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Application of Sobolev inequalities for higher order fractional derivatives

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Abstract
in this paper we study the general uncertainty principal , we obtain the best constant as application of Sobolev inequalities for higher order fractional derivatives .

1. Introduction

Sobolev inequality has many applications in mathematics , and it is important to estimates constants in these inequalities.

$k \in \mathbb{N}$ is an integer of Sobolev space which can be defined in $H^k(\mathbb{R}^n)$ as a function $f \in l^2(\mathbb{R}^n)$ satisfying $\left|\nabla^\ell f\right| \in L^2(\mathbb{R}^n), 1 \leq \ell \leq k.$

The sobolev imbedding theorem asserts that $H^k(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ for
$q = 2n/(n-2k).$
For example, let $k = 1, n \geq 3$ and $q = 2n/(n-2),$
Then we have inequality

$$\|f\|^2_{\frac{2n}{n-2}} \leq C_n \|\nabla f\|_2^2, \quad f \in C_0^\infty(\mathbb{R}^n) \qquad (1)$$

(C_n is constant)

The best value for the constant C_n in inequality (1) has been estimated to be

$$C_n = \pi^{-1}\, n^{-1}(n-2)^{-1}\left[\frac{\Gamma(n)}{\Gamma(^n/_2)}\right]^{2/n} \qquad (2)$$

If we take the formula

$$\frac{\Gamma\,(n)}{\Gamma(n/2)} = \frac{2^{n-1}}{\pi^{1/2}}\Gamma((n+1)/2)$$

Then

$$C_n = \frac{4}{n(n-2)}|\mathbb{S}^n|^{-2/n} = 2^{-2/n}\pi^{-(n+1)/n}\frac{4}{n(n-2)}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^{2/n}$$

Since S^n is the n- dimential unit sphere and $|S^n|$ is the surface area.

Proposition 1: for every $f \in H^s(\mathbb{R}^n)$ we have

$$\left\|f - e^{-t(-\Delta)^s} f\right\|_2 = \sqrt{t}\,\left\|(-\Delta)^{s/2} f\right\|_2$$

Proof: let $f \in H^s(\mathbb{R}^n)$ then

$$\left\|f - e^{-t(-\Delta)^s} f\right\|_2^2 = \int \left|\hat{f}(k)\right|^2 \left(1 - e^{-t(2\pi|k|)^{2s}}\right)^2 dk$$

Let $1 - e^{-x} \leq x$ for $x \geq 0$

Hence

$$\left\|f - e^{-t(-\Delta)^s} f\right\|_2^2 \leq t \int (2\pi|k|)^{2s} \left|\hat{f}(k)\right|^2 dk = t\left\|(-\Delta)^{s/2} f\right\|_2^2$$

By taking square root of both sides we have

$$\left\|f - e^{-t(-\Delta)^s} f\right\|_2 = \sqrt{t}\,\left\|(-\Delta)^{s/2} f\right\|_2$$

Theorem 1: let $n > 2s$ and $q = 2n/(n-2s)$ then

$$\|f\|_q^2 \leq S(n,s)\left\|(-\Delta)^{s/2}\right\|_2^2 \qquad , f \in H^s(\mathbb{R}^n). \tag{3}$$

where

$$S(n,s) = 2^{-2s}\pi^{-s} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2s/n} \tag{4}$$

We have equality in (3) if and only if

$$f(x) = C\left(\mu^2 + (x - x_0)^2\right)^{-\frac{n-2s}{2}} \qquad , x \in \mathbb{R}^n.$$

where $C \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ are fixed constants.

Also we have

$$S(n,s) = \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} |S^n|^{-\frac{2s}{n}}$$

Proof:

First :if $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ and $f \in C_0^\infty(\mathbb{R}^n)$ then the relation (3) is true.

Now , let $f, g \in C_0^\infty(\mathbb{R}^n)$ then

$$(f,g) = (\hat{f},\hat{g}) = \int |K|^s \overline{\hat{f}(k)} |k|^{-s} \hat{g}(k) dk = \int \widehat{(-\Delta)^{s/2} f}(k) \widehat{(-\Delta)^{-s/2} g}(k) dk$$

$$= \left((-\Delta)^{s/2} f \cdot (-\Delta)^{-s/2} g\right). \tag{5}$$

Hence

$$|(f,g)| \leq \left\|(-\Delta)^{s/2} f\right\|_2 \left\|(-\Delta)^{-s/2} g\right\|_2 \tag{6}$$

Hardy-Littlewood-Sobolev inequality defined as :

$$\left\|(-\Delta)^{-s/2}(g)\right\|_2 \leq 2^{-s}\pi^{-s/2}\left(\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\right)^{\frac{1}{2}}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{s/n}\|g\|_p \qquad (7)$$

where $\frac{1}{p}+\frac{1}{q}=1$, i.e $p=\frac{2n}{(n+2s)}$

combining (6) and (7) we have

$$|(f,g)| \leq \big(S(n,s)\big)^{\frac{1}{2}}\left\|(-\Delta)^{s/2}(f)\right\|_2\|g\|_p. \qquad (8)$$

Let us take $g=f^{q-1}$, there for we have

$$|(f,g)| = |(f,f^{q-1})| = \|f\|_q^q$$

$$\|g\|_p = \|f^{q-1}\|_p = \|f\|_q^{q-1}$$

Hence (8) be comes

$$\|f\|_q^2 \leq S(n,s)\left\|(-\Delta)^{s/2}\right\|_2^2$$

Notation :

If $x=(x_1,\ldots,x_n)$, $k=(k_1,\ldots,k_n)\in\mathbb{R}^n$, then we denote $(k,x)=k_1x_1+\cdots+x_nk_n$ and $|x| = (x,x)^{1/2}$.

If $f,g \in L^2(\mathbb{R}^n)$, then we denote $(f,g)=\int f(x)g(x)dx$.

Theorem 2: Sobolev's inequality For $n\geq 3$ let f is a function in $C^1(\mathbb{R}^n)$ with compact support. A constant C_n exists depending on the dimension rather than f so that

$$\|f\|_p \leq S_n\|\nabla f\|_2$$

Where

$$p=\frac{2n}{n-2}$$

Which is denoted as Sobolev index.

Remark 1: if $n\geq 3$, does not make a statement in 2 and 3 dimensions.

Remark 2: The Sobolev index can be understood as follows. assuming the inequality holds, pick any function f and consider its as a scaled verion $f(\lambda x)$ with $\lambda > 0$ arbitrary. Then, by changing variables

$$\left(\int_{\mathbb{R}^n}|f(\lambda x)|^p\,dx\right)^{1/p} = \lambda^{-n/p}\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p}$$

Which is

$$\leq C_n\left(\int_{\mathbb{R}^n}|\nabla f(\lambda x)|^2\,dx\right)^{1/2} = \lambda^{1-n/2}C_n\left(\int_{\mathbb{R}^n}|\nabla f(x)|^2\,dx\right)^{1/2}$$

Thus, the λ exponents must necessarily be the same, i.e., $n/p = n/2-1$.

Remark 3:The best possible constant in Sobolev's inequality is defined and it has the value.

$$\frac{n(n-2)}{4}|S^n|^{n/2}$$

where $|S^n|$ is the surface area of the unit n-sphere in $\mathbb{R}^{n+1}$, i.e.,

$$|S^n| = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

The functions which yield equality are of the form

$$\frac{k}{(\mu^2 + |x-a|^2)^{(n-2)/2}}. \qquad k \text{ is constant}.$$

See [1] , [3] and [4] for other proofs.

Theorem 3 : The values of p is a possible for the inequality

$$\|f\|_p \le C_{n,q}\|\nabla f\|_q. \qquad (9)$$

to hold is

$$p = \frac{qn}{n-q}.$$

In particular for $q = 1$, $p = n/(n-1)$.

Proof: the proof of inequality (9) is due to Gagliardo and Nirenberg, we prove it in 3-space .

Using the fundamental theorem of calculus

$$f(x,y,z) = \int_{-\infty}^{x} \partial_x f(r,y,z)dr$$

and in particular

$$|f(x,y,z)| \le \int_{-\infty}^{\infty} |\partial_x f(r,y,z)|\, dr = g_1(y,z).$$

Similarly, repeating the same argument in the other variables

$$|f(x,y,z)|^3 \le g_1(y,z)g_2(x,z)g_3(x,y),$$

and hence

$$\|f\|_{3/2} \le \left(\int \sqrt{g_1(y,z)}\sqrt{g_2(x,z)}\sqrt{g_3(x,y)}dxdydz\right)^{2/3},$$

Using Schwarz' inequality on the x- variable yields the upper bound

$$\left(\int \sqrt{g_1(y,z)}\sqrt{\int g_2(x,z)dx}\sqrt{\int g_3(x,y)dx}\, dydz\right)^{2/3}$$

Applying Schwarz' inequality once more in the y-variable yields

$$\left(\int\sqrt{\int g_1(y,z)dy}\sqrt{\int g_2(x,z)dx}\sqrt{\int g_3(x,y)dxdy}\,dz\right)^{2/3},$$

$$\left(\sqrt{g_1(y,z)dydz}\sqrt{g_2(x,z)dxdz}\sqrt{g_3(x,y)dxdy}\right)^{2/3},$$

$$= \left(\int g_1(y,z)dydz\int g_2(x,z)dxdz\int g_3(x,y)dxdy\right)^{1/3},$$

$$= \left(\|\partial_x f\|_1\|\partial_y f\|_1\|\partial_z f\|_1\right)^{1/3},$$

$$\leq \|\nabla f\|_1.$$

Thus it is established that

$$\|f\|_{3/2} \leq \|\nabla f\|_1 \ . \qquad (10)$$

To arrive at the general inequality, replace f by $|f|^s$ for a number $s > 0$ to be chosen later and calculate

$$\|f^s\|_{3/2} \leq s\||\nabla f||f|^{s-1}\|_1$$

Using Hölder's inequality on the right side yields the estimate

$$\|f^s\|_{3/2} \leq s\||\nabla f|\|_q\||f|^{s-1}\|_{q'} \qquad (11)$$

where $1/q + 1/q' = 1$ or $q' = q/(q - 1)$. Now if we choose $s = 2q/(3 - q)$ so that

$$\frac{3s}{2} = \frac{(s-1)q}{q-1} = \frac{3q}{3-q} = p,$$

we get from (11)

$$\|f\|_p^{2p/3} \leq 2q/(3-q)\left\||\nabla f|\right\|_q\|f\|_p^{p(q-1)/q}$$

and upon dividing both sides by $\|f\|_p^{p(q-1)/q}$ we obtain

$$\|f\|_p^{2p/3-p(q-1)/q} \leq 2q/(3-q)\,\||\nabla f|\|_q,$$

which is our desired inequality. Note, as a check, that

$$p[2/3 - (q - 1)/q] = 1 .$$

Remark 4: The sharp constant in (10) is strongly related to the isoperimetric inequality. This is a substantial subject all by itself and we just touch it with a few remarks. The inequality (10) on $\mathbb{R}^n$ in its sharp form reads as

$$\|f\|_{\frac{n}{n-1}} \leq n^{\frac{-(n-1)}{n}} |S^{n-1}|^{-1/n} \|\nabla f\|_1.$$

In other words, we claim that

$$sup_{f\neq 0} \frac{\|f\|_{\frac{n}{n-1}}}{\|\nabla f\|_1} = n^{\frac{-(n-1)}{n}} |s^{n-1}|^{-1/n}.$$

The constant is exactly the surface area of a ball divided by the $(n - 1)/n$ – th power of its volume. The constant is not attained by any function whose gradient is integrable but can be obtained arbitrarily close as the following calculation shows.

Consider Sobolev inequality (10), in $\mathbb{R}^n$, A among all functions the spherically symmetric functions delivers the worst constant. Thus, it is assumed that all the level sets are rearranged into balls with radius

$$\left[\frac{n}{|S^{n-1}|}\right]^{\frac{1}{n}} |\{x : |f(x)| > \alpha\}|^{1/n}$$

and hence this inequality reads

$$C_n \geq \left[\frac{1}{n-1}\right]^{\frac{n-1}{n}} |S^{n-1}|^{-1/n} sup_{f\neq 0} \frac{\left[\int_0^\infty \alpha^{1/(n-1)} (\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}}{\int_0^\infty \lambda(\alpha) d\alpha} \qquad (12)$$

where $\lambda(\alpha) = |\{x : |f(x)| > \alpha\}|^{\frac{n-1}{n}}$. Two observations about the function $\lambda(\alpha)$: it is a non increasing function and can be assumed that

$\int_0^\infty \lambda(\alpha) d\alpha = 0$ as well as $\lambda(0) = 1$, since the scaling $\lambda(\alpha) \to C\lambda(D\alpha)$ leaves the ratio in (12) fixed. To maximize

$$\left[\int_0^\infty \alpha^{1/(n-1)} (\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}$$

over all such functions $\lambda(\alpha)$ we proceed as follows. The functional

$$\lambda(\alpha) \longmapsto \mathcal{F}(\lambda) = \left[\int_0^\infty \alpha^{1/(n-1)} (\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}$$

is convex. Now restrict the set over which to maximize in order to consist of non-increasing functions haveing the value 1 at $\alpha = 0$, whose integral equals 1 and are 0 outside the interval $[0, N]$ for some N larger values. This set is called T_N and that T_N is a convex set and

$$F(N) = sup_{\lambda \in T_N} \mathcal{F}(\lambda)$$

is a non- decreasing function of N.

Suppose that our functional is convex it attains the maximum on the set T_N at the extreme points which consists of functions which have only 0 and 1 values. Since the function is a non-increasing, has the value 1 at $\alpha = 0$ and integrates to 1.
It must be

$$\lambda_{opt}(\alpha) = \lambda_{[0,1]}(\alpha)\,. \qquad (13)$$

which does not depend on the value of N as long as $N > 1$, and hence inserting the inequality (12), the result is:

$$C_n \geq |S^{n-1}|^{-1/n} n^{-(n-1)/n}\,,$$

which demonstrates our claim.

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