On Higher Derivations of Lattices

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Abstract

In this paper, as a generalization of derivation on a lattice, the notion of higher derivation is introduced and some fundamental properties are investigated for the higher derivation on a lattice. Furthermore it is shown that the image of an ideal and the set of fixed points under higher derivation are ideals under certain conditions.

Keywords: derivation, higher derivation, lattice

1. Introduction

Lattices play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis (Bell 2003, Carpineto & Romano 1996, Durfee 2002, Sandhu 1996). Recently the properties of lattices were widely researched (Abbott 1969, Balbes & Dwinger 1974, Bell 2003, Birkhoff 1940, Carpineto & Romano 1996, Degang et al. 2006, Durfee 2002, Honda & Grabisch 2006). The notion of lattice derivation have been introduced and developed by Szasz (1975), in which he established the main properties of derivations of lattices. To this day, many researchers studied on derivations and generalizations of derivations on a lattice and discussed some related properties (Alshehri 2010, Aşçı et al. 2011, Aşçı & Ceran 2013, Balogun 2014, Chaudhry & Allah 2011, Çeven 2009, Çeven & Öztürk 2008, Ferrari 2001, Xin et al. 2008).

In this paper, we inspired from some papers about higher derivations on rings (Nakai 1970, Nakajima 2000). As a generalization of derivation on a lattice, the notion of higher derivation of a lattice is introduced and some related properties are investigated for the higher derivation on a lattice.

2. Preliminaries

Definition 2.1 (Birkhoff 1940) Let \( L \) be a nonempty set endowed with operations “\( \wedge \)” and “\( \vee \)”. If \((L, \wedge, \vee)\) satisfies the following conditions: for all \( x, y, z \in L \)

(A) \( x \wedge x = x, x \vee x = x \),

(B) \( x \wedge y = y \wedge x, x \vee y = y \vee x \),

(C) \( (x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z) \)

(D) \( (x \wedge y) \vee x = x, (x \vee y) \wedge x = x \)

then \( L \) is called a lattice.

Definition 2.2 (Birkhoff 1940) A lattice \( L \) is distributive if the identity (E) or (F) holds

(E) \( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \),

(F) \( x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \).

In any lattice, the conditions (E) and (F) are equivalent.

Definition 2.3 (Birkhoff 1940) Let \((L, \wedge, \vee)\) be a lattice. A binary relation “\( \leq \)” is defined by \( x \leq y \) if and only if \( x \wedge y = x \) and \( x \vee y = y \).

Lemma 2.4 Let \((L, \leq)\) be a lattice. Define the binary relation “\( \leq \)” as Definition 2.3. Then
\( (L, \leq) \) is a poset and for any \( x, y \in L \), \( x \wedge y \) is the g.l.b. of \( x, y \) and \( x \vee y \) is the l.u.b. of \( x, y \).
From Lemma 2.4, we can see that a lattice is not only an algebraical system but also an order structure.

**Definition 2.5** (Szasz 1975) Let $L$ be a lattice and $d : L \to L$ be a function. We call $d$ a derivation on $L$, if it satisfies the following condition

$$d(x \wedge y) = (x \wedge d(y)) \vee (d(x) \wedge y).$$

(2.1)

**Proposition 2.6** (Xin et al. 2008) Let $L$ be a lattice and $d$ be a derivation on $L$. Then the following hold:

1. $d(x) \leq x$,
2. $d(x) \wedge d(y) \leq d(x \wedge y) \leq d(x) \wedge d(y)$,
3. If $I$ is an ideal of $L$, then $d(I) \subseteq I$,
4. If $L$ has a least element $0$ and a greatest element $1$, then $d(0) = 0$ and $d(1) \leq 1$.

**Definition 2.7** (Bikhoof 1940) An ideal is a non-void subset $A$ of a lattice $L$ with the properties

1. $x \leq y$ and $y \in A \Rightarrow x \in A$,
2. $x, y \in I \Rightarrow x \vee y \in I$.

3. **Higher Derivations in Lattices**

Let $L$ be a lattice, $I=\{0,1,2,…,t\}$ or $I=\mathbb{N}=\{0,1,2,…\}$ (with $t \to \infty$ in this case) and $D = \{d_n\}_{n \in I}$ be a family of mappings of $L$ such that $d_0 = id_L$.

The following definition introduces the notion of higher derivation for a lattice.

**Definition 3.1** $D$ is said to be a higher derivation of length $t$ on $L$ if for every $n \in I$ and $a, b \in L$ we have

$$d_n(a \wedge b) = \bigvee_{i=0}^{n} (d_i(a) \wedge d_i(b)).$$

(3.1)

Let $D$ is a higher derivation of length $t$ on $L$. Then we have $d_0(a \wedge b) = a \wedge b$ for all $a, b \in L$ since $d_0 = id_L$. Furthermore, $d_1$ is a derivation on $L$ since

$$d_1(a \wedge b) = (d_0(a) \wedge d_1(b)) \vee (d_1(a) \wedge d_0(b))$$

$$= (a \wedge d_1(b)) \vee (d_1(a) \wedge b)$$

and $d_2$ is a mapping such that $d_2(a \wedge b) = (a \wedge d_2(b)) \vee (d_1(a) \wedge d_1(b)) \vee (d_2(a) \wedge b)$ for all $a, b \in L$.

Generally, by the Eq.(3.1), we can write

$$d_n(a \wedge b) = (d_0(a) \wedge d_n(b)) \vee (d_1(a) \wedge d_{n-1}(b)) \vee \ldots \vee (d_n(a) \wedge d_0(b)).$$

(3.2)

**Example 3.1** Let $L=\{0,a,b,1\}$ be the lattice with the following figure:

```
  0
   |
   a
   |
   b
   |
  1
```

Let $d_0 = id_L$. We define the functions $d_1, d_2, d_3$ on $L$ as the following
Then it is easily seen that the the family \( D = \{d_0, d_1, d_2, d_3\} \) is a higher derivation of length 3 on \( L \).

**Proposition 3.2** Let \( L \) be a lattice with least element 0 and \( D = \{d_n\}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). Then \( d_n(0) = 0 \) for all \( n \in I \).

**Proof.** We have \( d_n(0) = 0 \) since \( d_0 = id_L \) and \( d_1(0) = 0 \) since \( d_1 \) is a derivation on \( L \). Then we get

\[
d_2(0) = d_2(0 \wedge 0)
= (0 \wedge d_2(0)) \lor (d_1(0) \wedge d_1(0)) \lor (d_2(0) \wedge 0)
= 0 \lor 0 \lor 0
= 0.
\]

Now we assume that \( d_n(0) = 0 \) for \( n=3,4,...,t-1 \). Then we have

\[
d_t(0) = d_t(0 \wedge 0)
= (d_0(0) \wedge d_t(0)) \lor (d_1(0) \wedge d_{t-1}(0)) \lor (d_t(0) \wedge d_0(0))
= 0 \lor ... \lor 0
= 0.
\]

Hence it is obtained that \( d_n(0) = 0 \) for all \( n \in I \).

**Proposition 3.3** Let \( L \) be a lattice and \( D = \{d_n\}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). Then \( d_t \leq d_n \) for all \( n \in I \).

**Proof.** Let \( a \) be any element of \( L \). Then \( d_1(a) \leq a = d_0(a) \) is clear by Proposition 2.6 (i) since \( d_1 \) is a derivation of \( L \). Since

\[
d_2(a) = d_2(a \wedge a)
= (d_0(a) \wedge d_2(a)) \lor (d_1(a) \wedge d_1(a)) \lor (d_2(a) \wedge d_0(a))
= (a \wedge d_2(a)) \lor d_1(a)
\]
we have \( d_1(a) \leq d_2(a) \). Similarly since

\[
d_3(a) = d_3(a \wedge a)
= (d_0(a) \wedge d_3(a)) \lor (d_1(a) \wedge d_2(a)) \lor (d_2(a) \wedge d_1(a)) \lor (d_3(a) \wedge d_0(a))
= (a \wedge d_3(a)) \lor (d_1(a) \wedge d_2(a))
= (a \wedge d_3(a)) \lor d_1(a)
\]
we have \( d_1 \ a \leq d_3 \ a \). Now we assume that \( d_1 \ a \leq d_n \ a \) for \( n=4,...,t-1 \). From the equality

\[
d_t(a) = d_t(a \wedge a)
= (a \wedge d_t(a)) \lor (d_1(a) \wedge d_{t-1}(a)) \lor ... \lor (d_t(a) \wedge a)
\]
we see that if \( t \) is an even number,

\[
d_t(a) = (a \wedge d_t(a)) \lor (d_1(a) \wedge d_{t-3}(a)) \lor ... \lor d_{t/2}(a) \quad (3.3)
\]
and if \( t \) is an odd number

\[
d_t(a) = (a \wedge d_t(a)) \lor (d_1(a) \wedge d_{t-3}(a)) \lor ... \lor (d_{t/2}(a) \wedge d_{t/2}(a)). \quad (3.4)
\]

In every case, we have \( d_t(a) \leq d_n(a) \). So we obtain \( d_t(a) \leq d_n(a) \) for all \( a \in L \) and \( n \in I \).

**Corollary 3.4** Let \( L \) be a lattice and \( D = \{d_n\}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). Then
(i) \( d_n(a) \land d_j(b) \leq d_n(a \land b), \) (i + j = n)
(ii) if \( n \) is even number, \( d_{n/2}(a) \leq d_n(a), \)
(iii) \( d_i(a) \land d_j(a) \leq d_n(a), \) (i + j = n)
for all \( n \in I \) and \( a, b \in L. \)

**Proof.** It is clear from the Eq. (3.2), Eq. (3.3) and (3.4).

**Proposition 3.5** Let \( L \) be a lattice and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L. \) Then

(i) \( d_n(a) \leq a, \)
(ii) \( d_n(a) \land d_n(b) \leq d_n(a \land b) \)
for all \( n \in I \) and \( a, b \in L. \)

**Proof.** (i) It is clear that \( d_0(a) \leq a \) and \( d_1(a) \leq a \) by Proposition 3.3, we have \( d_1(a) \leq a \) and \( d_2(a) \leq a. \) Then \( d_2(a) = a \land d_1(a) \lor d_1(a) = a \land d_2(a). \) Hence \( d_2(a) \leq a. \) Similarly we get \( d_3(a) \leq a. \) Now we assume that \( d_n(a) \leq a \) for \( n = 1, \ldots, t. \) Then, using Corollary 3.4 and Eq. (3.2), we obtain \( d_n(a) \leq a. \) Therefore \( d_n(a) \leq a \) for all \( a \in L \) and \( n \in I. \)

(ii) Using Corollary 3.4 (i), we have \( d_0(a) \land d_n(b) = d_n(a \land b). \) That is \( a \land d_n(b) \leq d_n(a \land b). \) By (i), since \( d_n(a) \leq a, \) we get \( d_n(a) \land d_n(b) \leq d_n(a \land b). \) Hence we have \( d_n(a) \land d_n(b) \leq d_n(a \land b) \) for all \( a, b \in L \) and \( n \in I. \)

**Proposition 3.6** Let \( L \) be a lattice and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L. \)

(i) \( d_n \) for all \( n \in I \) is an increasing mapping if and only if for all \( n \in I \) and \( a, b \in L \)

(ii) If \( L \) is a distributive lattice then \( d_n \) for all \( n \in I \) is an increasing mapping if and only if \( a \land b = d_n(a) \land d_n(b) \) for all \( n \in I \) and \( a, b \in L. \)

**Proof.** (i) Since \( a \land b \leq a \land b \) and \( d_n \) for all \( n \in I \) is an increasing mapping, we have \( d_n(a \land b) \leq d_n(a) \land d_n(b) \) for all \( n \in I \) and \( a, b \in L. \)

(ii) Since \( a \land b \leq a \land b \) and \( d_n \) for all \( n \in I \) is an increasing mapping, we have \( d_n(a \land b) \leq d_n(a) \land d_n(b) \) for all \( n \in I \) and \( a, b \in L. \)

**Theorem 3.7** Let \( L \) be a lattice, \( A \) be an ideal of \( L \) and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L. \) Then \( d_n(A) \subseteq A \) for all \( n \in I. \)

**Proof.** We know that \( d_n(a) \leq a \) for all \( a \in A \) and \( n \in I. \) Then, \( d_n(a) \leq a \) and \( a \in A \) implies \( d_n(A) \subseteq A \) for all \( n \in I. \)

**Theorem 3.8** Let \( L \) be a lattice with the greatest element \( 1 \) and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L. \) Then \( d_n(1) = 1 \) for all \( n \in I \) if and only if \( d_n(a) = a \) for all \( n \in I \) and \( a \in L. \)

**Proof.** Let \( d_n(1) = 1 \) for all \( n \in I. \) Then we have, for all \( n \in I \) and \( a \in L. \)
The converse part is clear.

**Theorem 3.9** If a lattice \( L \) has a least element \( 0 \) and a greatest element \( 1 \) and \( D = \{ d_n \}_{n \in I} \) is a higher derivation of length \( t \) on \( L \) then \( d_n 1 = 0 \) for all \( n \in I \) if and only if \( d_n a = 0 \) for all \( n \in I \) and \( a \in L \).

**Proof.** Let \( d_n 1 = 0 \) for all \( n \in I \). Then we have, for all \( n \in I \) and \( a \in L \),
\[
d_n(a) = d_n(a \land 1)
\]
\[
= d_0 a \land d_n 1 \lor d_1 a \land d_{n-1} 1 \lor \cdots \lor d_n a \land d_0 1
\]
\[
= a \lor d_1 a \lor \cdots \lor d_n a
\]
\[
= a.
\]

The converse part is clear.

**Theorem 3.10** Let \( L \) be a lattice and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). Then \( d_n^2 = d_n \) for all \( n \in I \).

**Proof.** Using Proposition 3.5. (i), we have
\[
d_n^2 a = d_n d_n a \leq d_n a \leq a.
\]
Also since
\[
d_i a \land d_j a \leq d_i a \land d_n a \leq d_n a
\]
where \( i+j=n \), we get
\[
d_n^2 a = d_n d_n a
\]
\[
= d_n a \land d_n a
\]
\[
= a \land d_n^2 a \lor d_1 a \land d_{n-1} a \lor \cdots \lor d_n a
\]
\[
= d_n(a).
\]

**Theorem 3.11** Let \( L \) be a lattice with the greatest element \( 1 \) and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( n \) on \( L \). If \( a \leq d_n 1 \) then \( d_n a = a \) for all \( n \in I \) and \( a \in L \).

**Proof.** Since \( d_1 a \leq a \leq d_j 1 \) where \( i+j=n \) for \( a \in L \), we obtain
\[
d_n a = d_n a \land 1
\]
\[
= a \land d_n(1) \lor d_1(a) \land d_{n-1}(1) \lor \cdots \lor (d_n a) \land 1
\]
\[
= a \lor d_1(a) \lor \cdots \lor d_n a
\]
\[
= a.
\]

Let \( L \) be a lattice and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). Let \( \text{Fix}_D \ L = x \in L : d_n x = x \) for all \( n \in I \). If \( L \) has a least element 0, then \( 0 \in \text{Fix}_D(L) \) by Proposition 3.1. Hence \( \text{Fix}_D \ L \neq \emptyset \).

**Theorem 3.12** Let \( L \) be a distributive lattice and \( D = \{ d_n \}_{n \in I} \) be a higher derivation of length \( t \) on \( L \). If \( d_n \) is an increasing function for all \( n \in I \), then \( \text{Fix}_D(L) \) is an ideal of \( L \).

**Proof.** Let \( x \in \text{Fix}_D(L) \) and \( y \leq x \). Then we have
\[ d_n y = d \ y \land x \]
\[ = d_0 y \land d_n x \lor d_1 y \land d_{n-1} x \lor \ldots \lor d_n y \land d_0 x \]
\[ = y \land x \lor d_1 y \land x \lor \ldots \lor d_n y \land x \]
\[ = y \lor d_1 y \lor \ldots \lor d_n y \lor x \]
\[ = y \lor x \]
\[ = y. \]

Hence we get \( y \in \text{Fix}_D(L) \). Now let \( x, y \in \text{Fix}_D(L) \). By Proposition 3.5, we have \( d_n x \lor y \leq x \lor y \). Since \( x \leq x \lor y, y \leq x \lor y \) and \( d_n \) is an increasing function for all \( n \in I \), we obtain \( d_n x \leq d_n (x \lor y) \) and \( d_n y \leq d_n (x \lor y) \). So we have \( x \lor y = d_n x \lor d_n y \leq d_n x \lor y \). Consequently we have \( d_n x \lor y = x \lor y \). Hence \( x \lor y \in \text{Fix}_D(L) \).

4. Conclusion

In this paper, some new definitions and results about higher derivations in a lattice are presented. The results are important in the study of the derivations of the lattice theory. The results also extend the derivation theory in a lattice.

References

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