On Higher Derivations of Lattices

Yılmaz Çeven

Süleyman Demirel University, Faculty of Sciences, Department of Mathematics, Isparta/TURKEY

E-mail: yilmazceven@sdu.edu.tr

Abstract

In this paper, as a generalization of derivation on a lattice, the notion of higher derivation is introduced and some fundamental properties are investigated for the higher derivation on a lattice. Furthermore it is shown that the image of an ideal and the set of fixed points under higher derivation are ideals under certain conditions.

Keywords: derivation, higher derivation, lattice

1. Introduction

Lattices play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis (Bell 2003, Carpineto &Romano 1996, Durfee 2002, Sandhu 1996). Recently the properties of lattices were widely researched (Abbott 1969, Balbes &Dwinger 1974, Bell 2003, Birkhoof 1940, Carpineto &Romano 1996, Degang *et al.* 2006, Durfee 2002, Honda &Grabisch 2006). The notion of lattice derivation have been introduced and developed by Szasz (1975), in which he established the main properties of derivations of lattices. To this day, many researchers studied on derivations and generalizations of derivations on a lattice and discussed some related properties (Alshehri 2010, Aşçı *et al.* 2011, Aşçı&Ceran 2013, Balogun 2014, Chaudhry&ullah 2011, Çeven 2009, Çeven&Öztürk 2008, Ferrari 2001, Xin *at al*. 2008).

In this paper, we inspired from some papers about higher derivations on rings (Nakai 1970, Nakajima 2000). As a generalization of derivation on a lattice, the notion of higher derivation of a lattice is introduced and some related properties are investigated for the higher derivation on a lattice.

2. Preliminaries

Definition 2.1 (Bikhoof 1940) Let L be a nonempty set endowed with operations "$\wedge$" and "$\vee$". If $(L,\wedge,\vee)$ satisfies the following conditions: for all $x,y,z\in L$

(A) $x\wedge x=x, x\vee x=x$,

(B) $x\wedge y=y\wedge x, x\vee y=y\vee x$,

(C) $(x\wedge y)\wedge z=x\wedge(y\wedge z), (x\vee y)\vee z=x\vee(y\vee z)$

(D) $(x\wedge y)\vee x=x, (x\vee y)\wedge x=x$

then L is called a lattice.

Definition 2.2 (Bikhoof 1940) A lattice L is distributive if the identity (E) or (F) holds

(E) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$,

(F) $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$.

In any lattice, the conditions (E) and (F) are equivalent.

Definition 2.3 (Bikhoof 1940) Let $(L,\wedge,\vee)$ be a lattice. A binary relation "$\le$" is defined by $x\le y$ if and only if $x\wedge y=x$ and $x\vee y=y$.

Lemma 2.4 Let $(L,\wedge,\vee)$ be a lattice. Define the binary relation "$\le$" as Definition 2.3. Then $(L, \le)$ is a poset and for any $x,y\in L, x\wedge y$ is the g.l.b. of x,y and $x\vee y$ is the l.u.b. of x,y.

From Lemma 2.4, we can see that a lattice is not only a algebrical system but also an order structure.

Definition 2.5 (Szasz 1975) Let L be a lattice and $d: L \to L$ be a function. We call d a derivation on L, if it satisfies the following condition

$$d(x \wedge y) = (x \wedge d(y)) \vee (d(x) \wedge y). \tag{2.1}$$

Proposition 2.6 (Xin *at al.* 2008) Let L be a lattice and d be a derivation on L. Then the following hold:

(1) $d(x) \leq x$,

(2) $d(x) \wedge d(y) \leq d(x \wedge y) \leq d(x) \wedge d(y)$,

(3) If I is a ideal of L, then $d(I) \subseteq I$,

(4) If L has a least element 0 and a greatest element 1, then $d(0) = 0$ and $d(1) \leq 1$.

Definition 2.7 (Bikhoof 1940) An ideal is a non-void subset A of a lattice L with the properties

(1) $x \leq y$ and $y \in A \Rightarrow x \in A$,

(2) $x, y \in I \Rightarrow x \vee y \in I$.

3. Higher Derivations in Lattices

Let L be a lattice, I={0,1,2,…,t} or $I = \mathbb{N} = \{0,1,2,\ldots\}$ (with $t \to \infty$ in this case) and $D = \{d_n\}_{n \in I}$ be a family of mappings of L such that $d_0 = id_L$.

The following definition introduces the notion of higher derivation for a lattice.

Definition 3.1 D is said to be a higher derivation of length t on L if for every $n \in I$ and $a, b \in L$ we have

$$d_n(a \wedge b) = \bigvee_{i+j=n} (d_i(a) \wedge d_j(b)) \tag{3.1}$$

Let D is a higher derivation of length t on L. Then we have $d_0(a \wedge b) = a \wedge b$ for all $a, b \in L$ since $d_0 = id_L$. Furthermore, d_1 is a derivation on L since

$$\begin{aligned} d_1(a \wedge b) &= (d_0(a) \wedge d_1(b)) \vee (d_1(a) \wedge d_0(b)) \\ &= (a \wedge d_1(b)) \vee (d_1(a) \wedge b) \end{aligned}$$

and d_2 is a mapping such that $d_2(a \wedge b) = (a \wedge d_2(b)) \vee (d_1(a) \wedge d_1(b)) \vee (d_2(a) \wedge b)$ for all $a, b \in L$. Generally, by the Eq.(3.1), we can write

$$d_n(a \wedge b) = (d_0(a) \wedge d_n(b)) \vee (d_1(a) \wedge d_{n-1}(b)) \vee \ldots \vee (d_n(a) \wedge d_0(b)) \tag{3.2}$$

Example 3.1 Let L={0,a,b,1} be the lattice with the following figure:

Let $d_0 = id_L$. We define the functions d_1, d_2, d_3 on L as the following

$$d_1(x)=\begin{cases}0, & \text{if } x=0,1\\ b, & \text{if } x=a,b\end{cases},\quad d_2(x)=\begin{cases}0, & \text{if } x=0\\ b, & \text{if } x=a,b,1\end{cases},\quad d_3(x)=\begin{cases}0, & \text{if } x=0\\ b, & \text{if } x=b\\ a, & \text{if } x=a,1\end{cases}$$

Then it is easily seen that the the family $D = \{d_0, d_1, d_2, d_3\}$ is a higher derivation of lenght 3 on L.

Proposition 3.2 Let L be a lattice with least element 0 and $D=\{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then $d_n(0)=0$ for all $n\in I$.

Proof. We have $d_n(0)=0$ since $d_0=id_L$ and $d_1(0)=0$ since d_1 is a derivation on L. Then we get

$$\begin{aligned} d_2(0) &= d_2(0\wedge 0)\\ &= (0\wedge d_2(0))\vee(d_1(0)\wedge d_1(0))\vee(d_2(0)\wedge 0)\\ &= 0\vee 0\vee 0\\ &= 0.\end{aligned}$$

Now we assume that $d_n(0)=0$ for n=3,4,...,t-1. Then we have

$$\begin{aligned} d_t(0) &= d_t(0\wedge 0)\\ &= (d_0(0)\wedge d_t(0))\vee(d_1(0)\wedge d_{t-1}(0))\vee\ldots\vee(d_t(0)\wedge d_0(0))\\ &= 0\vee\ldots\vee 0\\ &= 0\end{aligned}$$

Hence it is obtained that $d_n(0)=0$ for all $n\in I$.

Proposition 3.3 Let L be a lattice and $D=\{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then $d_1\le d_n$ for all $n\in I$.

Proof. Let a be any element of L. Then $d_1(a)\le a=d_0(a)$ is clear by Proposition 2.6 (i) since d_1 is a derivation of L. Since

$$\begin{aligned} d_2(a) &= d_2(a\wedge a)\\ &= (d_0(a)\wedge d_2(a))\vee(d_1(a)\wedge d_1(a))\vee(d_2(a)\wedge d_0(a))\\ &= (a\wedge d_2(a))\vee d_1(a)\end{aligned}$$

we have $d_1(a)\le d_2(a)$. Similarly since

$$\begin{aligned} d_3(a) &= d_3(a\wedge a)\\ &= (d_0(a)\wedge d_3(a))\vee(d_1(a)\wedge d_2(a))\vee(d_2(a)\wedge d_1(a))\vee(d_3(a)\wedge d_0(a))\\ &= (a\wedge d_3(a))\vee(d_1(a)\wedge d_2(a))\\ &= (a\wedge d_3(a))\vee d_1(a)\end{aligned}$$

we have $d_1(a)\le d_3(a)$. Now we assume that $d_1(a)\le d_n(a)$ for n=4,...,t-1. From the equality

$$\begin{aligned} d_t(a) &= d_t(a\wedge a)\\ &= (a\wedge d_t(a))\vee(d_1(a)\wedge d_{t-1}(a))\vee\ldots\vee(d_t(a)\wedge a)\end{aligned}$$

we see that if t is an even number,

$$d_t(a)=(a\wedge d_t(a))\vee d_1(a)\vee(d_3(a)\wedge d_{t-3}(a))\vee\ldots\vee d_{t/2}(a) \tag{3.3}$$

and if t is an odd number

$$d_t(a)=(a\wedge d_t(a))\vee d_1(a)\vee(d_3(a)\wedge d_{t-3}(a))\vee\ldots\vee(d_{\frac{t-1}{2}}(a)\wedge d_{\frac{t+1}{2}}(a)). \tag{3.4}$$

In every case, we have $d_1(a)\le d_t(a)$. So we obtain $d_1(a)\le d_n(a)$ for all $a\in L$ and $n\in I$.

Corollary 3.4 Let L be a lattice and $D=\{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then

(i) $d_i(a) \wedge d_j(b) \leq d_n(a \wedge b)$, $(i + j = n)$

(ii) if n is even number, $d_{n/2}(a) \leq d_n(a)$,

(iii) $d_i(a) \wedge d_j(a) \leq d_n(a)$, $(i + j = n)$

for all $n \in I$ and $a, b \in L$.

Proof. It is clear from the Eq. (3.2), Eq. (3.3) and (3.4).

Proposition 3.5 Let L be a lattice and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then

(i) $d_n(a) \leq a$,

(ii) $d_n(a) \wedge d_n(b) \leq d_n(a \wedge b)$

for all $n \in I$ and $a, b \in L$.

Proof. (i) It is clear that $d_0(a) \leq a$ and $d_1(a) \leq a$. Since $d_1(a) \leq a$ and $d_1(a) \leq d_2(a)$ by Proposition 3.3, we have $d_1(a) \leq a \wedge d_2(a)$. Then $d_2(a) = (a \wedge d_2(a)) \vee d_1(a) = a \wedge d_2(a)$. Hence $d_2(a) \leq a$. Similarly we get $d_3(a) \leq a$. Now we assume that $d_n(a) \leq a$ for n=4,…,t-1. Then, using Corollary 3.4 and Eq.(3.2), we obtain $d_t(a) = a \wedge d_t(a)$. Hence $d_t(a) \leq a$. Therefore $d_n(a) \leq a$ for $a \in L$ and $n \in I$.

(ii) Using Corollary 3.4 (i), we have $d_0(a) \wedge d_n(b) \leq d_n(a \wedge b)$. That is $a \wedge d_n(b) \leq d_n(a \wedge b)$. By (i), since $d_n(a) \leq a$, we get $d_n(a) \wedge d_n(b) \leq a \wedge d_n(b)$. Hence we have $d_n(a) \wedge d_n(b) \leq d_n(a \wedge b)$ for all $a, b \in L$ and $n \in I$.

Proposition 3.6 Let L be a lattice and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L.

(i) d_n for all $n \in I$ is an increasing mapping if and only if for all $d_n(a \wedge b) = d_n(a) \wedge d_n(b)$ for all $n \in I$ and $a, b \in L$.

(ii) If L is a distributive lattice then d_n for all $n \in I$ is an increasing mapping if and only if $d_n(a \vee b) = d_n(a) \vee d_n(b)$ for all $n \in I$ and $a, b \in L$.

Proof. (i) Since $a \wedge b \leq a, a \wedge b \leq b$ and d_n for all $n \in I$ is an increasing mapping, we have $d_n(a \wedge b) \leq d_n(a), d_n(a \wedge b) \leq d_n(b)$. Hence we get $d_n(a \wedge b) \leq d_n(a) \wedge d_n(b)$. Together with Proposition 3.5, we obtain $d_n(a \wedge b) = d_n(a) \wedge d_n(b)$. Conversely, let $a \leq b$ and $d_n(a \wedge b) = d_n(a) \wedge d_n(b)$ for all $a, b \in L$ and $n \in I$. Then since $d_n(a) = d_n(a \wedge b) = d_n(a) \wedge d_n(b)$, we have $d_n(a) \leq d_n(b)$.

(ii) Since $a \leq a \vee b, b \leq a \vee b$ and d_n for all $n \in I$ is an increasing mapping, we have $d_n(a) \leq d_n(a \vee b)$ and $d_n(b) \leq d_n(a \vee b)$. Hence we get $d_n(a) \vee d_n(b) \leq d_n(a \vee b)$. Furthermore, by Corollary 3.4(i), since $a \wedge d_n(a \vee b) = d_0(a) \wedge d_n(a \vee b) \leq d_n(a \wedge (a \vee b)) = d_n(a)$ and similarly since $b \wedge d_n(a \vee b) \leq d_n(b)$, we have $(a \wedge d_n(a \vee b)) \vee (b \wedge d_n(a \vee b)) \leq d_n(a) \vee d_n(b)$. Since L is distributive, we get $(a \vee b) \wedge d_n(a \vee b) \leq d_n(a) \vee d_n(b)$. By Proposition 3.5 (i), we obtain $d_n(a \vee b)) \leq d_n(a) \vee d_n(b)$. Hence it is obtained that $d_n(a \vee b) = d_n(a) \vee d_n(b)$.

Conversely, let $a \leq b$ and $d_n(a \vee b) = d_n(a) \vee d_n(b)$. Then since $d_n(b) = d_n(a \vee b) = d_n(a) \vee d_n(b)$, we have $d_n(a) \leq d_n(b)$.

Theorem 3.7 Let L be a lattice, A be an ideal of L and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then $d_n(A) \subseteq A$ for all $n \in I$.

Proof. we know that $d_n(a) \leq a$ for all $a \in A$ and $n \in I$. Then, $d_n(a) \leq a$ and $a \in A$ implies $d_n(a) \in A$. Hence $d_n(A) \subseteq A$ for all $n \in I$.

Theorem 3.8 Let L be a lattice with the greatest element 1 and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then $d_n(1) = 1$ for all $n \in I$ if and only if $d_n(a) = a$ for all $n \in I$ and $a \in L$.

Proof. Let $d_n(1) = 1$ for all $n \in I$. Then we have, for all $n \in I$ and $a \in L$

$$d_n(a) = d_n(a \wedge 1)$$
$$= (d_0(a) \wedge d_n(1)) \vee (d_1(a) \wedge d_{n-1}(1)) \vee \ldots \vee (d_n(a) \wedge d_0(1))$$
$$= a \vee d_1(a) \vee \ldots \vee d_n(a)$$
$$= a.$$

The converse part is clear.

Theorem 3.9 If a lattice L has a least element 0 and a greatest element 1 and $D = \{d_n\}_{n\in I}$ is a higher derivation of lenght t on L then $d_n(1) = 0$ for all $n \in I$ if and only if $d_n(a) = 0$ for all $n \in I$ and $a \in L$.

Proof. Let $d_n(1) = 0$ for all $n \in I$. Then we have, for all $n \in I$ and $a \in L$,

$$d_n(a) = d_n(a \wedge 1)$$
$$= (d_0(a) \wedge d_n(1)) \vee (d_1(a) \wedge d_{n-1}(1)) \vee \ldots \vee (d_n(a) \wedge d_0(1))$$
$$= 0 \vee 0 \vee \ldots \vee 0$$
$$= 0.$$

The converse part is clear.

Theorem 3.10 Let L be a lattice and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Then $d_n^2 = d_n$ for all $n \in I$.

Proof. Using Proposition 3.5. (i), we have $d_n^2(a) = d_n(d_n(a)) \le d_n(a) \le a$. Also since $d_i(a) \wedge d_j(d_n(a)) \le d_i(a) \wedge d_n(a) \le d_n(a)$ where i+j=n, we get

$$d_n^2(a) = d_n(d_n(a))$$
$$= d_n(a \wedge d_n(a))$$
$$= (a \wedge d_n^2(a)) \vee (d_1(a) \wedge d_{n-1}(d_n(a))) \vee \ldots \vee d_n(a)$$
$$= d_n(a).$$

Theorem 3.11 Let L be a lattice with the greatest element 1 and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght n on L. If $a \le d_n(1)$ then $d_n(a) = a$ for all $n \in I$ and $a \in L$.

Proof. Since $d_i(a) \le a \le d_j(1)$ where i+j=n for $a \in L$, we obtain

$$d_n(a) = d_n(a \wedge 1)$$
$$= (a \wedge d_n(1)) \vee (d_1(a) \wedge d_{n-1}(1)) \vee \ldots \vee (d_n(a) \wedge 1)$$
$$= a \vee d_1(a) \vee d_2(a) \vee \ldots \vee d_n(a)$$
$$= a.$$

Let L be a lattice and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. Let $Fix_D(L) = \{x \in L : d_n(x) = x \text{ for all } n \in I\}$. If L has a least element 0, then $0 \in Fix_D(L)$ by Proposition 3.1. Hence $Fix_D(L) \neq \varnothing$.

Theorem 3.12 Let L be a distributive lattice and $D = \{d_n\}_{n\in I}$ be a higher derivation of lenght t on L. If d_n is an increasing function for all $n \in I$, then $Fix_D(L)$ is an ideal of L.

Proof. Let $x \in Fix_D(L)$ and $y \le x$. Then we have

$$
\begin{aligned}
d_n(y) &= d(y \wedge x) \\
&= (d_0(y) \wedge d_n(x)) \vee (d_1(y) \wedge d_{n-1}(x)) \vee \ldots \vee (d_n(y) \wedge d_0(x)) \\
&= (y \wedge x) \vee (d_1(y) \wedge x) \vee \ldots \vee (d_n(y) \wedge x) \\
&= (y \vee d_1(y) \vee \ldots \vee d_n(y)) \wedge x \\
&= y \wedge x \\
&= y.
\end{aligned}
$$

Hence we get $y \in Fix_D(L)$. Now let $x, y \in Fix_D(L)$. By Proposition 3.5, we have $d_n(x \vee y) \leq x \vee y$. Since $x \leq x \vee y, y \leq x \vee y$ and d_n is an increasing function for all $n \in I$, we obtain $d_n(x) \leq d_n(x \vee y)$ and $d_n(y) \leq d_n(x \vee y)$. So we have $x \vee y = d_n(x) \vee d_n(y) \leq d_n(x \vee y)$. Consequently we have $d_n(x \vee y) = x \vee y$. Hence $x \vee y \in Fix_D(L)$.

4. Conclusion

In this paper, some new definitions and results about higher derivations in a lattice are presented. The results are important in the study of the derivations of the lattice theory. The results also extend the derivation theory in a lattice.

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