

Unit Mean and Constant Variance of the Generalized Gamma Distribution after Square Root Transformation in Statistical modeling

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Abstract

In this paper, we studied the effect of square root transformation on the error component of the multiplicative error model whose distribution belongs to the generalized gamma family. The purpose of the study is to determine the effect of the said transformation on the basic assumptions; unit mean and constant variance required for statistical modeling. The special cases of the Generalized Gamma Distribution considered are the three-parameter Gamma distribution error component, the Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributed error components. From the results of the study, the unit mean assumption is approximately maintained for all the distributions. It was also found that there were reduction in the variances of all the square root transformed distributions under study except those of the Gamma(a, b, 1), when a > 1, Rayleigh and Maxwell distributions that increased. Therefore we conclude that square root transformation is not appropriate for multiplicative error models with either a Gamma (a, b, 1), for a > 1 or Rayleigh or Maxwell distributed error component. Finally square root transformations where applicable are successful for the distributions under study if the variance of the transformed data < 0.5.

Keywords: Generalized gamma distribution; Square root transformation; Mean; Variance**;** multiplicative error Model; Error component

1. Introduction

The Gamma distribution is often used to describe variables bounded on one side. An even more flexible version of this distribution is obtained by adding a third parameter to obtain a generalized Gamma distribution (GGD) given in Walck (2000) as

$$f\left(x;a,b,c\right)=\frac{ac\left(ax\right)^{bc-1}e^{-\left(ax\right)^{c}}}{\Gamma\left(b\right)},\ x>0,a>0,b>0 \tag{1}$$

where a (a shape parameter) and b are real positive parameters. c can in principle take any real value but normally we consider the case where c > 0 or even c ≥ 1. The GGD is the most popular model for analyzing skewed data and is suitable for modeling data with different types of hazard rate function: increasing, decreasing, in the form of bathtub and unimodal (Pascoa et al., (2011). The characteristics is useful for estimating individual hazard rate functions and both relative hazards and relative times (Cox (2008)).The GGD was introduced by Stacy (1962) and includes as special sub-models: the exponential, Weibull, gamma and Rayleigh distributions, among

others. Recently the GGD has been used in several research areas such as engineering, hydrology and survival analysis. Ortega et al., (2003) discussed influence diagnostics in GG regression models, Nadarajah and Gupta (2007) used the distribution with application to drought.

In practice the familiar application of the normal linear model involves a response variable that is assumed normally distributed with constant variance. In other applications a response variable may occur in a form that suppresses an underlying normal linear structure (Fraser (1967)). Sometimes in these applications the context may suggest a logarithm or inverse or square root transformation and so on, which reveals the normal linear form. Transformation may also be necessary to either stabilize the variance component of a model or to normalize it. Details on the reasons for transformations are found in; Box and Jenkings (1964); Iwueze et al., (2011).

Suppose the model of interest is a multiplicative error model (MEM) given as

$$X_{t,t\in N} = \Psi\left(X_{t-1}\right)\xi_t \tag{2}$$

where $X_{t,t\in N}$ is a discrete time series process defined on [0, ∞), $\Psi\left(X_{t-1}\right)$, the information available for forecasting $X_{t,t\in N}$ and ξ_t, a random variable defined over a [0, +∞) support with unit mean and unknown constant variance, σ_1^2. That is

$$\xi_t \sim V^+\left(1, \sigma_1^2\right) \tag{3}$$

In principle the distribution of ξ_t in (2) can be specified by means of any probability density function (pdf) having the characteristics in (3) of which the GGD is a major subclass. Examples are Gamma, Log-Normal, Weibull, Inverted-Gamma and mixtures of them (Brownlees et al., (2011)). Engle and Gallo (2006) favor a Gamma (ϕ, ϕ) (which implies $\sigma^2 = 1/\phi$); Bauwens and Giot (2000), in ACD framework considered a Weilbull $\Gamma\left(\left(1+\phi\right)^{-1}, \phi\right)$ (in this case, $\sigma^2 = \Gamma\left(1+2\phi\right)/\Gamma\left(\left(1+\phi\right)^2 - 1\right)$).

Sometimes in practice data transformation on Model (2) may be necessary or strongly recommended. Data transformation is a mathematical operation that changes the measurement scale of a variable. Reasons for data transformation include stabilizing variance, normalizing, reducing the effect of outliers, making a measurement scale more meaningful and to linearize relationship (Iwueze et al., (2011). The popular and common transformations are the power transformation such as $Log_e X_t$, $\sqrt{X_t}$, $1/X_t$, $1/\sqrt{X_t}$, X_t^2, and $1/X_t^2$. For further details on transformation see [Bartlett (1947); Box and Cox, (1964); Akpanta and Iwueze (2009)] .

Studies on the effects of transformation on the error component of the multiplicative time series model

whose error component is classified under the characteristics given in (2) are not new in the statistical literature. The overall aim of such studies is to establish the conditions for successful transformation. A successful transformation is achieved when the desirable properties of a data set remains unchanged after transformation. These basic properties or assumptions of interest for this study are; (i) Unit mean and (ii) constant variance. In this end, Iwueze (2007) investigated the effect of logarithmic transformation on the error component (e_t) of a multiplicative time series model where ($e_t \sim N\left(1, \sigma^2\right)$) and discovered that the logarithm transform; Y = Log e_t can be assumed to be normally distributed with mean, zero and the same variance, σ^2 for $\sigma < 0.1$. Similarly Nwosu et al., (2010) and Otuonye et al., (2011) had studied the effects of inverse and square root transformations on the error component of the same model .

Nwosu et al., (2010) discovered that the inverse transform $Y = \dfrac{1}{e_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma \leq 0.07$ Similarly Otuonye et al., (2011) discovered that the square root transform; $Y = \sqrt{e_t}$ can be assumed to be normally distributed with unit mean and variance, 4 σ^2 for $\sigma \leq 0.3$, where σ^2 is the variance of the original error component before transformation. Furthermore, Ohakwe et al.,(2012) has studied the implication of square root transformation on a two-parameter Gamma distributed error component of a MEM whose distributional characteristics is given in (3) and discovered that the unit mean assumption is approximately maintained, but the variance of the transformed distributions is one-quarter of the original variance.

The generalized Gamma distribution given in (1) is a general form for which for certain parameter combinations gives many other distributions as special cases. Some of such relations are given in Table 1(Walck, (2000)).

The application of a square root transformation to model (2) gives

$$X^*_{t,t\in N} = \Psi^*\left(X_{t-1}\right)\xi^*_t \tag{4}$$

where $X^*_{t,t\in N} = \sqrt{X_{t,t\in N}}$, $\Psi^*\left(X_{t-1}\right) = \sqrt{\Psi\left(X_{t-1}\right)}$ and $\xi^*_t = \sqrt{\xi_t}$. Model (4) is still a

multiplicative error model and therefore ξ^*_t must also be characterized with unit mean and some constant variance, σ^2_2 which may or may not be equal to σ^2_1 . It is on these premises that we want to study the effect of a square root transformation on a non-normal distributed error component of a multiplicative error model whose distributional characteristics belong to the generalized Gamma family given in (3). The purpose is to determine if the assumed fundamental structure (unit mean and constant variance) is maintained after square root transformation and also to investigate what happens to σ^2_2 and σ^2_1 in terms of equality or non-equality. The overall reason for concentrating on the error component of model (2) is as plane as the nose on the face: the reason is that the assumptions for model analysis are always placed on the error component, ξ_t .

The paper is organized into Five Sections. The introductory part of this paper in contained in Section one while the probability density function of the generalized gamma distribution under square root transformation and its moments are contained in Section two. The comparison of the means and variances of the distributions before and after square root transformation are contained in Section three while the discussions of the results is contained in Section four. Finally the conclusion, reference and appendix are respectively contained in Sections five, six and seven.

2. Distribution of the Generalized Gamma Distribution under Square Root Transformation

In this section, we would first obtain the probability density function (pdf) of the GGD under square root transformation and secondly obtain an expression for its kth moment (k = 1, 2, 3, . . .) from where the mean and variance would be obtained.

2.1 Probability Density Function of the Square Root Transformed Generalized Gamma Distribution

Given that the distribution of ξ_t given in (2) is of the generalized Gamma family, therefore the probability density function (pdf) of ξ_t denoted as $f\left(\xi_t\right)$ is given as

$$f(\xi;a,b,c)=\frac{ac(a\xi)^{bc-1}e^{-(a\xi)^c}}{\Gamma(b)},\ x>0,a>0,b>0 \tag{5}$$

where a, b and c are as defined in (1). Having applied square root transformation on model (2) and obtained model (4), we now proceed as follows to obtain the the pdf of ξ_t^* denoted as $f(\xi_t^*)$.

Since $\xi_t^*=\sqrt{\xi_t}$, it implies that

$$(\xi_t^*)^2=\xi_t\,. \tag{6}$$

In what follows, let $\xi_t=\xi$ and $\xi_t^*=\xi_*$, hence

$$\xi=\xi_*^{\,2};\frac{d\xi}{d\xi_*}=2\xi_* \tag{7}$$

but

$$f(\xi_*)=f(\xi)\left|\frac{d\xi}{d\xi_*}\right| \tag{8}$$

where $\left|\frac{d\xi}{d\xi_*}\right|=2\xi_*$ is the Jacobian of the transformation (Hogg and Craig (1978)), thus

$$f(\xi_*)=f(\xi)\left|\frac{d\xi}{d\xi_*}\right|\ =\frac{a^{bc}c}{\Gamma(b)}(\xi_*^{\,2})^{bc-1}e^{-(a\xi_*^{\,2})^c}\,2\xi_*\ =\frac{2a^{bc}c}{\Gamma(b)}(\xi_*)^{2bc-1}e^{-(a\xi_*^{\,2})^c}$$

hence

$$f(\xi_*)=\begin{cases}\frac{2a^{bc}c}{\Gamma(b)}(\xi_*)^{2bc-1}e^{-(a\xi_*^{\,2})^c} & 0<\xi_*<\infty\\ 0 & otherwise\end{cases} \tag{9}$$

2.2 Moments of the Generalized Gamma Distribution (GGD) Under Square root Transformation.

In this Section, we would obtain an expression for the kth moment, $E(\xi_*^{\,k})$ (k=1,2,3,…) of the GGD under square root transformation from where the mean and variance of the distribution would be obtained

k$^{\text{th}}$ Moment, $E(\xi_*^{\,k})$

By definition the the k$^{\text{th}}$ moment, $E(\xi_*^{\,k})$ (k=1,2,3,…) denoted as $E(\xi_*^{\,k})$ is given by www.iiste.org

$$E\left(\xi_*^{\,k}\right)=\int_0^\infty \xi_*^{\,k} f\left(\xi_*\right) d\xi_* = \frac{2a^{bc}c}{\Gamma(b)}\int_0^\infty \xi_*^{\,k}\,\xi_*^{(2bc-1)} e^{-\left(a\xi_*^2\right)^c} d\xi_* = \frac{2a^{bc}c}{\Gamma(b)}\int_0^\infty \xi_*^{(2bc+k)-1} e^{-\left(a\xi_*^2\right)^c} d\xi_* \quad (10)$$

If we let

$$\left(a\xi_*^2\right)^c = p \quad (11)$$

The following results are true

$$\xi_*^2 = a^{-1} p^{\frac{1}{c}} \quad (12)$$

$$\xi_* = a^{-\frac{1}{2}} p^{\frac{1}{2c}} \quad (13)$$

and

$$d\xi_* = \frac{a^{-\frac{1}{2}} p^{\frac{1}{2c}-1}}{2c} dp \quad (14)$$

If we substitute the results of equations (11) through (14) into (10), we obtain

$$\int_0^\infty \xi_*^{\,k} f\left(\xi_*\right) d\xi_* = \frac{2a^{bc}c}{\Gamma(b)}\int_0^\infty \left(a^{-\frac{1}{2}} p^{\frac{1}{2c}}\right)^{2bc+k-1} e^{-p} \frac{a^{-\frac{1}{2}} p^{\frac{1}{2c}-1} dp}{2c}$$

$$\frac{a^{-\frac{k}{2}}c}{\Gamma(b)}\int_0^\infty p^{\left(b+\frac{k}{2c}\right)-1} e^{-p} dp = \frac{a^{-\frac{k}{2}}}{\Gamma(b)}\Gamma\left(b+\frac{k}{2c}\right)$$

thus

$$E\left(\xi_*^{\,k}\right) = \frac{1}{a^{\frac{k}{2}}\Gamma(b)}\Gamma\left(b+\frac{k}{2c}\right) \quad (15)$$

When k =1 in (15), we obtain the mean of the GGD under square root transformation as

$$E\left(\xi_*\right) = \frac{1}{\Gamma(b)\sqrt{a}}\Gamma\left(b+\frac{1}{2c}\right) \quad (16)$$

Similarly, when k =2, we also obtain from (15) the second moment, $E\left(\xi_*^{\,2}\right)$ as

$$E\left(\xi_*^{\,2}\right) = \frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right) \tag{17}$$

hence the variance of ξ_* $\left(Var\left(\xi_*\right)\right)$ is given by

$$Var\left(\xi_*\right) = E\left(\xi_*^2\right) - \left(E\left(\xi_*\right)\right)^2 = \frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right) - \left[\frac{1}{\Gamma(b)\sqrt{a}}\Gamma\left(b + \frac{1}{2c}\right)\right]^2 \tag{18}$$

The Kth mean of GGD given in model (1) as contained in Walck (2000) is given as

$$E\left(\xi^k\right) = \frac{1}{a^k\Gamma(b)}\Gamma\left(b + \frac{k}{c}\right) \tag{19}$$

with

$$E\left(\xi\right) = \frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right) \tag{20}$$

$$E\left(\xi^2\right) = \frac{1}{a^2\Gamma(b)}\Gamma\left(b + \frac{2}{c}\right) \tag{21}$$

and

$$Var\left(\xi\right) = E\left(\xi^2\right) - \left(E\left(\xi\right)\right)^2 = \frac{1}{a^2\Gamma(b)}\Gamma\left(b + \frac{2}{c}\right) - \left[\frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right)\right]^2 \tag{22}$$

3. Comparison of the Mean and Variance of the GGD Before and After Inverse Transformation

In this Section, we would compare the means and variances of the GGD before and after square root transformation with respect to the special cases given in Table 1 with the aim of determining the conditions where the assumptions of unit mean and equality of variances, if any, are satisfied. For this purpose, we obtain the values of the means and variances of the special cases given in Table 1 before and after the square root transformation. We would use (20) and (22) for the untransformed distribution while (16) and (18) would be used for the square-root-transformed distribution. The results for the untransformed and the transformed are respectively given in Tables 2 and 3.

Furthermore, the conditions that give rise to the unit mean of the distributions under study and their corresponding variances are given in Table 4. In Table 5, the unit mean condition of the untransformed distribution are applied to the transformed distributions in order to determine its impact on the means and variances of the transformed distributions. The reason for applying the unit mean condition on the transformed distributions is due to the fact that a square root transformation on the MEM of (2) also gives a MEM of (3) whose error component, ξ_t^* is also defined on $[0, +\infty]$ support with unit mean and constant variance, σ_2^2. That is

$$\xi_t^* \sim V^+\left(1, \sigma_2^2\right) \tag{23}$$

and as a result, all the assumptions made on ξ_t are expected to be carried over to ξ_t^*. Also given in table 5 is the difference between the variances of the transformed and untransformed distributions subject to the unit mean conditions.

4. Discussion

Comparing the results of equations (17) and (20), it is obvious that the mean of the untransformed distribution,

$E\left(\xi_t\right)$ is same as the second uncorrected moment $E\left(\xi_t^{*2}\right)$ of the transformed distribution. This very fact is also evidenced by the results given in Tables 2 and 3.

On the unit mean assumption one can confidently say that the distributions under study maintained it after the square root transformation as can be seen in Table 5 and this agrees with Ohakwe et al.,(2012). While the variances of the untransformed distributions (Table 4) are at most ≤ 2.0 even as that of the Gamma (a, b, 1) depends on the shape parameter, a, those of the square-root transformed distributions (Table 5) are < 1.0 even also as that of the Gamma (a, b, 1) depends on a. This reduction in variance are in line with the suggestion of Osborne(2002) that a successful transformation should be able to reduce the variance of the distribution. However, the reduction in variance was not true for the Gamma (a, b, 1), when $a > 1$, Rayleigh and Maxwell distributions. For the Rayleigh and Maxwell distributions the variances increased by factors of 0.0531 and 0.1480 respectively as shown by the results under variance differences in Table 4. For the Gamma (a, b, 1), when $a > 1$, the difference between the variances of the untransformed and that of the transformed becomes negative, which implies an increase in variance. Furthermore, except the variance of the Gamma (a, b, 1) that depends on the shape parameter, a, it is clear from table 5 that the variances of the square-root transformed distributions are all < 0.5.

5. Conclusion

In this paper, we studied the effect of square root transformation on the error component of the multiplicative error model whose distribution belongs to the generalized gamma family. The purpose of the study is to determine the effect of the said transformation on the basic assumptions; unit mean and constant variance. The special cases considered are the three-parameter Gamma distribution error component, the Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributed error components. From the results of the study, the unit means assumption is approximately maintained for all the distributions. It was also found that there reduction in the variances of all the square root transformed distributions under study except those of the Gamma (a, b, 1), when $a > 1$, Rayleigh and Maxwell distributions that increased. Therefore we conclude that square root transformation is not appropriate for multiplicative error models with either a Gamma (a, b, 1), for $a > 1$or Rayleigh or Maxwell distributed error components. Finally square root transformations are successful for the distributions under study if the variance of the transformed data < 0.5.

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Table 1: Relation of the GGD to other Distributions

S/n	Distribution	a	b	c
1	Gamma $\left(Gamma\left(a,b,1\right)\right)$	a	b	1
2	Chi- square $\left(\chi^2\left(\frac{1}{2},\frac{n}{2},1\right)\right)$	$\frac{1}{2}$	$\frac{n}{2}$	1
3	Exponential $\left(Exp\left(\frac{1}{\alpha},1,1\right)\right)$	$\frac{1}{\alpha}$	1	1
4	Weibull $\left(Wb\left(\frac{1}{\sigma}, 1, \alpha\right)\right)$	$\frac{1}{\sigma}$	1	α
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}}, 1, 2\right)\right)$	$\frac{1}{\sigma\sqrt{2}}$	1	2
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}}, \frac{3}{2}, 2\right)\right)$	$\frac{1}{\sigma\sqrt{2}}$	$\frac{3}{2}$	2

Table 2: Mean and Variance of the special cases of the original GGD

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	Gamma $\left(Gamma(a,b,1)\right)$	$\frac{b}{a}$	$\frac{b}{a^2}$
2	Chi- square $\left(\chi^2\left(\frac{1}{2},\frac{n}{2},1\right)\right)$	n	$2n$
3	Exponential $\left(Exp\left(\frac{1}{\alpha},1,1\right)\right)$	α	α^2
4	Weibull $\left(Wb\left(\frac{1}{\sigma}, 1, \alpha\right)\right)$	$\frac{\sigma}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$	$\sigma^2\left[\frac{2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)-\left(\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)^2\right)\right]$
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}}, 1, 2\right)\right)$	$\sigma\sqrt{\frac{\pi}{2}}$	$\frac{\sigma^2}{2}(4-\pi)$
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}}, \frac{3}{2}, 2\right)\right)$	$2\sigma\sqrt{\frac{2}{\pi}}$	$\frac{\sigma^2}{\pi}(3\pi-8)$

Table 3: Mean and Variance of the special cases of the GGD under Square root Transformation

S/n	Distribution	Mean $E(\xi_*)$	$E(\xi_*^2)$	Variance $Var(\xi_*)$
1	Gamma $(Gamma(a,b,1))$	$\frac{1}{\sqrt{a}\,\Gamma(b)}\Gamma\left(b+\frac{1}{2}\right)$	$\frac{b}{a}$	$\frac{b}{a}-\frac{1}{a}\left[\frac{\Gamma\left(b+\frac{1}{2}\right)}{\Gamma(b)}\right]^2$
2	Chi- square $\left(\chi^2\left(\frac{1}{2},\frac{n}{2},1\right)\right)$	$\sqrt{2}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)}$	n	$n-2\left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2$
3	Exponential $\left(Exp\left(\frac{1}{\alpha},1,1\right)\right)$	$\frac{\sqrt{\pi\alpha}}{2}$	α	$\frac{\alpha}{4}(4-\pi)$
4	Weibull $\left(Wb\left(\frac{1}{\sigma},1,\alpha\right)\right)$	$\sqrt{\sigma}\frac{1}{2\alpha}\Gamma\left(\frac{1}{2\alpha}\right)$	$\frac{\sigma}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$	$\sigma\left[\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)-\left(\frac{1}{2\alpha}\Gamma\left(\frac{1}{2\alpha}\right)\right)^2\right]$
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}},1,2\right)\right)$	$2^{\frac{1}{4}}\sqrt{\sigma}\,\Gamma\left(\frac{5}{4}\right)$	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma\left[\left(\frac{\pi}{2}\right)-\sqrt{2}\left(\Gamma\left(\frac{1}{2\alpha}\right)\right)^2\right]$
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}},\frac{3}{2},2\right)\right)$	$2\sqrt{\frac{2\sigma}{\pi}}\Gamma\left(\frac{7}{4}\right)$	$2\sigma\sqrt{\frac{2}{\pi}}$	$2\sigma\left[\sqrt{\frac{2}{\pi}}-\frac{2}{\pi}\left(\Gamma\left(\frac{7}{4}\right)\right)^2\right]$

Table 4: Condition for Unit Mean and its implication on the Variance of the special cases of the original GGD

S/n	Distribution	Mean	Condition for Unit Mean	Variance
1	Gamma $\left(Gamma\left(a,b,1\right)\right)$	$\frac{b}{a}$	$a = b$	$\frac{1}{a}$
2	Chi- square $\left(\chi^2\left(\frac{1}{2},\frac{n}{2},1\right)\right)$	n	$n = 1$	2
3	Exponential $\left(Exp\left(\frac{1}{\alpha},1,1\right)\right)$	α	$\alpha = 1$	1
4	Weibull $\left(Wb\left(\frac{1}{\sigma},1,\alpha\right)\right)$	$\frac{\sigma}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$	$\alpha = 1 = \sigma$	1
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}},1,2\right)\right)$	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma = \sqrt{\frac{2}{\pi}}$	$\frac{1}{\pi}\left(4-\pi\right) = 0.2732$
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}},\frac{3}{2},2\right)\right)$	$2\sigma\sqrt{\frac{2}{\pi}}$	$\sigma = \frac{1}{2}\sqrt{\frac{\pi}{2}}$	$\frac{3\pi}{8} - 1 = 0.1781$

Table 5: Application of the Unit mean Condition of the Untransformed GGD and its Implication on the Mean and Variance of the Special Cases of the GGD Under Square Root Transformation

S/n	Distribution	Mean	Variance	$Var(\xi) - Var(\xi_*)$
1	Gamma $(Gamma(a,b,1))$	$\frac{1}{\sqrt{a}\,\Gamma(a)}\Gamma\left(a+\frac{1}{2}\right)$	$1-\frac{1}{a}\left[\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\right]^2$	$\frac{1}{a}\left\{1+\left[\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\right]^2\right\}-1$
2	Chi- square $\left(\chi^2\left(\frac{1}{2},\frac{n}{2},1\right)\right)$	$\frac{1}{\sqrt{2}} = 0.7071$	$1-\frac{2}{\pi} = 0.3634$	1.6366
3	Exponential $\left(Exp\left(\frac{1}{\alpha},1,1\right)\right)$	$\frac{\sqrt{\pi}}{2} = 0.8864$	$1-\frac{\pi}{4} = 0.2146$	0.7854
4	Weibull $\left(Wb\left(\frac{1}{\sigma},1,\alpha\right)\right)$	$\frac{\sqrt{\pi}}{2} = 0.8864$	$1-\frac{\pi}{4} = 0.2146$	0.7854
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}},1,2\right)\right)$	$\frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\Gamma\left(\frac{5}{4}\right) = 0.9628$	$\sqrt{\frac{\pi}{2}}-\frac{2}{\sqrt{\pi}}\left(\Gamma\left(\frac{5}{4}\right)\right)^2 = 0.3263$	-0.0531
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}},\frac{3}{2},2\right)\right)$	$\left(\frac{8}{\pi}\right)^{\frac{1}{4}}\Gamma\left(\frac{7}{4}\right) = 1.1610$	$1-\sqrt{\frac{2}{\pi}}\left(\Gamma\left(\frac{7}{4}\right)\right)^2 = 0.3261$	-0.1480