y, \ y(0) = 3$

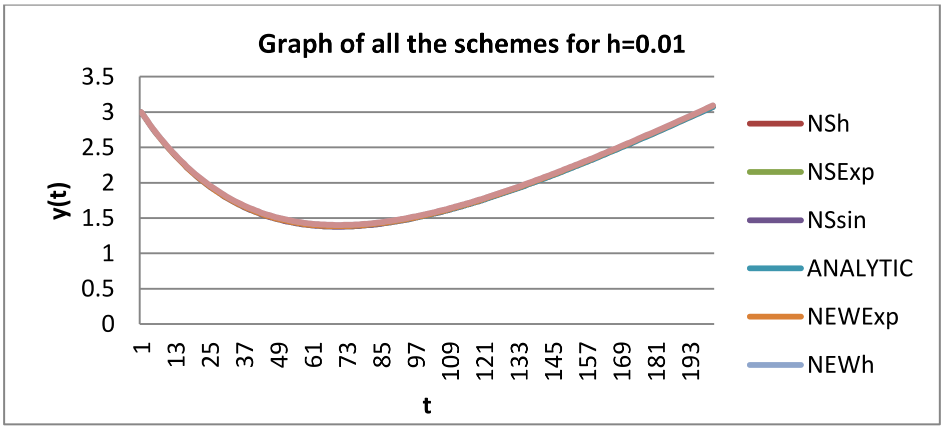

Fig 3: solution curves for the standard, hybrid and Nonstandard schemes of example 2

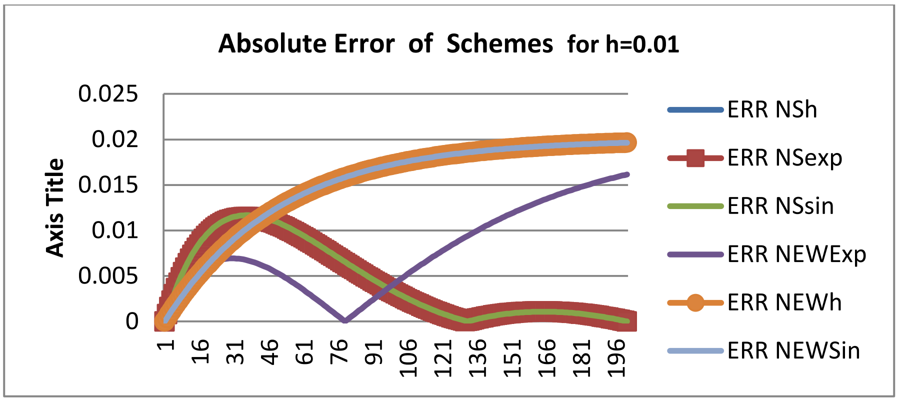

Fig 4: Graph of absolute Error for the standard, hybrid and Nonstandard schemes of example 2

Example 3 schemes of $\frac{dy}{dx}(e^{2x}y) = (2x - e^{2x}y^2)$, y(0)=2

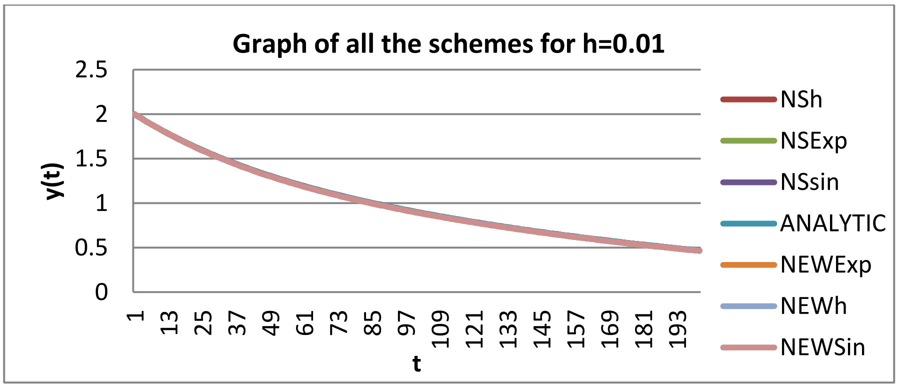

Fig 5: solution curves for the standard, hybrid and Nonstandard schemes of example 3

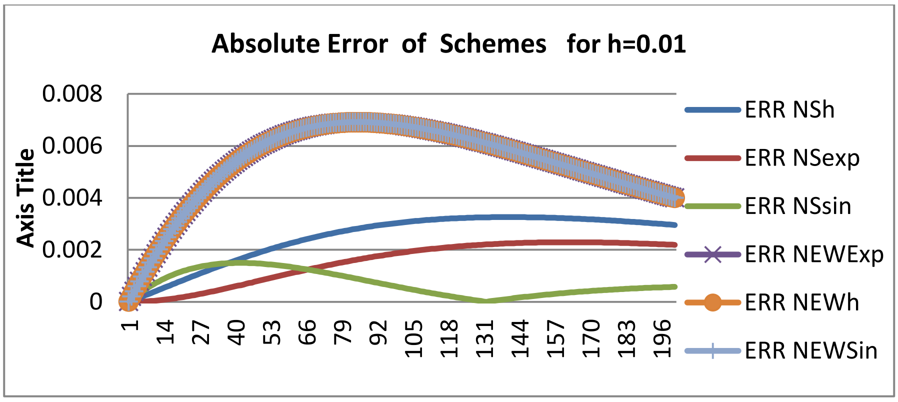

Fig 6: Graph of absolute Error for the standard, hybrid and Nonstandard schemes of example 3

Discussion Conclusion

Mickens non-standard modeling rules remains a very powerful tool for discrete modeling of dynamic systems. It has ones again prove to be very useful tool for building numerically stable finite difference schemes. This example shows that a lot of improvement can be obtained by apply the renormalization techniques to finite difference schemes. It will be recalled that only the derived scheme NEW h is a standard finite difference scheme . Even though this New h scheme also possess the consistency, stability and convergence qualities, the renormalized schemes produced lower absolute error of deviation from the Analytic solution . It can be observed that schemes with normalized step-size (Sin, Exp) instead of standard step-size h have very low absolute error and are relatively closer to the Analytic solution. All the schemes have been found to be consistent with literature and compared favorably with the dynamics of the original equation.

References

[1]. Obayomi A. A. (2015). On the NonStandard Approach to the Numerical Solution of Ordinary Differential Equations . International Journal of Mathematics Trends and Technology. 17(2),9-24

[2]. Obayomi A. A. and M.O Oke (2016). Development of new Non Standard Denominator function for Finite Difference Schemes. Journal of the Nigerian Association of Mathematical Physics Applied Science and Mathematical Theory. 33(1), 51-60 Nigeria.

[3]. R.E. Mickens (1994). Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore. Pp 1-115, 144-162

[4]. D.G. Zill and R.M. Cullen (2005) Differential Equations with boundary value problems (sixth Edition) Brooks /Cole Thompson Learning Academic Resource Center. Pp40 -112, 367-387

[5]. Anguelov R, Lubuma JMS (2003). Nonstandard finite difference method by nonlocal approximation, Mathematics and Computers in simulation 6: 465-475.

[6]. Lambert J. D. (1991). Numerical Methods for Ordinary Differential System; Wiley, New York

[7]. Mickens R.E. (2000). Applications of Nonstandard Methods for initial value problems, World Scientific, Singapore.

[8]. Henrici P. (1962). Discrete Variable Methods in ODE. John Willey &Sons , New York.

[9]. Fatunla S. O. (1988), Numerical Methods for Initial Values Problems on Ordinary Differential Equations. Academic Press New York.

[10]. Shepley L. Ross (1984), Differential Equations 3RD Edition; John Wiley &Sons, Canada

[11]. Dalquist G. (1959), Stability and Error Bounds in the Numerical Integration of ODEs. Transcript, 130, Royal Institute of Technology Stockholm