

Parameter Estimation of Weighted Erlang Distribution Using R Software

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Abstract

The Erlang distribution belongs to a group of continuous probability distributions with universal relevance primarily due to its relation to the exponential and Gamma distributions. If the time period of individual telephone calls is exponentially distributed, then the duration of the successive calls follows the Erlang distribution. In this paper, we take into account the weighted version of Erlang distribution known as weighted Erlang distribution. We obtain the posterior mean and posterior variance of the model. Maximum likelihood method of estimation is discussed. Bayes estimates of the scale parameter of Weighted Erlang distribution is offered for consideration under Squared Error Loss Function (SELF), Quadratic Loss Function (QLF) and entropy Loss Function (ELF) using Jeffrey's, extension of Jeffrey's and Quasi priors.

Keywords: Erlang distribution, Weighted Erlang distribution, Loss function, Bayesian estimation.

1. Introduction

The Erlang distribution developed by A.K.Erlang (1909) belongs to a group of continuous probability distributions with universal relevance primarily due to its relation to the exponential and Gamma distributions. The distribution was developed by A.K.Erlang to analyze the number of telephone calls that could be made simultaneously to switching station operators. This work on telephone traffic engineering has been extended to take into deliberation waiting times in queuing systems. Queuing theory came into existence on the publication of Erlang's fundamental paper (1909) pertaining to the study of telephone traffic congestion. If the time period of individual telephone calls is exponentially distributed, then the duration of the successive calls follows the Erlang distribution. A random variable X has an Erlang distribution with parameters λ and β if its probability density function is of the form

$$f(x; \lambda, \beta) = \frac{1}{(\lambda - 1)! \beta^\lambda} x^{\lambda-1} \exp\left(-\frac{x}{\beta}\right), x > 0; \lambda = 1, 2, 3, \dots; \beta > 0 \quad (1)$$

Where λ and β are the shape and the scale parameters, respectively, such that λ is a positive integer number.

The Mean and variance of Erlang distribution is given as

$$\text{Mean} = \lambda\beta \quad \text{and} \quad \text{Variance} = \lambda\beta^2$$

1.1 Weighted Erlang Distribution

The observations produced from a stochastic process if not given equal chances of being recorded leads to the emergence of weighted distributions; rather they are recorded in accordance with some weighted function. The concept of weighted distributions is traceable to the work of Fisher (1934) in respect of his studies on how methods of ascertainment can

affect the form of distribution of recorded observations. Later it was introduced and formulated in a more general way by Rao (1965) with respect to modeling statistical data where the routine practice of using standard distributions for the purpose was dismissed as inappropriate. In Rao's paper he spotlighted a number of situations that can be modeled by weighted distributions. These situations refer to such instances where the recorded observations cannot be considered as a random sample from the original distributions. This may happen as a result of non-observability of some events or damage inflicted on the original observations leading to the reduced value or adoption of a sampling procedure which provides unequal chances to the units in the original. The usefulness and applications of weighted distributions to biased samples in various areas including medicine, ecology, reliability and branching processes can be seen in Patil and Rao (1978), Gupta and Keating (1985), Gupta and Kirmani (1990), Oluyede (1999). Within the context of cell kinetics and the early detection of disease, Zelen (1974) introduced weighted distributions to represent what he broadly perceived as length biased sampling (introduced earlier in Cox, D.R. (1962)). For additional and important results on weighted distributions, see Zelen and Feinleib (1969), Patil and Ord (1976), Rao (1997). Applications examples for weighted distributions as El-Shaarawi and Walter (2002), and there are many researches for weighted distributions as Castillo and Perez-Casany (1998) introduced new exponential families that come from the concept of weighted distribution that include and generalize the Poisson distribution, Jing (2010) introduced the weighted inverse Weibull distribution and beta inverse Weibull distribution and theoretical properties of them, Das and Roy (2011) discussed the length biased weighted generalized Rayleigh distribution with its properties. Das and Roy (2011) also develop the length biased form of the weighted Weibull distribution. Priyadarshani (2011) introduced a new class of weighted generalized gamma distribution and related distributions, Sofi Mudasir and S.P. Ahmad (2015) study the length biased Nakagami distribution.

In order to introduce the concept of weighted distribution, let us suppose that X is a non-negative random variable with probability density function (p.d.f.) $f(x)$, then the p.d.f. of the weighted random variable X_w is given as

$$f_w(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)dx}, x > 0 \quad (2)$$

Here the recording (weight) function $w(x)$ is a non-negative function representing the recording (sighting) mechanism. The random variable X_w is called the weighted version of X , and its distribution in relation to that of X is called the weighted distribution with weight function $w(x)$. Note that the weight function $w(x)$ need not lie between zero and one, and actually may exceed unity.

Let $w(x) = x^\theta$ (3)

Also,
$$\int_0^{\infty} w(x)f(x)dx = \int_0^{\infty} \frac{1}{(\lambda - 1)! \beta^\lambda} x^{\lambda + \theta - 1} \exp\left(-\frac{x}{\beta}\right) dx$$

$$= \frac{\beta^\theta \Gamma(\theta + \lambda)}{(\lambda - 1)!} \quad (4)$$

Substitute the value of equations (1), (3), and (4) in equation (2), we get

$$f_w(x) = \frac{1}{\Gamma(\lambda + \theta) \beta^{\lambda + \theta}} x^{\lambda + \theta - 1} \exp\left(-\frac{x}{\beta}\right) \quad (5)$$

This is the required probability density function of weighted Erlang distribution.

The cumulative distribution function corresponding to (5) is given as

$$F_X(x) = \frac{1}{\Gamma(\lambda + \theta)} \gamma\left(\lambda + \theta, \frac{x}{\beta}\right)$$

Also the mean and variance of weighted Erlang distribution is

$$\text{mean} = \beta(\lambda + \theta)$$

$$\text{variance} = \beta^2(\lambda + \theta)$$

Special cases

1) If $\theta=0$ in (5), we get Erlang distribution

$$f(x) = \frac{1}{\Gamma(\lambda) \beta^\lambda} x^{\lambda-1} \exp\left(-\frac{x}{\beta}\right)$$

2) If $\theta=1$ in (5), we get length biased Erlang distribution

$$f_l(x) = \frac{1}{\Gamma(\lambda + 1) \beta^{\lambda+1}} x^\lambda \exp\left(-\frac{x}{\beta}\right)$$

3) If $\theta=2$ in (5), we get area biased Erlang distribution

$$f_a(x; \lambda, \beta) = \frac{1}{\Gamma(\lambda + 2) \beta^{\lambda+2}} x^{\lambda+1} \exp\left(-\frac{x}{\beta}\right)$$

4) If $\theta=0, \lambda=1$ in (5), we get exponential distribution

$$f(x) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right)$$

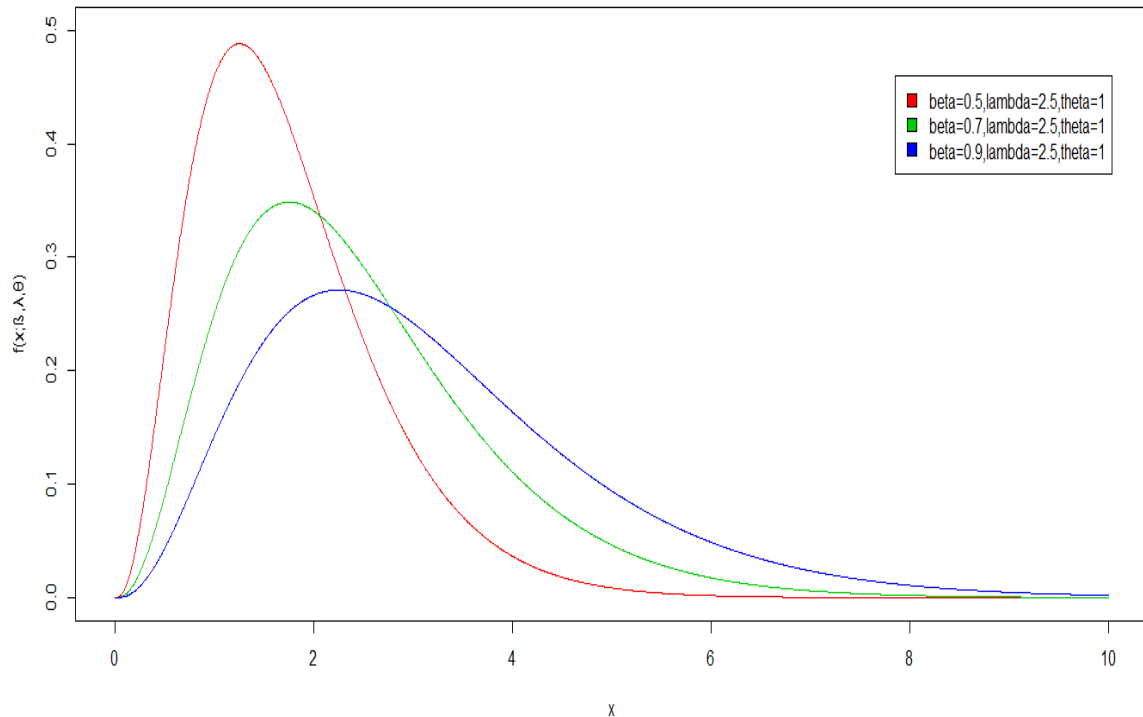


Figure 1.1: The pdf of weighted Erlang distribution under different values of Scale Parameter β

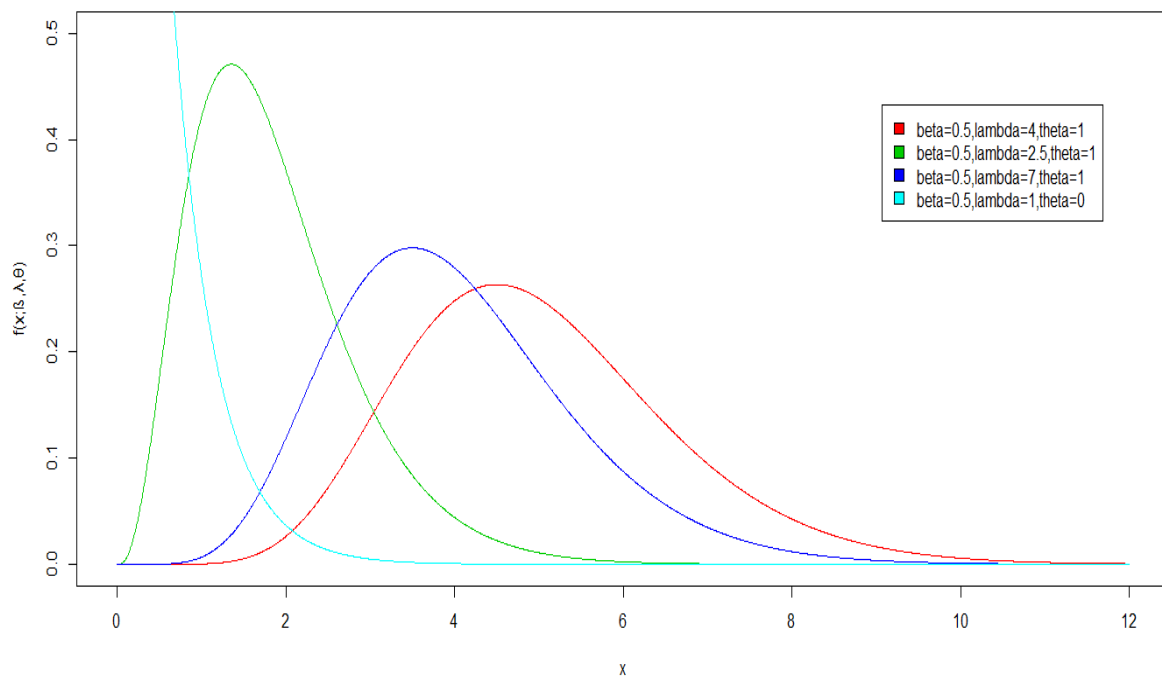


Figure 1.2: The pdf of weighted Erlang distribution under different values of shape Parameter λ

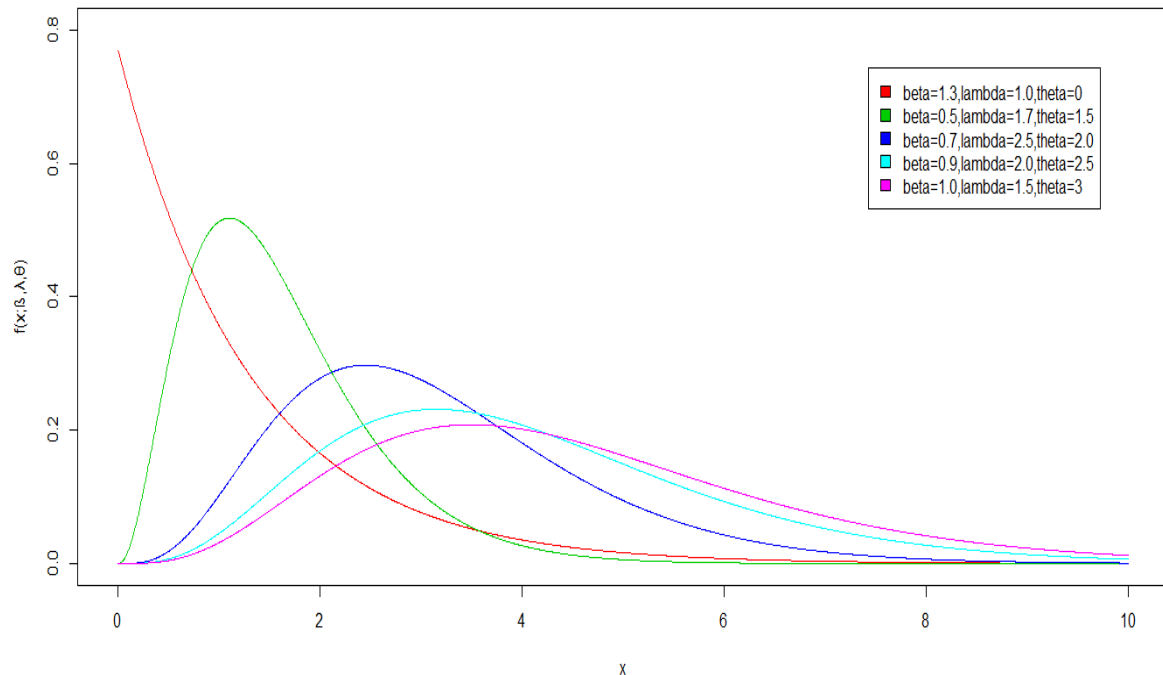


Figure 1.3: The pdf of the weighted Erlang distribution under different values of Parameters

2. Estimation of Scale Parameter

In this section we provide the maximum likelihood and Bayes estimates of the scale parameter of weighted Erlang distribution.

2.1 Maximum likelihood estimation

Let x_1, x_2, \dots, x_n be a random sample from weighted Erlang distribution. Then the likelihood function is given by

$$L(x; \lambda, \beta, \theta) = \left(\frac{1}{\Gamma(\lambda + \theta)\beta^{\lambda + \theta}} \right)^n \prod_{i=1}^n x_i^{\lambda + \theta - 1} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta} \right) \quad (6)$$

Using equation (6), the log likelihood function is given by

$$\log L(x; \alpha, \beta, \theta) = -n \log(\Gamma(\lambda + \theta)) - n(\lambda + \theta) \log \beta + (\lambda + \theta - 1) \sum_{i=1}^n \log x_i - \frac{\sum_{i=1}^n x_i}{\beta}$$

Now differentiate the above equation with respect to β and equate to zero, we get

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n(\lambda + \theta)} = \frac{\bar{x}}{(\lambda + \theta)}$$

This is the required MLE of β

2.2 Bayesian method of estimation

In this section Bayesian estimation of the scale parameter of weighted Erlang distribution is obtained by using Jeffery's, extension of Jeffrey's and quasi priors under different loss functions.

2.2.1 Posterior distribution under Jeffrey's prior $\pi_1(\beta)$

Let x_1, x_2, \dots, x_n be a random sample of size n having the probability density function (5) and the likelihood function (6).

The Jeffery's prior relating to the scale parameter β is given by

$$\pi_1(\beta) \propto \frac{1}{\beta}; \beta > 0 \quad (7)$$

According to Bayes theorem, the posterior distribution is given by

$$P_1(\beta | \underline{x}) \propto L(x; \lambda, \beta, \theta) \pi_1(\beta) \quad (8)$$

Using equations (6) and (7) in equation (8), we get

$$P_1(\beta | \underline{x}) = k \frac{1}{\beta^{n\lambda+n\theta+1}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (9)$$

Where k is independent of β and is given by

$$k = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta}}{\Gamma(n\lambda+n\theta)}$$

Thus, from equation (9), posterior distribution is given by

$$P_1(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta}}{\Gamma(n\lambda+n\theta)\beta^{n\lambda+n\theta+1}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (10)$$

2.2.2 Posterior distribution under extension of Jeffrey's prior $\pi_2(\beta)$

Let x_1, x_2, \dots, x_n be a random sample of size n having the probability density function (5) and the likelihood function (6).

The extension of Jeffery's prior relating to the scale parameter β is given by

$$\pi_2(\beta) \propto \frac{1}{\beta^{2C}}; \beta > 0 \quad (11)$$

By using the Bayes theorem, we have

$$P_2(\beta | \underline{x}) \propto L(x; \lambda, \beta, \theta) \pi_2(\beta) \quad (12)$$

Using equations (6) and (11) in equation (12), we get

$$P_2(\beta | \underline{x}) = k \frac{1}{\beta^{n\lambda+n\theta+2C}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (13)$$

Where k is independent of β and is given by

$$k = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta+2C-1}}{\Gamma(n\lambda+n\theta+2C-1)}$$

Thus, from equation (13), posterior distribution is given by

$$P_2(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta+2C-1}}{\Gamma(n\lambda+n\theta+2C-1)\beta^{n\lambda+n\theta+2C}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (14)$$

2.2.3 Posterior distribution under Quasi prior $\pi_3(\beta)$

Let x_1, x_2, \dots, x_n be a random sample of size n having the probability density function (5) and the likelihood function (6).

The quasi prior relating to the scale parameter β is given by

$$\pi_3(\beta) \propto \frac{1}{\beta^d}; \beta > 0, d > 0 \quad (15)$$

According to Bayes theorem, the posterior distribution is given by

$$P_3(\beta | \underline{x}) \propto L(x; \lambda, \beta, \theta) \pi_3(\beta) \quad (16)$$

Using equations (6) and (15) in equation (16), we get

$$P_3(\beta | \underline{x}) = k \frac{1}{\beta^{n\lambda+n\theta+d}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (17)$$

Where k is independent of β and is given by

$$k = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta+d-1}}{\Gamma(n\lambda+n\theta+d-1)}$$

Thus, from equation (17), posterior distribution is given by

$$P_3(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda+n\theta+d-1}}{\Gamma(n\lambda+n\theta+d-1)\beta^{n\lambda+n\theta+d}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (18)$$

2.3. Bayesian estimation by using Jeffrey's' prior under different loss functions

Theorem 1: Assuming the square error loss function (SELF), $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta - 1}$$

Proof: By using the square error loss function $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} (\beta - \hat{\beta})^2 P_1(\beta | \underline{x}) d\beta \quad (19)$$

Using (10) in (19), we get

$$R(\hat{\beta}) = \hat{\beta}^2 + \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta - 1)(n\lambda + n\theta - 2)} - 2\hat{\beta} \frac{\sum_{i=1}^n x_i}{(n\lambda + n\theta - 1)}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta - 1}$$

Theorem 2: Assuming the quadratic loss function (QLF), $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 1}$$

Proof: By using the quadratic loss function $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2 P_1(\beta | \underline{x}) d\beta \quad (20)$$

Substitute the value of equation (10) in equation (20), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta + 1)(n\lambda + n\theta)}{\left(\sum_{i=1}^n x_i\right)^2} \hat{\beta}^2 + 1 - 2\hat{\beta} \frac{(n\lambda + n\theta)}{\sum_{i=1}^n x_i}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 1}$$

Theorem 3: Assuming the Entropy loss function (ELF), $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right)$, the

bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta}$$

Proof: By using the Entropy loss function $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right)$, the risk function is

given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right) P_1(\beta | \underline{x}) d\beta \quad (21)$$

Substitute the value of equation (10) in equation (21), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta)}{\sum_{i=1}^n x_i} \hat{\beta} + \log\left(\frac{\sum_{i=1}^n x_i}{\hat{\beta}}\right) - \frac{\Gamma'(n\lambda + n\theta)}{\Gamma(n\lambda + n\theta)}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta}$$

2.4. Bayesian estimation by using extension of Jeffrey's' prior under different loss functions

Theorem 4: Assuming the square error loss function (SELF), $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C - 2}$$

Proof: By using the square error loss function $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} (\beta - \hat{\beta})^2 P_2(\beta | \underline{x}) d\beta \quad (22)$$

Substitute the value of equation (14) in equation (22), we get

$$R(\hat{\beta}) = \hat{\beta}^2 + \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + 2C - 2)(n\lambda + n\theta + 2C - 3)} - 2\hat{\beta} \frac{\sum_{i=1}^n x_i}{(n\lambda + n\theta + 2C - 2)}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C - 2}$$

Theorem 5: Assuming the quadratic loss function (QLF) $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C}$$

Proof: By using the quadratic loss function $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2 P_2(\beta | \underline{x}) d\beta \quad (23)$$

Substitute the value of equation (14) in equation (23), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta + 2C)(n\lambda + n\theta + 2C - 1)}{\left(\sum_{i=1}^n x_i\right)^2} \hat{\beta}^2 + 1 - 2\hat{\beta} \frac{(n\lambda + n\theta + 2C - 1)}{\sum_{i=1}^n x_i}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C}$$

Theorem 6: Assuming the Entropy loss function (ELF), $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right)$, the

bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C - 1}$$

Proof: By using the Entropy loss function $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right)$, the risk function is

given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right) P_2(\beta | \underline{x}) d\beta \quad (24)$$

Substitute the value of equation (14) in equation (24), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta + 2C - 1)}{\sum_{i=1}^n x_i} \hat{\beta} - \log(\hat{\beta}) + \frac{\Gamma'(n\lambda + n\theta)}{\Gamma(n\lambda + n\theta)} - 1$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{(n\lambda + n\theta + 2C - 1)}$$

Remark: If $C = \frac{1}{2}$, estimates obtained by using extension of Jeffrey's prior coincides with the estimates obtained by using Jeffrey's prior.

2.5. Bayesian estimation by using Quasi' prior under different loss functions

Theorem 1: Assuming the square error loss function (SELF), $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d - 2}$$

Proof: By using the square error loss function $L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} (\beta - \hat{\beta})^2 P_3(\beta | \underline{x}) d\beta \quad (25)$$

Using (18) in (25), we get

$$R(\hat{\beta}) = \hat{\beta}^2 + \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + d - 2)(n\lambda + n\theta + d - 3)} - 2\hat{\beta} \frac{\sum_{i=1}^n x_i}{(n\lambda + n\theta + d - 2)}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d - 2}$$

Theorem 2: Assuming the quadratic loss function (QLF) $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d}$$

Proof: By using the quadratic loss function $L(\beta, \hat{\beta}) = \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2$, the risk function is given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2 P_3(\beta | \underline{x}) d\beta \quad (26)$$

Substitute the value of equation (18) in equation (26), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta + d)(n\lambda + n\theta + d - 1)}{\left(\sum_{i=1}^n x_i\right)^2} \hat{\beta}^2 + 1 - 2\hat{\beta} \frac{(n\lambda + n\theta + d - 1)}{\sum_{i=1}^n x_i}$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d}$$

Theorem 3: Assuming the Entropy loss function (ELF), $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1\right)$, the

bayes estimate of the scale parameter β , when the shape parameter λ is known, is of the form

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d - 1}$$

Proof: By using the Entropy loss function $L(\beta, \hat{\beta}) = \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1\right)$, the risk function is

given by

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\hat{\beta}}{\beta} - \log\left(\frac{\hat{\beta}}{\beta}\right) - 1 \right) P_3(\beta | \underline{x}) d\beta \quad (27)$$

Substitute the value of equation (18) in equation (27), we get

$$R(\hat{\beta}) = \frac{(n\lambda + n\theta + d - 1)}{\sum_{i=1}^n x_i} \hat{\beta} - \log(\hat{\beta}) + \log\left(\sum_{i=1}^n x_i\right) - \frac{\Gamma'(n\lambda + n\theta + d - 1)}{\Gamma(n\lambda + n\theta + d - 1)} - 1$$

Minimization of risk function with respect to $\hat{\beta}$ gives us the optimal estimator as

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d - 1}$$

Remark: If $d = 1$, estimates obtained by using quasi prior coincides with the estimates obtained by using Jeffrey's prior.

3. Posterior Mean and Posterior Variance of Scale Parameter β under Different Priors

In this section, the posterior mean and posterior variance of the scale parameter β under Jeffrey's, extension of Jeffrey's and quasi priors are obtained.

3.1 Posterior Mean and Posterior Variance of β under Jeffrey's prior

Posterior distribution under Jeffrey's prior is given as

$$P_1(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda + n\theta}}{\Gamma(n\lambda + n\theta)\beta^{n\lambda + n\theta + 1}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (28)$$

$$\text{Now, } E(\beta^r) = \int_0^{\infty} \beta^r P_1(\beta | \underline{x}) d\beta \quad (29)$$

By using equation (28) in equation (29), we get

$$E(\beta^r) = \frac{\left(\sum_{i=1}^n x_i\right)^r}{\Gamma(n\lambda + n\theta)} \Gamma(n\lambda + n\theta - r) \quad (30)$$

Put $r=1$ in equation (30), we get

$$E(\beta) = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta - 1}$$

This is the posterior mean.

Put $r=2$ in equation (30), we get

$$E(\beta^2) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta - 1)(n\lambda + n\theta - 2)}$$

Thus, the posterior variance is given as

$$v(\beta) = E(\beta^2) - (E(\beta))^2$$

$$\Rightarrow v(\beta) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta - 1)^2(n\lambda + n\theta - 2)}$$

3.2 Posterior Mean and Posterior Variance of β under extension of Jeffrey's prior

Posterior distribution under extension of Jeffrey's prior is given as

$$P_2(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda + n\theta + 2C - 1}}{\Gamma(n\lambda + n\theta + 2C - 1)\beta^{n\lambda + n\theta + 2C}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (31)$$

$$\text{Now, } E(\beta^r) = \int_0^{\infty} \beta^r P_2(\beta | \underline{x}) d\beta \quad (32)$$

By using equation (31) in equation (32), we get

$$E(\beta^r) = \frac{\left(\sum_{i=1}^n x_i\right)^r}{\Gamma(n\lambda + n\theta + 2C - 1)} \Gamma(n\lambda + n\theta + 2C - r - 1) \quad (33)$$

Put $r=1$ in equation (33), we get

$$E(\beta) = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + 2C - 2}$$

This is the posterior mean.

Put $r=2$ in equation (33), we get

$$E(\beta^2) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + 2C - 2)(n\lambda + n\theta + 2C - 3)}$$

Thus, the posterior variance is given as

$$v(\beta) = E(\beta^2) - (E(\beta))^2$$

$$\Rightarrow v(\beta) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + 2C - 2)^2 (n\lambda + n\theta + 2C - 3)}$$

Remark: If $C = \frac{1}{2}$, the posterior mean and posterior variance obtained under extension of Jeffrey's prior coincides with the posterior mean and posterior variance obtained under Jeffrey's prior.

3.3 Posterior Mean and Posterior Variance of β under Quasi prior

Posterior distribution under Quasi prior is given as

$$P_3(\beta | \underline{x}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\lambda + n\theta + d - 1}}{\Gamma(n\lambda + n\theta + d - 1)\beta^{n\lambda + n\theta + d}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\beta}\right) \quad (34)$$

$$\text{Now, } E(\beta^r) = \int_0^{\infty} \beta^r P_3(\beta | \underline{x}) d\beta \quad (35)$$

By using equation (34) in equation (35), we get

$$E(\beta^r) = \frac{\left(\sum_{i=1}^n x_i\right)^r}{\Gamma(n\lambda + n\theta + d - 1)} \Gamma(n\lambda + n\theta + d - 1 - r) \quad (36)$$

Put $r=1$ in equation (36), we get

$$E(\beta) = \frac{\sum_{i=1}^n x_i}{n\lambda + n\theta + d - 2}$$

This is the posterior mean.

Put $r=2$ in equation (36), we get

$$E(\beta^2) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + d - 2)(n\lambda + n\theta + d - 3)}$$

Thus, the posterior variance is given as

$$v(\beta) = E(\beta^2) - (E(\beta))^2$$

$$\Rightarrow v(\beta) = \frac{\left(\sum_{i=1}^n x_i\right)^2}{(n\lambda + n\theta + d - 2)^2 (n\lambda + n\theta + d - 3)}$$

Remark: If $d = 1$, the posterior mean and posterior variance obtained under Quasi prior coincides with the posterior mean and posterior variance obtained under Jeffrey's prior.

4. Real Data

For analysis the strength data, reported by Badar and Priest (1982), It may be noted that Raqab et al.(2008) fitted the 3-parameter generalized exponential distribution to the same data set. Badar and Priest (1982) reported strength data measured in GPA for single carbon fibre and impregnated 1000 carbon fibre tows. Single fibres were tested at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibres were tested at gauge lengths of 20, 50, 150 and 300 mm. The transformed data sets that were considered by Raqab and Kundu (2005) are used here.

0.031, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585

Programs have been developed in R software to obtain the bayes estimates and posterior mean and variance and are presented in the tables below

Table 1. Bayes estimates of β under Jeffrey’s prior

λ	θ	MLE	SELF	QLF	ELF
0.97	1.0	0.7345	0.7399	0.7291	0.7345
	3.2	0.3469	0.3482	0.3457	0.3469
	4.0	0.2911	0.2919	0.2902	0.2911
1.77	1.0	0.5223	0.5251	0.5196	0.5223
	3.2	0.2911	0.2919	0.2902	0.2911
	4.0	0.2507	0.2514	0.2501	0.2507
3.97	1.0	0.2911	0.2919	0.2902	0.2911
	3.2	0.2018	0.2022	0.2014	0.2018
	4.0	0.1815	0.1818	0.1812	0.1815

MLE=Maximum Likelihood Estimator, SELF= Square Error Loss Function, QLF=Quadratic Loss Function, ELF=Entropy Loss Function

Table 2. Bayes estimates of β under extension of Jeffrey's prior

λ	θ	C	MLE	SELF	QLF	ELF
0.97	1.0	0.5	0.7345	0.7399	0.7291	0.7345
	3.2	3.0	0.3469	0.3422	0.3399	0.3410
	4.0	5.5	0.2911	0.2836	0.2820	0.2828
1.77	1.0	0.5	0.5223	0.5251	0.5196	0.5223
	3.2	3.0	0.2911	0.2877	0.2861	0.2869
	4.0	5.5	0.2507	0.2452	0.2440	0.2446
3.97	1.0	0.5	0.2911	0.2919	0.2902	0.2911
	3.2	3.0	0.2018	0.2001	0.1993	0.1997
	4.0	5.5	0.1815	0.1786	0.1779	0.1783

Table 3. Bayes estimates of β under Quasi prior

λ	θ	d	MLE	SELF	QLF	ELF
0.97	1.0	1.0	0.7345	0.7399	0.7291	0.7345
	3.2	3.0	0.3469	0.3457	0.3434	0.3446
	4.0	4.5	0.2911	0.2890	0.2873	0.2881
1.77	1.0	1.0	0.5223	0.5251	0.5196	0.5223
	3.2	3.0	0.2911	0.2902	0.2886	0.2894
	4.0	4.5	0.2507	0.2492	0.2480	0.2486
3.97	1.0	1.0	0.2911	0.2919	0.2902	0.2911
	3.2	3.0	0.2018	0.2014	0.2005	0.2009
	4.0	4.5	0.1815	0.1807	0.1800	0.1804

Table 4. Posterior mean and variance under Jeffrey's prior

λ	θ	Posterior mean	Posterior variance
0.97	1.0	0.7399466	0.0040881134
	3.2	0.3482056	0.0004243417
	4.0	0.2919925	0.0002500796
1.77	1.0	0.5251197	0.0014579953
	3.2	0.2919925	0.0002500796
	4.0	0.2514063	0.0001595566
3.97	1.0	0.2919925	2.7978683220
	3.2	0.2022178	17.5592301980
	4.0	0.1818829	11.4116890570

Table 5. Posterior mean and variance under extension of Jeffrey's prior

λ	θ	C	Posterior mean	Posterior variance
0.97	1.0	0.5	0.7399466	0.0040881134
	3.2	3.0	0.3422377	0.0004028708
	4.0	5.5	0.2836956	0.0002293426
1.77	1.0	0.5	0.5251197	0.0014579953
	3.2	3.0	0.2877843	0.0002394120
	4.0	5.5	0.2452313	0.0001480766
3.97	1.0	0.5	0.2919925	2.797868322
	3.2	3.0	0.2001905	16.887064341
	4.0	5.5	0.1786288	10.545967942

Table 6. Posterior mean and variance under Quasi prior

λ	θ	d	Posterior mean	Posterior variance
0.97	1.0	1.0	0.7399466	0.0040881134
	3.2	3.0	0.3457936	0.0004155745
	4.0	4.5	0.2890340	0.0002425475
1.77	1.0	1.0	0.5251197	0.0014579953
	3.2	3.0	0.2902945	0.0002457380
	4.0	4.5	0.2492100	0.0001554078
3.97	1.0	1.0	0.2919925	2.797868322
	3.2	3.0	0.2014020	17.287105723
	4.0	4.5	0.1807306	11.101541255

Conclusion: In this research work, we have primarily estimate the scale parameter of the new model known as weighted Erlang distribution under different prior distributions assuming different loss functions. For comparison, we use the real life data set and the results are shown in the tables above.

Table 1, table 2 and table 3 shows maximum likelihood and the bayes estimates of the scale parameter β for different values of the shape parameter λ under the Jeffrey's, extension of Jeffrey's and Quasi priors. The values of the shape parameter λ are obtained through R-software. From the tables it is clear that as we increase the value of λ , the value of estimates of β decreases. The bayes estimates have minimum value for extension of Jeffrey's prior as compared to Jeffrey's and quasi priors. The estimates under the quadratic loss function have overall the minimum value than the square error loss function and entropy loss function. The estimates obtained under extension of Jeffrey's prior and quasi prior coincides with the estimates obtained under Jeffrey's prior when the value of hyper-parameters c and d is 0.5 and 1. The posterior mean and posterior variance are presented in table 4, table 5 and table 6 for different values of θ, λ, c and d . From these tables, it is clear that the posterior mean and variance obtained under extension of Jeffrey's prior and Quasi

prior coincides with the posterior mean and variance obtained under Jeffrey's prior when $c = 0.5$ and $d = 1$. We also infer that the posterior variance using extension of Jeffrey's prior have minimum value as compared to Jeffrey's and Quasi priors. So we conclude that the extension of Jeffrey's prior is more efficient as compared to other priors which we have used in this paper.

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