Transmuted Inverse Loglogistic Model: Properties and Application in Medical Sciences and Engineering

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Abstract: In this article, we introduced a two parameter Transmuted model of Inverse Log-logistic Distribution (TILLD) using the quadratic rank transmutation map technique studied by Shaw and Buckley [1]. We provide a comprehensive description of the statistical properties of the TILLD. Robust measures of skewness and kurtosis of the proposed model have also been derived along with the moment generating function, characteristic function, reliability function and hazard rate function of the said model. The estimation of the model parameters is performed by maximum likelihood method followed by a Monte Carlo simulation procedure. The applicability of this distribution to modeling real life data is illustrated by two real life examples and the results of comparison to base distribution in modeling the data are also exhibited.

Key Words: Transmuted Inverse Log-logistic Distribution (TILLD), Survival Analysis, Robust measures, Monte Carlo Simulation.

1.Introduction

Statistical distributions are widely applied to describe real world phenomena. Sometimes typical and complicated situations arise in the field of Statistical analysis, as a result of which the already existing models does not fit much accurately to the complex data arising in such situations. For the purpose of dealing with the complex data, we sometimes require a different type of model for its fitting. These models already do not have existence in the statistical literature. So in order to surmount such requirements, we approach to develop some new models. These newly introduced classes of models provide greater flexibility in modeling complex data and the results drawn from them seems quite sound and genuine. Thus our main concern becomes, to give importance especially to model specification and the data interpretation. In the current article, we have used the transmutation technique for the construction of Transmuted Inverse Log-logistic Distribution with the help of Quadratic Rank Transmutation Map (QRTM) technique given by Shaw and Buckley [1]. Recently, a lot of research has been done in the field of transmutation. Ashour and Eltehiwy [2,16] introduced the transmuted model of the exponentiated modified Weibull and exponentiated Lomax distributions as a new generalized distributions. Aryal and Tsokos [3] developed the transmuted Extreme valve Distribution. Hussain [4] studied the transmuted Exponentiated Gamma Distribution. Merovci [5] proposed the transmuted Lindley Distribution. Now we are going to study the transmuted Inverse Loglogistic Distribution as a new lifetime model using the quadratic rank transmutation map technique studied by Shaw and Buckley [1], as there is a need to find more plausible probability models or survival models in medical sciences and other fields, to fit to various lifetime data sets. It is well known in general that a transmuted model is more flexible than the ordinary model and it is preferred by many data analysts in analyzing statistical data. Moreover, it presents beautiful mathematical exercises and broadened the scope of the concerned model being transmuted.

According to the Quadratic Rank Transmutation Map,(QRTM), approach the cumulative distribution function(cdf) satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda[F_1(x)]^2 \tag{1.1}$$

which on differentiation yields,

$$f_2(x) = f_1(x)[1+\lambda-2\lambda F_1(x)] \tag{1.2}$$

Where $f_1(x)$ and $f_2(x)$ and the probability density functions corresponding to $F_1(x)$ and $F_2(x)$ respectively and $|\lambda| \le 1$. Using above formulation for the purpose of generalization of a probability distribution. Therefore, a random variable X is said to have transmuted distribution if its cumulative distribution function is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \quad ,|\lambda| \le 1 \tag{1.3}$$

where $G(x)$ is the cdf of the base distribution. If we put $\lambda = 0$, we get the base distribution.

In probability theory, the log-logistic distribution is a continuous probability distribution used in survival analysis as a parametric model for events whose rate increases initially and decreases later, for example mortality rate from cancer following diagnosis or treatment. The inverse version of log-logistic model also provide a greater flexibility in survival data sets.
The probability density function (pdf) of the inverse Log-logistic (ILL) distribution is defined as

$$g(x;\alpha) = \frac{\alpha}{x^{(\alpha+1)}\left(1+x^{-\alpha}\right)^2} \qquad \mathrm{x} > 0, \alpha > 0 \tag{1.4}$$

and its corresponding cumulative distribution function (cdf) is given by

$$G(x;\alpha) = \frac{1}{1+(x)^{-\alpha}} \qquad \mathrm{x} > 0, \alpha > 0 \tag{1.5}$$

where α is the shape parameters.

The rest of this paper is organized as follows. In Section 2 we demonstrate transmuted Inverse Log-logistic distribution. In Section 3, various statistical properties, the distributions of order statistics, moment generating function and the quantile function are summarized. The maximum likelihood estimates (MLE) of the distribution parameters are demonstrated in Section 4 followed by Monte Carlo simulation procedure. Robust measures of skewness and Kurtosis along with graphical overview is presented in section 5. Simulation procedure for model comparison is given in section 6. Real life application part of the article is presented in section 7.

2. Transmuted Inverse Log-Logistic Distribution

In this section we studied the transmuted Inverse Log-logistic distribution and the sub-models of this distribution. Now using (1.3) and (1.4), we have the cdf of TILLD given by

$$F(x,\alpha,\lambda)=\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^2} \qquad x>0,\alpha>0,-1\le\lambda\le 1 \qquad (2.1)$$

Hence the pdf of TILLD with parameters $\alpha\ and\ \lambda$ is given as

$$f(x,\alpha,\lambda)=\frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^3} \qquad x>0,\alpha>0,|\lambda|\le 1 \qquad (2.2)$$

Fig.1 to Fig.4 gives the pdf plot for (2.2) for different values of parameters. It is evident that the distribution of the transmuted inverse Log-logistic random variable X is right skewed.

3. Statistical Properties of Transmuted Inverse Loglogistic Distribution

In this section we shall discuss structural properties of transmuted Inverse Log-logistic distribution. Specially moments, order statistics, maximum likelihood estimation, moment generating function.

3.1 Moments: The following theorem gives the rth moment of the transmuted Inverse Log-logistic distribution.

Theorem 3.1: If X has the TILLD(α,λ) distribution with $|\lambda|\le 1$, then the rth non-central moments are given by

$$\mu_r{}'=(1+\lambda)\beta\left(1-\frac{r}{\alpha},\frac{r}{\alpha}+1\right)-2\lambda\beta\left(1-\frac{r}{\alpha},\frac{r}{\alpha}+2\right)$$

Proof: $\mu_r{}'=\int_0^\infty x^r\,\frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^3}dx$

$$\mu_r{}'=\int_0^\infty\frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha-r+1}\left(1+x^{-a}\right)^3}dx$$

$$=\int_0^\infty\frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)\right]}{x^{\alpha-r+1}\left(1+x^{-a}\right)^3}dx-\int_0^\infty x^r\,\frac{2\lambda\alpha}{x^{\alpha-r+1}\left(1+x^{-a}\right)^3}dx$$

Substituting $x^{-a} = t$, we get $x = t^{-\frac{1}{\alpha}}$, and $dx = -\frac{1}{\alpha} t^{-\frac{1}{\alpha}-1}$

$As \quad x \to 0, \quad t \to \infty \quad and \ As \ x \to \infty\,,\ t \to 0$

After simplification we get

$$\mu_r' = (1+\lambda)\beta\left(1-\frac{r}{\alpha}, 1+\frac{r}{\alpha}\right) - 2\lambda\beta\left(1-\frac{r}{\alpha}, 2+\frac{r}{\alpha}\right)$$

$$where \quad \beta(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

$$\mu_r' = (1+\lambda)\Gamma\left(\frac{\alpha-r}{\alpha}\right)\Gamma\left(\frac{\alpha+r}{\alpha}\right) - 2\lambda\Gamma\left(\frac{\alpha-r}{\alpha}\right)\Gamma\left(\frac{2\alpha+r}{\alpha}\right) \tag{3.1}$$

The first two moments about origin for transmuted Inverse Log-logistic distribution given by

$$\mu_1' = (1+\lambda)\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(\frac{\alpha+1}{\alpha}\right) - 2\lambda\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(\frac{2\alpha+1}{\alpha}\right) \tag{3.2}$$

$$\mu_2' = (1+\lambda)\Gamma\left(\frac{\alpha-2}{\alpha}\right)\Gamma\left(\frac{\alpha+2}{\alpha}\right) - 2\lambda\Gamma\left(\frac{\alpha-2}{\alpha}\right)\Gamma\left(\frac{2(\alpha+1)}{\alpha}\right) \tag{3.3}$$

Thus the variance of TILL distribution is given by

$$\mu_2 = \left[(1+\lambda)\Gamma\left(\frac{\alpha-2}{\alpha}\right)\Gamma\left(\frac{\alpha+2}{\alpha}\right) - 2\lambda\Gamma\left(\frac{\alpha-2}{\alpha}\right)\Gamma\left(\frac{2(\alpha+1)}{\alpha}\right)\right] - \left\{(1+\lambda)\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(\frac{\alpha+1}{\alpha}\right) - 2\lambda\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(\frac{2\alpha+1}{\alpha}\right)\right\}^2$$

For the convergence of rth moment, $\left(1-\frac{r}{\alpha}\right) > 0 \Rightarrow \alpha > r$. Thus, for the existence of mean, α should be greater than 0 and for convergence of variance α should be greater than 2. Similarly, for skewness and kurtosis measures α needs to be greater than 3 and 4 respectively. Under the situation of divergence of any of the statistical measures, the problem will be approached through robust measures which we will discuss in section 5.

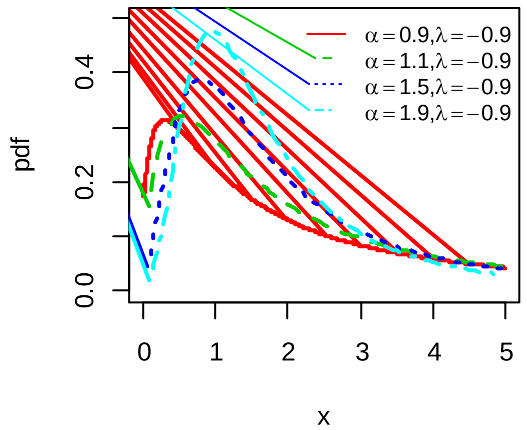


Fig.1. pdf plot for TILLD(α,λ)

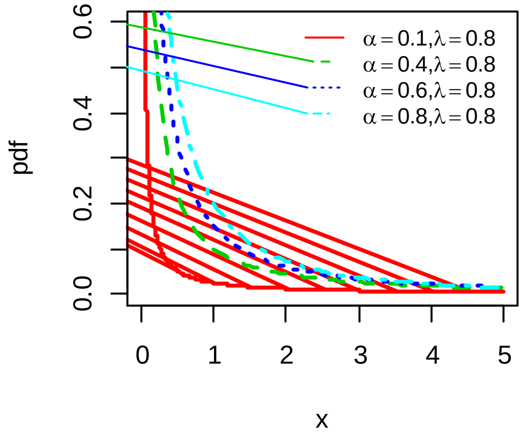


Fig.2. pdf plot for TILLD(α,λ)

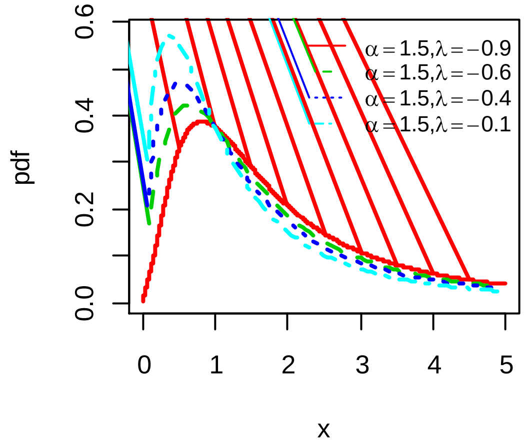


Fig.3. pdf plot for TILLD(α,λ)

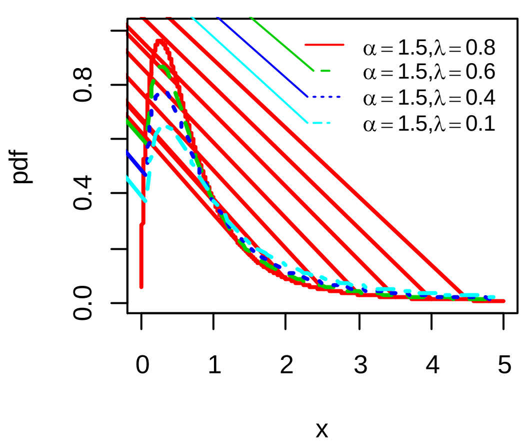


Fig.4. pdf plot for TILLD(α,λ)

The CDF plot of Transmuted Inverse Loglogistic distribution for different values of parameters is given in fig.5. The initial rise of the CDF curve increases as the shape parameter increases.

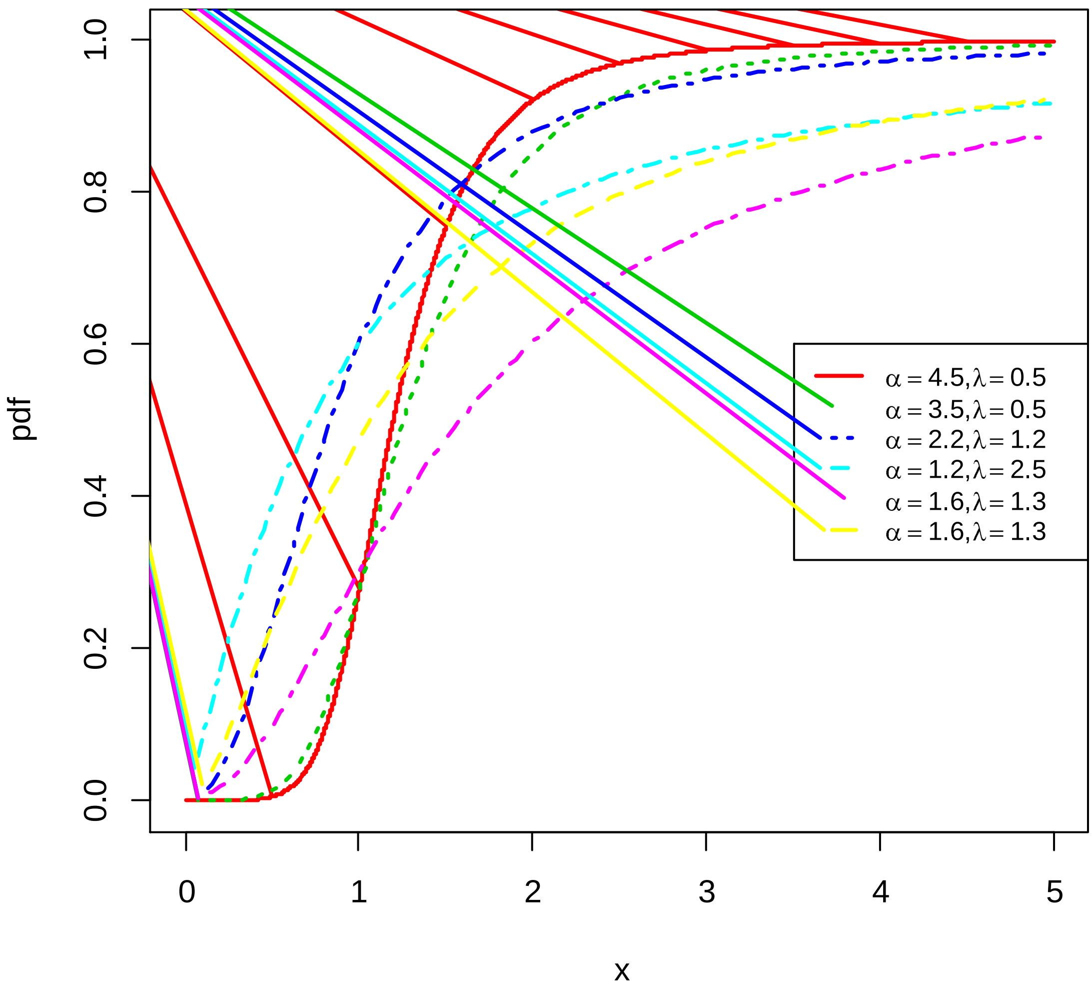


Fig.5. CDF plot for TILLD(α,λ)

3.2 Moment generating function and Characteristic function of TILLD

We will derive moment generating function and characteristic function of TILLD(α,λ) in this sub section.

Theorem 3.2: If X has the TILL(α,λ) distribution with $|\lambda|\le 1$, then the moment generating function $M_X(t)$ and the characteristic function $\psi_X(t)$ has the following form

$$M_X(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\left[(1+\lambda)\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{\alpha+j}{\alpha}\right)-2\lambda\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{2\alpha+j}{\alpha}\right)\right]$$

and

$$\psi_X(t)=\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left\{(1+\lambda)\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{\alpha+j}{\alpha}\right)-2\lambda\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{2\alpha+j}{\alpha}\right)\right\}\ \textit{respectively.}$$

Proof: We begin with the well known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}\,f(x;\alpha,\lambda)\,dx$$

$$=\int_0^{\infty}\left[1+t\,x+\frac{(t\,x)^2}{2!}+....\right]f(x;\alpha,\lambda)\,dx$$

$$=\int_0^{\infty}\sum_{j=0}^{\infty}\frac{t^j}{j!}x^j\,f(x;\alpha,\lambda)\,dx$$

$$=\sum_{j=0}^{\infty}\frac{t^j}{j!}\,\mu_j{}'$$

$$\Rightarrow M_X(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\left[(1+\lambda)\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{\alpha+j}{\alpha}\right)-2\lambda\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{2\alpha+j}{\alpha}\right)\right] \tag{3.2.1}$$

Also we know that $\psi_X(t)=M_X(it)$

Therefore,

$$\psi_X(t)=\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left\{(1+\lambda)\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{\alpha+j}{\alpha}\right)-2\lambda\Gamma\left(\frac{\alpha-j}{\alpha}\right)\Gamma\left(\frac{2\alpha+j}{\alpha}\right)\right\} \tag{3.2.2}$$

3.3. Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)},X_{(2)},...,X_{(n)}$ denotes the order statistics of a random sample $X_1,X_2,...,X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{X\,(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r}$$

For r = 1, 2, . . . , n

The pdf of the rth order statistic for a transmuted Inverse Log-logistic distribution is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^{3}}$$
$$\left[\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^{2}}\right]^{r-1}\left[1-\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^{2}}\right]^{n-r} \qquad (3.3.1)$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X(n)}(x)=\frac{n\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^{3}}\left[\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^{2}}\right]^{n-1} \qquad (3.3.2)$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X(1)}(x)=\frac{n\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^{3}}\left[1-\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^{2}}\right]^{n-1} \qquad (3.3.3)$$

Note that $\lambda=0$ yields the order statistics of the Inverse Log-logistic distribution.

3.4 Quantile and Random Number Generation from TILLD

Inverse CDF Method is one of the methods used for the generation of random numbers from a particular distribution. In this method the random numbers from a particular distribution are generated by solving the equation obtained on equating the CDF of a distribution to a number *u*. The number *u* is itself being generated from $U(0,1)$. Thus following the same procedure for the generation of random numbers from the TILLD we will proceed as:

$$F(x,\alpha,\lambda)=u$$

$$\frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^{2}}=u \;\Rightarrow x_{u}=\left[\frac{-(2u-1-\lambda)+\sqrt{(\lambda+1)^{2}-4u\lambda}}{2u}\right]^{-\frac{1}{\alpha}} \qquad (3.4.1)$$

On solving the equation (3.4.1) for x, we will obtain the required random number from the TILLD. If $p=0.25$, $p=0.5$ and $p=0.75$, the resulting solutions will be the first quartile (Q_1), Median (Q_2) and third quartile (Q_3) respectively. Similarly we will find out the deciles and percentiles of different orders by simply assigning the different values to *u*. Now, the main problem which is being faced while using this method of generating the random numbers is to solve the equations which are usually complex and complicated. In order to overcome such hindrances we use statistical softwares like R for solving such a complex equation.

3.5 Reliability Measures of TILLD

In this sub section, we present the reliability function and the hazard function for the proposed transmuted inverse log-logistic distribution. The reliability function is otherwise known as the survival or survivor function. It is the probability that a system will survive beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (cdf).

The survivor function is given by

$$s(x) = 1 - F(x)$$

$$s(x,\alpha,\beta,\lambda) = 1 - \frac{1+(1+\lambda)x^{-a}}{\left(1+x^{-a}\right)^2} \qquad x>0, \alpha>0, -1\le\lambda\le 1 \qquad (3.5.1)$$

The hazard function also known as the hazard rate, failure rate or force of mortality. This is an important quantity characterizing life phenomenon. It can be interpreted as the conditional probability of failure, given it has survived to time *x*.

The hazard rate function of Transmuted Inverse Loglogistic distribution is given by

$$h(x) = \frac{f(x)}{s(x)} \qquad \Rightarrow h(x) = \frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)\left[\left(1+x^{-a}\right)^2 - 1 + (1+\lambda)x^{-a}\right]}$$

$$x>0, \alpha>0, -1\le\lambda\le 1 \qquad (3.5.2)$$

Fig.5 to Fig.9 exhibits the hazard rate function plot for (3.5.2) for different values of parameters.

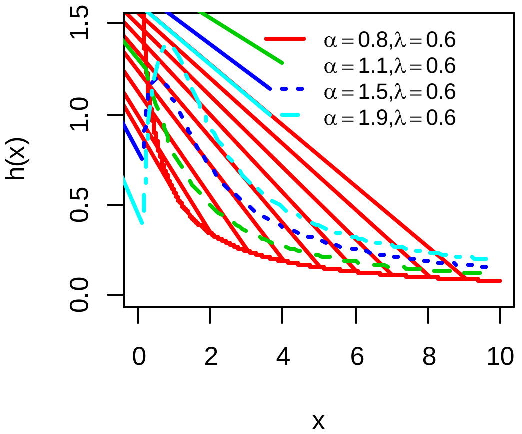


Fig.6. h(x) plot for TILLD(α,λ)

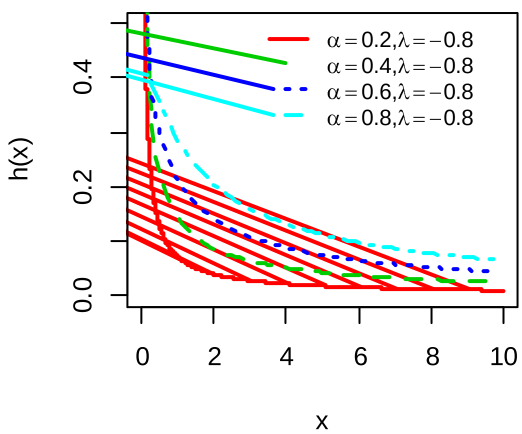


Fig.7. h(x) plot for TILLD(α,λ)

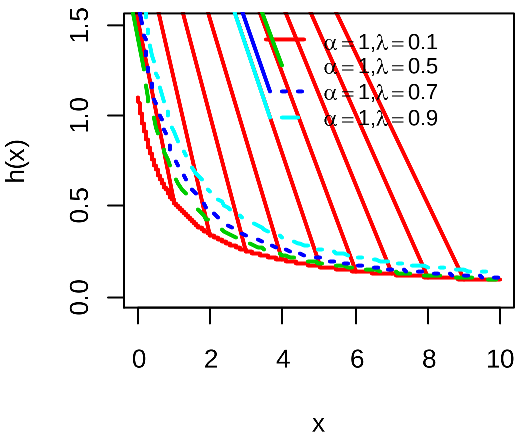


Fig.8. h(x) plot for TILLD(α,λ)

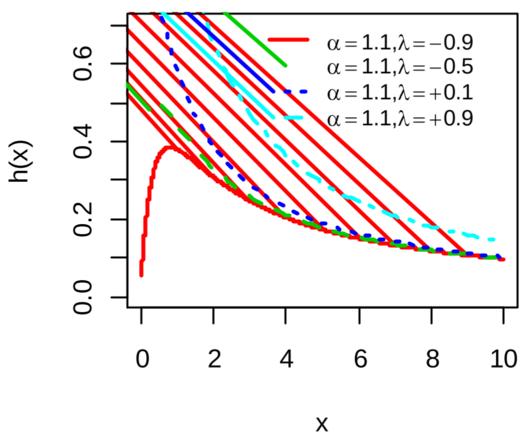


Fig.9. h(x) plot for TILLD(α,λ)

4. Maximum Likelihood Estimation

We estimate the parameters of the TILL distribution using the method of maximum likelihood estimation (MLE) as follows;

Let $X_1, X_2, ..., X_n$ be a random sample of size n from TILL distribution. Then the likelihood function is given by

$$L(x \mid \alpha, \lambda) = \prod_{i=1}^{n} \frac{\alpha\left[(1+\lambda)\left(1+x^{-a}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-a}\right)^{3}} \tag{4.1}$$

By taking logarithm of (4.1), we find the log likelihood function

$$l = n\log\alpha + \sum_{i=1}^{n}\log\left((1+\lambda)\left(1+x_i^{-\alpha}\right)-2\lambda\right)-(\alpha+1)\sum_{i=1}^{n}\log(x_i)-3\sum_{i=1}^{n}\log\left(1+x_i^{-\alpha}\right) \qquad (4.2)$$

To obtain the MLE's of α *and* λ, we differentiating loglikelihood with respect to α *and* λ

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\frac{\left((1+\lambda)x_i^{-\alpha}\log(x_i)\right)}{\left((1+\lambda)\left(1+(x_i)^{-\alpha}\right)-2\lambda\right)}+3\sum_{i=1}^{n}\frac{(x_i)^{-\alpha}\log(x_i)}{\left(1+(x_i)^{-\alpha}\right)} \qquad (4.3)$$

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{(x_i)^{-\alpha}-1}{\left((1+\lambda)\left(1+(x_i)^{-\alpha}\right)-2\lambda\right)x_i^{\alpha+1}} \qquad (4.4)$$

These two derivative equations cannot be solved analytically, therefore $\hat{\alpha}$ *and* $\hat{\lambda}$ will be obtained by maximizing the log likelihood function numerically using Newton-Raphson method which is a powerful technique for solving equations iteratively and numerically. We can compute the second partial derivatives, which are useful to obtain the Fisher's information matrix as follows.

$$I_x(\alpha,\lambda)=\begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) \\ -E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & -E\left(\frac{\partial^2 l}{\partial\lambda^2}\right)\end{bmatrix} \qquad (4.6)$$

One can show that the transmuted Inverse Loglogistic distribution satisfies the regularity conditions (see, e.g., [6]). Hence, the MLE vector $\hat{\Theta}=\left(\hat{\alpha},\hat{\lambda}\right)^T$ is consistent and asymptotically normal; that is, $\sqrt{n}\left[\left(\hat{\alpha},\hat{\lambda}\right)^T-(\alpha,\lambda)^T\right]$ converges in distribution to a normal distribution with the (vector) mean zero and the identity variance covariance matrix of unknown parameter $\Theta=(\alpha,\lambda)^T$. That is, $\sqrt{n}\left(\hat{\Theta}-\Theta\right)\rightarrow N_2(0,I_x^{-1}(\hat{\Theta}))$, Also, the Fisher's information matrix can be computed using the approximation

$$I_x\left(\hat{\alpha},\hat{\lambda}\right)=\begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial\alpha^2}\right)\Big|_{\left(\hat{\alpha},\hat{\lambda}\right)} & -E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right)\Big|_{\left(\hat{\alpha},\hat{\lambda}\right)} \\ -E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right)\Big|_{\left(\hat{\alpha},\hat{\lambda}\right)} & -E\left(\frac{\partial^2 l}{\partial\lambda^2}\right)\Big|_{\left(\hat{\alpha},\hat{\lambda}\right)}\end{bmatrix} \qquad (4.7)$$

where $\hat{\alpha}$ *and* $\hat{\lambda}$ are the MLEs of α *and* λ, respectively (see, e.g.,[7]). Using this approximation, we may construct confidence intervals for parameters of the transmuted inverse Log-logistic model. Approximate $100(1-\alpha)\%$ confidence intervals for α *and* λ are, respectively, given by

$\hat{\alpha} \pm z_{\frac{\alpha}{2}} \sqrt{I_{11}^{-1}(\hat{\Theta})}$ and $\hat{\lambda} \pm z_{\frac{\alpha}{2}} \sqrt{I_{22}^{-1}(\hat{\Theta})}$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}th$ percentiles of the standard normal distribution. Using R studio statistical software, we can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log likelihoods to construct the likelihood ratio (LR) statistics for testing the significance of transmuted parameter of the proposed model. For example, we can use LR test to check whether the fitted transmuted Inverse Loglogistic distribution for a given data set is statistically "superior" to the fitted Inverse Loglogistic distribution. In any case, hypothesis tests of the type $H_0 : \Theta = \Theta_0$ versus $H_1 : \Theta \neq \Theta_0$ can be performed using LR statistics. In this case, the LR statistic for testing H_0 versus H_1 is $\omega = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$ where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically $(as\, n \to \infty)$ distributed as χ_k^2, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.05 at 95% confidence level.

4.1 Monte Carlo Simulation for ML estimates:

In this section, we investigate the behavior of the ML estimators for a finite sample size n. Simulation study based on different $TILLD(x, \alpha, \lambda)$ is carried out. The random observations are generated by using the inverse cdf method presented in section 3.4 from TILLD(α, λ). Monte Carlo simulation study was carried out for four parameter combinations as $(\alpha = 0.8, \lambda = 0.6)$, $(\alpha = 1.5, \lambda = 0.8)$, $(\alpha = 2.5, \lambda = 1.0)$ and $(\alpha = 1.2, \lambda = -0.5)$.The process was repeated 2000 times by taking different sample sizes n = (25,50,75,100,150,200,300,500). We observe in table 1 that the agreement between theory and practice improves as the sample size *n* increases. MSE and Variance of the estimators suggest us that the estimators are consistent and the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

Table 1: Average Bias, MSE and Variance for simulated results of ML estimates.

sample size n		$\alpha=0.8, \lambda=0.6$			$\alpha=1.5, \lambda=0.8$		
	parameters	Bias	Variance	MSE	Bias	Variance	MSE
25	α	0.026917	0.020948	0.021672	0.058485	0.066683	0.070104
	λ	0.000855	0.087407	0.087408	-0.02165	0.058143	0.058612
50	α	0.01366	0.009804	0.00999	0.025411	0.032544	0.03319
	λ	0.001442	0.047641	0.047643	-0.00981	0.032229	0.032325
75	α	0.008365	0.006356	0.006426	0.018239	0.021671	0.022004
	λ	-0.00413	0.028813	0.02883	-0.00999	0.021504	0.021604
100	α	0.00727	0.004725	0.004778	0.010229	0.01565	0.015755
	λ	0.001204	0.024411	0.024413	-0.00227	0.017766	0.017771
150	α	0.004978	0.003113	0.003138	0.00723	0.010605	0.010658
	λ	-0.00088	0.015891	0.015891	-0.00659	0.012211	0.012255
200	α	0.005056	0.002199	0.002225	0.005293	0.007957	0.007985
	λ	-0.00283	0.012485	0.012493	-0.00054	0.00902	0.00902
300	α	0.000233	0.001493	0.001493	0.003153	0.005829	0.005838
	λ	-0.00231	0.00822	0.008225	-0.00081	0.006089	0.00609
500	α	-0.00033	0.000849	0.000849	0.001243	0.00311	0.003112
	λ	0.002101	0.004759	0.004763	0.001232	0.003568	0.00357
		$\alpha=2.5, \lambda=1.0$			$\alpha=1.2, \lambda=-0.5$		
25	α	0.118382	0.21104	0.225055	0.046584	0.045726	0.047896
	λ	-0.08558	0.023005	0.030329	-0.00189	0.102553	0.102556
50	α	0.077836	0.083471	0.08953	0.0166	0.020595	0.02087
	λ	-0.05536	0.010032	0.013097	-0.00719	0.050258	0.05031
75	α	0.045996	0.058048	0.060164	0.009597	0.014499	0.014591
	λ	-0.04195	0.005853	0.007613	0.000392	0.033607	0.033607
100	α	0.035689	0.040749	0.042023	0.011486	0.014284	0.014416
	λ	-0.03261	0.003627	0.00469	0.006135	0.035459	0.035496
150	α	0.028784	0.028163	0.028991	0.002757	0.006714	0.006722
	λ	-0.02682	0.00221	0.002929	0.003851	0.016898	0.016913
200	α	0.030127	0.021343	0.022251	0.005566	0.005196	0.005227
	λ	-0.02243	0.001572	0.002075	0.002313	0.012557	0.012563
300	α	0.011998	0.01351	0.013654	0.002676	0.003457	0.003464
	λ	-0.01882	0.001054	0.001408	-8.9E-05	0.008339	0.008339
500	α	0.010415	0.007595	0.007703	0.001843	0.002068	0.002071
	λ	-0.013	0.00051	0.000679	-0.00037	0.005037	0.005037

5. Robust Skewness and Kurtosis measures for TILLD

To illustrate the effect of the parameter α *and* λ on skewness and kurtosis, we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative. They are less sensitive to outliers and they exist for the distributions even without defined moments. The Galtons's skewness [8] is one of the earliest skewness measures based on the octiles, given by

$$G = \frac{E_6 + E_2 - 2E_4}{E_6 - E_2} = \frac{Q\left(\frac{6}{8}\right) + Q\left(\frac{2}{8}\right) - 2Q\left(\frac{4}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

and the Moors kurtosis [9] is based on octiles and is given by

$$M = \frac{(E_3 - E_1) + (E_7 - E_5)}{E_6 - E_2} = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

For any distribution symmetrical to 0 the Moors kurtosis reduces to

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)}$$

It is easy to calculate that for standard normal distribution $E_1 = -E_7 = -1.15$, $E_2 = -E_6 = -0.67$ and $E_3 = -E_5 = -0.32$. Therefore, M = 1.23. Hence, the centered Moor's coefficient is given by:

$$M = \frac{\left[Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)\right] + \left[Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)\right]}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} - 1.23$$

Fig. 10 and fig. 11 provides the graphical overview of robust measures of skewness and kurtosis for Transmuted Inverse Loglogistic distribution. Table 2 provides the numerically calculated Galtons Skewness and Moors kurtosis of two parameter Transmuted Inverse log-logistic distribution for different values of parameters using R studio statistical software. For fixed λ, the Galton's skewness and Moor's Kurtosis are decreasing functions of α. For fixed value of α, the Galton's skewness and Moor's Kurtosis exhibit both decreasing and increasing nature for different values of transmuted parameter λ.

Table 2: Galton's Skewness and Moors Kurtosis of Transmuted Inverse log-logistic distribution for different values of parameters

Parameters	Galton's Skewness of Transmuted ILLD									
		α								
		0.3	0.8	1.5	2.0	2.6	3.2	4.1	4.5	5.6
λ	-1.0	0.92880	0.56821	0.35254	0.28196	0.23130	0.19903	0.16795	0.15807	0.13808
	-0.8	0.93432	0.57220	0.34761	0.27360	0.22041	0.18651	0.15387	0.14348	0.12248
	-0.6	0.93982	0.57803	0.34499	0.26761	0.21192	0.17639	0.14218	0.13130	0.10929
	-0.3	0.94685	0.58916	0.34695	0.26564	0.20697	0.16951	0.13341	0.12192	0.09870
	0.1	0.94950	0.59473	0.34984	0.26726	0.20762	0.16950	0.13277	0.12108	0.09744
	0.6	0.93147	0.54953	0.31080	0.23227	0.17596	0.14012	0.10567	0.09472	0.07260
	0.7	0.92440	0.53332	0.29644	0.21913	0.16380	0.12864	0.09487	0.08414	0.06248
	0.8	0.91632	0.51577	0.28091	0.20485	0.15056	0.11609	0.08302	0.07252	0.05133
	1.0	0.89782	0.47888	0.24843	0.17494	0.12271	0.08964	0.05797	0.04792	0.02766
Moors Kurtosis of Transmuted ILLD										
Parameters		α								
		0.3	0.8	1.5	2.0	2.6	3.2	4.1	4.5	5.6
λ	-1.0	14.58998	2.61021	1.70238	1.54309	1.45566	1.41009	1.37303	1.36257	1.34330
	-0.8	15.03530	2.62154	1.69667	1.53808	1.45276	1.40924	1.37467	1.36512	1.34784
	-0.6	15.53027	2.63657	1.69010	1.53087	1.44675	1.40470	1.37209	1.36326	1.34763
	-0.3	16.29179	2.66658	1.68295	1.51987	1.43496	1.39324	1.36155	1.35315	1.33856
	0.1	16.64456	2.68350	1.68218	1.51655	1.43052	1.38837	1.35647	1.34803	1.33345
	0.6	12.60513	2.39974	1.59621	1.46441	1.39721	1.36507	1.34152	1.33550	1.32545
	0.7	11.16733	2.28146	1.55441	1.43612	1.37657	1.34859	1.32857	1.32358	1.31549
	0.8	9.73829	2.15318	1.50690	1.40314	1.35191	1.32847	1.31232	1.30847	1.30255
	1.0	7.28318	1.90062	1.40796	1.33269	1.29797	1.28362	1.27529	1.27375	1.27226

Fig.10: Galton's Skewness for two parameter TILLD

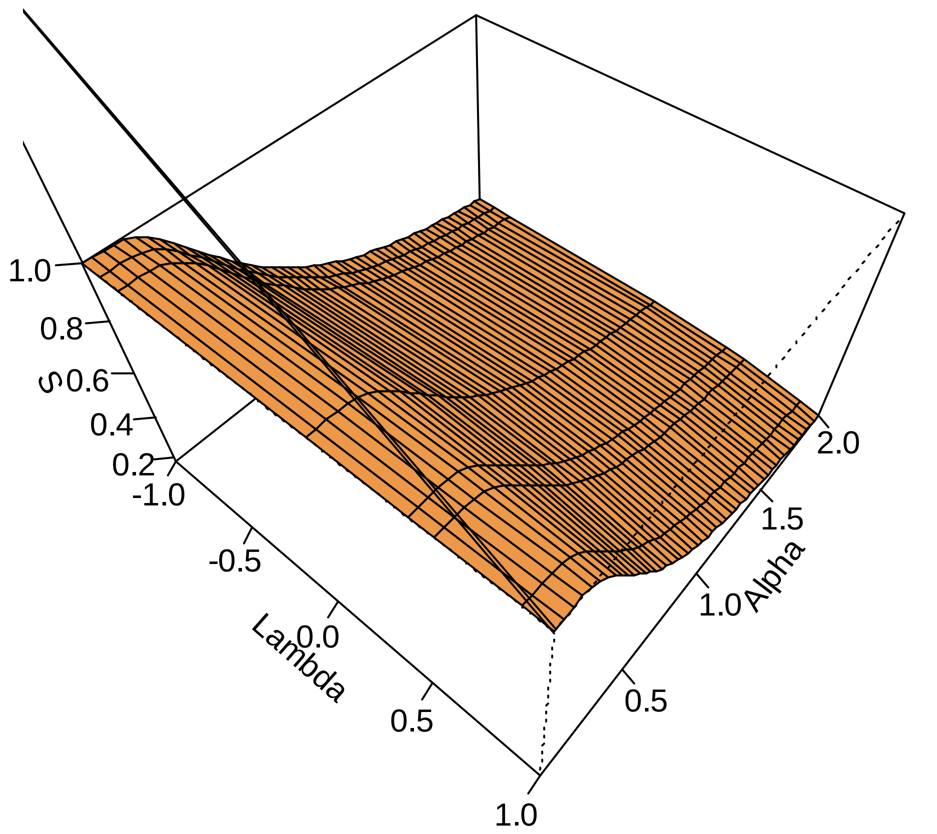
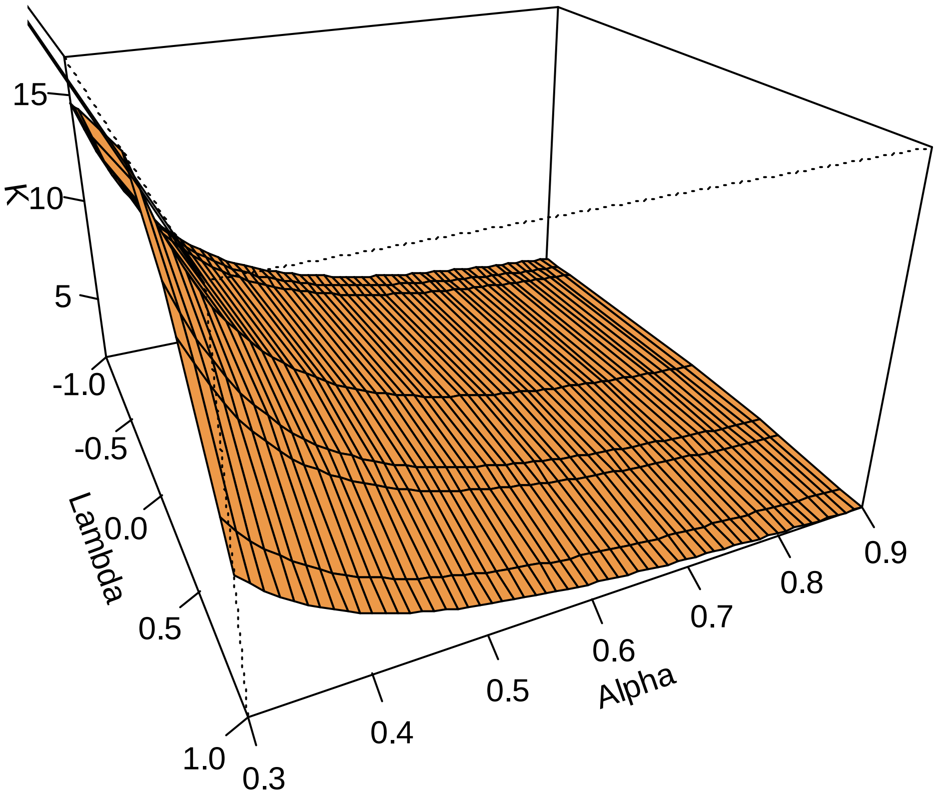


Fig.11: Moors Kurtosis for two parameter TILLD



6. Model comparison based on simulated data from TILLD

In order to compare the Transmuted Model with the base model on the basis of simulated data. We proceed by simulating a data from TILLD using data generation technique discussed in section 3.4. The data generation is based on two sets of parameter combinations $(\alpha = 0.5, \lambda = 0.2)$ and $(\alpha = 1.5, \lambda = 0.8)$ with sample sizes (n=10, 25, 75,150,300). It is clear from the table 3(a) and table 3(b) that transmuted parameter plays a significant role for the large samples. Even though in small as well as large samples the AIC, AICC, BIC and Negative Loglikelihood values are minimum in case of Transmuted model but the likelihood ratio test reveals that the role of transmuted parameter exhibits a significant role in case of large samples only.

Table 3(A): Model Comparison Based On Simulated Data From TILLD.

$\alpha = 0.5, \lambda = 0.2$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	Transmuted Distribution	Base Distribution	Sample Size (n)	TILLD	ILLD	
-logL	28.92071	31.68451	10	$\hat{\alpha} = 0.498(0.128)$ $\hat{\lambda} = -0.11(0.525)$	$\hat{\alpha} = 0.37(0.09)$	5.5276
AIC	61.84141	65.36902				
AICC	65.84141	69.36902				
BIC	62.44658	65.67161				
-logL	70.8628	77.69515	25	$\hat{\alpha} = 0.514(0.08)$ $\hat{\lambda} = 0.011(0.33)$	$\hat{\alpha} = 0.38(0.06)$	13.6647
AIC	145.7256	157.3903				
AICC	146.8685	158.53315				
BIC	148.1634	158.60917				
-logL	164.1412	177.0951	75	$\hat{\alpha} = 0.596(0.057)$ $\hat{\lambda} = 0.178(0.191)$	$\hat{\alpha} = 0.47(0.04)$	25.9078
AIC	332.2824	356.1902				
AICC	332.6205	356.5283				
BIC	336.9174	358.5077				
-logL	352.1891	382.0877	150	$\hat{\alpha} = 0.565(0.04)$ $\hat{\lambda} = 0.151(0.14)$	$\hat{\alpha} = 0.44(0.022)$	59.7972
AIC	708.3783	766.1754				
AICC	708.5426	766.3398				
BIC	714.3995	769.186				
-logL	638.5797	685.6495	300	$\hat{\alpha} = 0.558(0.02)$ $\hat{\lambda} = 0.27(0.09)$	$\hat{\alpha} = 0.45(0.021)$	94.1396
AIC	1281.1594	1373.2989				
AICC	1281.2405	1373.38				
BIC	1288.567	1377.0027				

Table 3(B): Model Comparison Based On Simulated Data From TILLD.

$\alpha = 1.5, \lambda = 0.8$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	Transmuted Distribution	Base Distribution	Sample Size (n)	TILLD	ILLD	
-logL	14.98281	17.05532	10	$\hat{\alpha} = 1.61(0.42)$ $\hat{\lambda} = 0.52(0.54)$	$\hat{\alpha} = 1.45(0.353)$	4.14502
AIC	33.96562	36.11065				
AICC	37.96562	40.11065				
BIC	34.57079	36.41323				
-logL	29.83482	31.86065	25	$\hat{\alpha} = 1.62(0.26)$ $\hat{\lambda} = 0.59(0.29)$	$\hat{\alpha} = 1.434(0.22)$	4.05166
AIC	63.66964	65.72129				
AICC	64.8125	66.86415				
BIC	66.1074	66.94017				
-logL	98.26392	106.6319	75	$\hat{\alpha} = 1.75(0.16)$ $\hat{\lambda} = 0.87(0.13)$	$\hat{\alpha} = 1.68(0.15)$	16.73596
AIC	200.52784	215.2639				
AICC	200.86587	215.6019				
BIC	205.16282	217.5813				
-logL	151.5693	156.6266	150	$\hat{\alpha} = 1.69(0.11)$ $\hat{\lambda} = 0.786(0.10)$	$\hat{\alpha} = 1.58(0.10)$	10.1146
AIC	307.1386	315.2531				
AICC	307.303	315.4175				
BIC	313.1599	318.2637				
-logL	336.4271	353.3758	300	$\hat{\alpha} = 1.457(0.06)$ $\hat{\lambda} = 0.63(0.08)$	$\hat{\alpha} = 1.30(0.0.05)$	33.8974
AIC	676.8542	708.7516				
AICC	676.9352	708.8327				
BIC	684.2617	712.4554				

7. Applications of Transmuted Inverse Log-logistic Distribution in Medical Science and Reliability

In this section, we compared the performance of the Transmuted Inverse Log-logistic distribution with the base model on some survival data sets already in literature. First of all, we apply the two parameter Transmuted Inverse Log-logistic distribution to the data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. For previous studies with the data sets, see Andrews and Herzberg [10], Barlow et al. [11], and Abdul-Moniem and Seham [12]. The data is given in table 4.

Table. 4 Lifetime of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed.

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.912	0.9836
1.0483	1.0596	1.0773	1.1733	1.257	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.488	1.5728	1.5733	1.7083	1.7263	1.746	1.763	1.7746	1.8275	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.133	2.21
2.246	2.2878	2.3203	2.347	2.3513	2.4951	2.526	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.096	

The parameter estimation of TILLD (Transmuted Model) and ILLD (Base Model) is done by using ML estimation technique using R studio statistical software. The ML estimates, model functions and standard errors of parameters are given in table 5 below.

Table 5: ML estimates with standard Errors of parameters for TILLD and ILLD.

Model	Model Function	ML Estimates	Standard Errors
Transmuted Model	$\dfrac{\alpha\left[(1+\lambda)\left(1+x^{-\alpha}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-\alpha}\right)^{3}}$	$\hat{\alpha}=1.856,\ \hat{\lambda}=-0.701$	$SE(\hat{\alpha})=0.17, SE(\hat{\lambda})=0.13$
Base Model	$\dfrac{\alpha}{x^{(\alpha+1)}\left(1+x^{-\alpha}\right)^{2}}$	$\hat{\alpha}=1.19$	$SE(\hat{\alpha})=0.11$

The Likelihood Ratio statistics to test the influence of the transmuted parameter, the hypotheses $\mathrm{H}_0:\lambda=0$ versus $\mathrm{H}_1:\lambda\neq 0$; $\xi=85.624>6.635=\chi_1^2(\alpha=0.01)$, so we reject the null hypotheses. And conclude that the parameter λ plays statistically a significant role.

We also compare the models using AIC (Akaike Information Criterion) given by Akaike [13], AICC (Akaike Information Criterion Corrected) and BIC (Bayesian information criterion) given by Schwarz [14]. Generic functions calculating AIC, AICC and BIC for the model having p number of parameters are given by

$$AIC=2p-2\log(l)$$

$$AICC=AIC+\frac{2p(p+1)}{n-p-1}$$

$$BIC=p\log(n)-2\log(l)$$

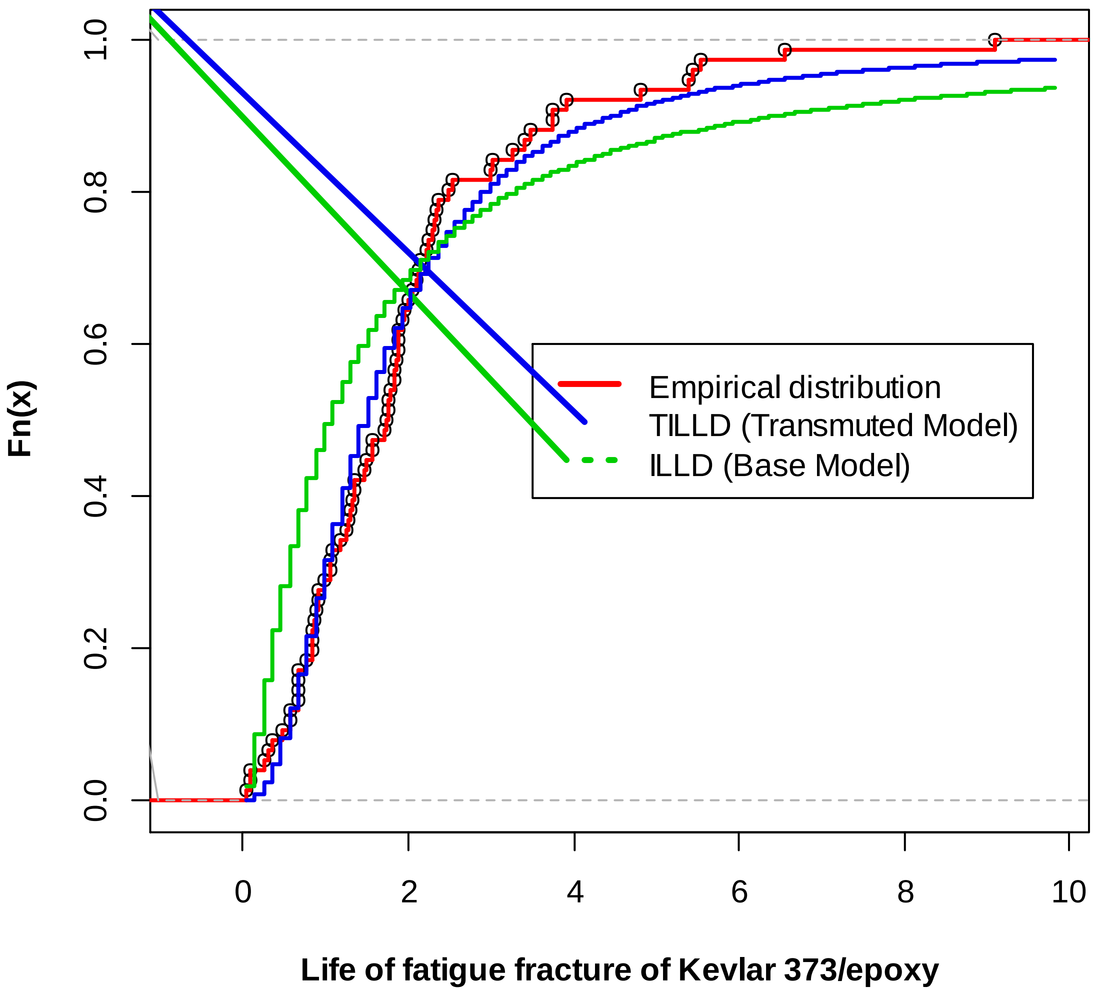
Table 6 exhibits the AIC, AICC, BIC and Negative Loglikelihood values for the models fitted to the data in table 4. It is obvious that AIC, AICC, and BIC criterion favors discrete Transmuted Inverse Log-logistic distribution in comparison with the one parameter inverse Loglogistic distribution, which is the base distribution for the proposed model.

Table 6: AIC, AICC ,BIC, K-S Statistic and Negative Loglikelihood values for fitted distributions

Model	*-logL*	AIC	AICC	BIC	K-S distance	K-S Stat p-value	LR-Stat
TILLD	125.0546	254.1093	254.4426	258.7707	0.26329	0.000039	85.624
ILLD	167.8668	337.7336	338.067	340.0644	0.11792	0.22290	

Fig. 12 provides the graphical overview of CDF plot of empirical, Transmuted Inverse Loglogistic and Inverse Loglogistic distribution.

Fig.12 : Empirical, fitted TILLD and ILLD CDF of Lifetime of fatigue fracture of Kevlar 373/epoxy



Data set II)

Here we consider a data set which represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia studied by Kotz and Johnson [17]. The data set is given in table 7.

Table 7: Survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia

0.0190	0.1290	0.1590	0.2030	0.4850	0.6360	0.7480	0.7810	0.8690
1.1750	1.2060	1.2190	1.2190	1.2820	1.3560	1.3620	1.4580	1.5640
1.5860	1.5920	1.7810	1.9230	1.9590	2.1340	2.4130	2.4660	2.5480
2.6520	2.9510	3.0380	3.6000	3.6550	3.7450	4.2030	4.6900	4.8880
5.1430	5.1670	5.6030	5.6330	6.1920	6.6550	6.8740		

Table 8. exhibits some descriptive statistical measures of survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia based on 1000 bootstrap samples. The data set has a skewed nature with mean lifetime of patient 2.53 years.

Table 8: Descriptive Statistics of Survival times (in years) after diagnosis of 43 patients

Descriptive measures	Statistic	Std. Error	Bootstrap[a]			
			Bias	Std. Error	95% Confidence Interval	
					Lower	Upper
Mean	2.5340	.29389	-.0078	.2859	1.9599	3.0894
Std. Deviation	1.92719		-.03247	.17500	1.53096	2.21376
Variance	3.714		-.094	.656	2.344	4.901
Skewness	.772	.361	-.003	.270	.239	1.326
Kurtosis	-.481	.709	.075	.668	-1.329	1.245
N	43		0	0	43	43

a. Bootstrap results are based on 1000 bootstrap samples

We have fitted Transmuted Inverse Log-logistic distribution and Inverse Loglogistic distribution to the data set in Table 7. For the purpose of parameter estimation, we employ the fitdistr procedure in R studio statistical software to find out the estimates of the parameters. The ML estimates and their standard errors provided by the fitdistr procedure are given in the table 9.

Table 9: ML estimates with standard Errors of parameters for fitted TILLD and ILLD for Survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia.

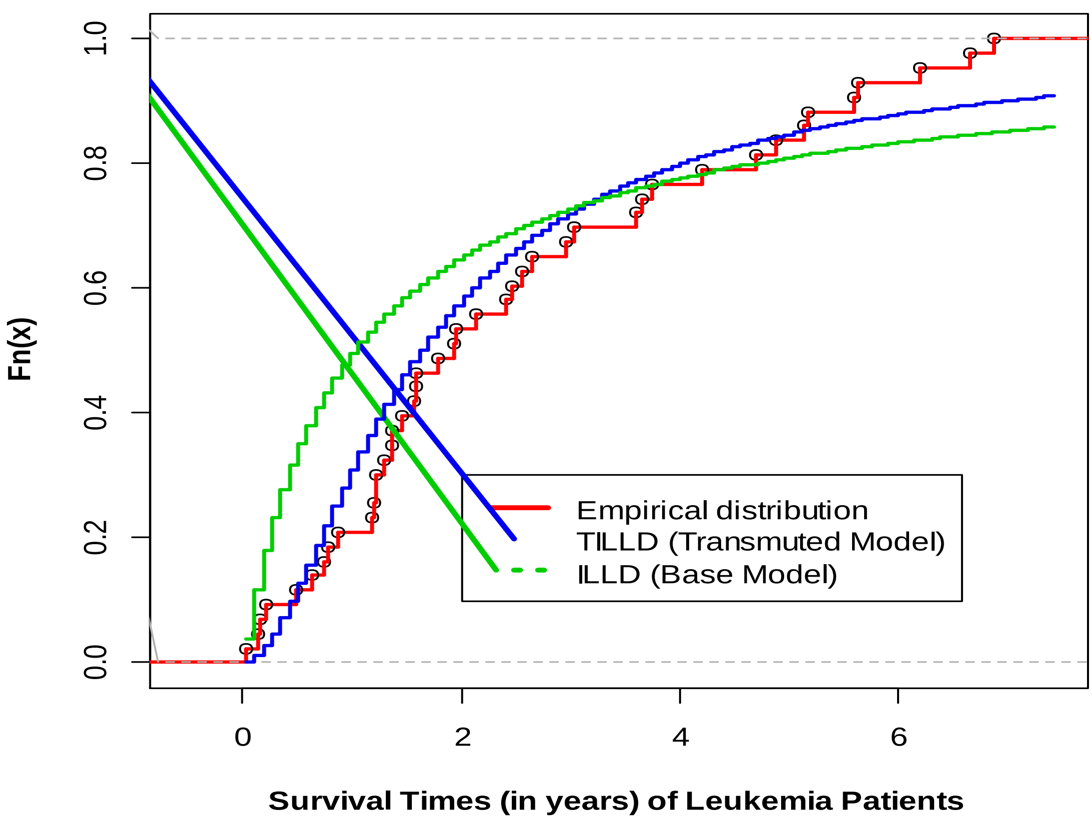
Model	Model Function	ML Estimates	Standard Errors
Transmuted Model	$\dfrac{\alpha\left[(1+\lambda)\left(1+x^{-\alpha}\right)-2\lambda\right]}{x^{\alpha+1}\left(1+x^{-\alpha}\right)^{3}}$	$\hat{\alpha}=1.432,\ \hat{\lambda}=-0.735$	$SE(\hat{\alpha})=0.18, SE(\hat{\lambda})=0.166$
Base Model	$\dfrac{\alpha}{x^{(\alpha+1)}\left(1+x^{-\alpha}\right)^{2}}$	$\hat{\alpha}=0.902$	$SE(\hat{\alpha})=0.112$

Table 10 exhibits the AIC, AICC, BIC and Negative Loglikelihood values for the models fitted to the data in table 7. It is obvious that AIC, AICC, and BIC criterion favors discrete Transmuted Inverse Log-logistic distribution in comparison with the one parameter inverse Loglogistic distribution, which is the base distribution for the proposed model. The Likelihood Ratio statistics to test the influence of the transmuted parameter in a data set studied by Kotz and Johnson [17] which represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia, the hypotheses $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$; $LR - Statistic = \omega = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0)) = 52.3785 > 6.635 = \chi_1^2(\alpha = 0.01)$, so we reject the null hypotheses and conclude that the parameter λ plays statistically a significant role.

Table 10: AIC, AICC, BIC , Negative Loglikelihood and LR statistic values for fitted distributions

Model	*-logL*	AIC	AICC	BIC	K-S distance	K-S Stat p-value	LR-Statistic
TILLD	87.52947	179.05895	179.67433	182.58135	0.16676	0.1829	52.3785
ILLD	113.7187	229.4374	230.0528	231.1986	0.32701	0.0002	

Fig.13 : Empirical, fitted TILLD and ILLD CDF of Survival Times (in years) of Leukemia Patients



Conclusion:

In the present study we have introduced a new generalization of the inverse Loglogistic distribution called the Transmuted inverse Loglogistic distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the inverse Loglogistic distribution as the base distribution. Some mathematical properties along with reliability measures are discussed. The hazard rate function and reliability behavior of transmuted inverse Loglogistic distribution exhibits that subject distribution can be used to model survival data from medical sciences and other fields of interest.

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