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# Closed Ideal with Respect a Binary Operation * On BCK-Algebra

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**Abstract**

In this paper, we define a new ideal of BCK-algebra, we call it a closed ideal with respect a binary operation ∗, and denoted by (∗ -closed ideal). We stated and proved some properties on closed ideal and give some examples on it.

**Indexing Terms/Keywords:** BCK-algebra, Closed Ideal, A Binary Operation ∗ on BCK-Algebra.

## 1) Introduction

The notion of BCK- algebras was introduced and formulated first in 1966 by Y.Imai and K.Iseki [Y.Imai and K.Iseki, 1966]. In the same year, K.Iseki [ K.Iseki , 1966] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras where the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The notion of a BCI-algebra is a generalization of a BCK-algebra. The general development of BCK/ BCI-algebra the ideal theory plays an important role. We introduce a new ideal of BCK-algebra is called a closed ideal with respect a binary operation ∗, then we study and prove some properties of them.

## 2) Preliminary

In this section we review some concepts we needed in this paper

**Definition 2.1 [**Z.M.Samaei , M.A.Azadani and L.N. Ranjbar, ,2011**]**

Let X be a non-empty set with binary operation "∗" and 0 is a constant an algebraic system (X, ∗, 0) is called a BCK-algebra if it satisfies the following conditions:

1) $((x * y) * (x * z)) * (z * y) = 0$,

2) $(x * (x * y)) * y = 0$,

3) $x * x = 0$,

4) If $x * y = 0$ and $y * x = 0$ then $x = y$, $\forall x, y, z \in X$

5) $0 * x = 0$.

**Remarks 2.2 [**A.A.A. Agboola1 and B. Davvaz2, 2015**]**

Let X be a BCK-algebra then:

a) A partial ordering" ≤" on X can be defined by $x \le y$ if and only if

$x * y = 0$.

b) A BCK-algebra X has the following properties:

1) $x * 0 = x$.
2) If $x * y = 0$ implies $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$.
3) $(x * y) * z = (x * z) * y$.
4) $(x * y) * (x * z) \le (x * z)$.

**Example 2.3**

The set X = {0, 1, 2} with binary operation " * " defined by the following table is a BCK-algebra.

Table 1. BCK-algebra

| * | 0 | 1 | 2 |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **1** | 1 | 0 | 0 |
| **2** | 2 | 2 | 0 |

**Definition 2.4 [**Sun Shin Ahn and Keumseong Bang, 2003**]**

Let (X, ∗, 0) and (X', ∗', 0') be two BCK-algebras. A mapping

f: X → Y is called a homomorphism from X to X' if for any x, y ∈ X, f (x ∗ y) = f (x) ∗' f (y).

Note that If f: X → Y is a homomorphism of BCK-algebras, then f (0) = 0.

**Definition 2.5:**

A mapping f: (X, *, 0) → (Y, *', 0) of BCK-algebras is called an epimorphism if f is a homomorphism and surjective.

**Definition 2.6 [** Young Bae Jun, and Kyoung Ja Lee, 2012**]**

A BCK-algebra is said to be commutative if x ∗ (x ∗ y) = y ∗ (y ∗ x) for any x, y ∈ X

**Example 2.7**

The set X = {0, 1, 2} with binary operation " * " defined by the following table is commutative BCK-algebra.

Table 2. commutative BCK-algebra

| * | 0 | 1 | 2 |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **1** | 1 | 0 | 1 |
| **2** | 2 | 2 | 0 |

**Definition 2.8 [**Young Bae Jun, and Kyoung Ja Lee, 2012**]**

A nonempty subset S of a BCK-algebra X is called a BCK sub algebra of X if x ∗ y ∈ S for all x, y ∈ S.

**Definition 2.9 [**Young Bae Jun, and Kyoung Ja Lee, 2012**]**

A nonempty subset A of a BCK-algebra X is called a BCK ideal of X if it satisfies:

1) 0 ∈ A
2) x ∗ y ∈ A, y ∈ A then x ∈ A and x, y ∈ X

**Proposition 2.10 [**Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015**]**

Let I and J are BCK-algebra of X, then I × J is BCK-algebra of X × X.

**Proposition 2.11 [**Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015**]**

Let A and B are BCK-algebra of X, then A ∩ B is BCK-algebra of X.

**Proposition 2.12 [**Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015**]**

Let A and B are BCK-algebra of X, then A∪B is BCK-algebra of X if A ⊆ B or B ⊆ A.

### 3) Main Results:

In this section, we define a closed ideal with respect a binary operation ∗ of BCK-algebra. We stated and proved some properties on closed ideal and give some examples on it.

**Definition 3.1**

Let X is a BCK-algebra. A non empty subset I of X is said closed ideal with respect a binary operation ∗ and denoted by (∗ -closed ideal) on X if satisfies the following conditions :

1) $a * b \in I \quad \forall\, a, b \in I$
2) $I * X \subseteq I$

**Example 3.2:**

Let X = {0, 1, 2} with binary operations '∗' defined by the following tables is BCK-algebra:

Table 3. (∗ -closed ideal)

| ∗ | **0** | **1** | **2** |
|---|---|---|---|
| **0** | **0** | **0** | **0** |
| **1** | **1** | **0** | **1** |
| **2** | **2** | **0** | **0** |

Then by usual calculation we can prove that I = {0, 1} ⊆ X is (∗ -closed ideal)

**Example 3.3:**

Let X = {0, 1, 2, 3} with binary operations '∗' defined by the following tables is BCK-algebra:

Table 4. is not (∗ -closed ideal)

| **∗** | **0** | **1** | **2** | **3** |
|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** |
| **1** | **1** | **0** | **3** | **2** |
| **2** | **2** | **0** | **0** | **0** |
| **3** | **3** | **0** | **0** | **0** |

Then I = {0, 1, 2} ⊆ X is not (∗ -closed ideal) since 1∈ I and 2∈ I but 1 ∗ 2 = 3 ∉ I

**Remark 3.4**

If I is (∗ -closed ideal) of BCK-algebra, then, 0 ∈ I

**Proof**

Let I be ($*$ -closed ideal) so $I \neq \emptyset$. Then $\exists\, a \in I$,

then $a * x \in I \ \forall x \in X$ [by 2 of definition 3.1]

So, $0 = a * a \in I$ , and therefore $0 \in I$.

**Remark 3.5**

If I is ($*$ -closed ideal) of BCK-algebra, then I is sub algebra.

**Proof**

Let I is ($*$ -closed ideal) of BCK-algebra and let $a, b \in I$

$\Rightarrow a * b \in I \ \Rightarrow$ I is sub algebra.

**Remark 3.6**

The converse of above remark in general is not true.

**Proof**

We will prove it by using the example (3.3):

Take $I = \{0, 1\} \subseteq X$ it is clear that is a sub algebra but I is not ($*$ -closed ideal)

since $I * x \not\subset I$ where $1 \in I$ and $3 \in X$ but $1 * 3 = 2 \notin I$ .

**Proposition 3.7**

Let X is BCK-algebra and let A, B ($*$ -closed ideal) of X Then $A \cap B$ is ($*$ -closed ideal) of X

**Proof**

Let X is BCK-algebra and since $A \cap B \neq \emptyset$ by (3.4)

1) Let $a , b \in A \cap B \ \Rightarrow\ a , b \in A$ and $a , b \in B$
   Since A, B are ($*$ -closed ideal) then $a * b \in A$ and $a * b \in B \ \Rightarrow\ a * b \in A \cap B$
2) Let $a \in A \cap B$ and $x \in X \ \Rightarrow\ a \in A$ and $a \in B$ and $x \in X$
   $\Rightarrow\ a * x \in A$ and $a * x \in B$; [since A and B ($*$ -closed ideal)]
   $\Rightarrow\ a * x \in A \cap B \Rightarrow (A \cap B) * X \subseteq (A \cap B)$,

then $A \cap B$ is ($*$ -closed ideal).

**Remark 3.8**

The converse of above remark is not true in general.

Take $A = \{0, 1\}$ and $B = \{0, 1, 2\}$ in (example 3.3) then:

$A \cap B = \{0, 1\}$ is ($*$ -closed ideal) but $B = \{0, 1, 2\}$ is not ($*$ -closed ideal); since $1 * 2 = 3 \notin B$

**Remark 3.9**

Let X is BCK-algebra and let A, B ($*$ -closed ideal) of X. Then $A \cup B$ is ($*$ -closed ideal) of X if $A \subseteq B$ or $B \subseteq A$, and the converse is not true in general.

**Proof**

Proof is clear now, we show that the converse is not true in general; since if we take A, B and $A \cup B$ are ($*$ -closed ideal) of X

Table 5. the converse is not true in general.

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 2 |
| 3 | 3 | 0 | 0 | 0 |

A = {0, 1} is ($*$ -closed ideal)

B = {0, 2} is ($*$ -closed ideal), $A \cup B$ = {0, 1, 2} is ($*$ -closed ideal),

but $A \not\subset B$ and $B \not\subset A$

**Proposition 3.10**

Let f: X → Y is BCK-algebra homomorphism. Then ker f is ($*$ -closed ideal) of X.

**Proof**

Let f: X → Y is BCK-algebra homomorphism. Then

1) $a, b \in \ker f \Rightarrow f(a) = 0$ and $f(b) = 0$
   $\Rightarrow f(a * b) = f(a) * f(b) = 0 * 0 = 0 \Rightarrow f(a * b) = 0 \Rightarrow a * b \in \ker f$
2) let $a \in \ker f$ and $x \in X \Rightarrow f(a) = 0$
   $\Rightarrow f(a * x) = f(a) * f(x)$; [since f is a homomorphism ]
   $= 0 * f(x) = 0$; [by 5 of definition 2.1]
   $\Rightarrow f(a * x) = 0 \Rightarrow a * x \in \ker f \;\; \forall a \in \ker f$ and $x \in X$
   $\Rightarrow \ker f * X \subseteq \ker f$

Then ker f is ($*$ -closed ideal)

**Proposition 3.11**

Let f: X → Y is BCK-algebra epimorphism if A is ($*$ -closed ideal) of X, then f(A) is ($*$ -closed ideal) of Y.

**Proof**

Let f: X → Y is BCK-algebra epimorphism. Let A be ($*$ -closed ideal) of X then:

1) Let $x', y' \in f(A)$, then $\exists\; x, y \in A$ such that $x'=f(x)$, $y'= f(y)$,
   since A is ($*$ -closed ideal) $\Rightarrow x * y \in A \Rightarrow f(x * y) \in f(A)$
   but $f(x * y) = f(x) * f(y) \Rightarrow f(x) * f(y) \in f(A)$ So $x' * y' \in f(A)$
2) Let $a' \in f(A)$ and $y \in Y$ since f is an epimorphism
   $\Rightarrow \exists\, a \in A$ and $x \in X$ such that $f(a) = a'$ and $f(x) = y$
   $\Rightarrow a * x \in A$; [since A is ($*$ -closed ideal)]
   $\Rightarrow f(a * x) \in f(A) \Rightarrow f(a) * f(x) \in f(A)$; [since f is a homomorphism]
   $\Rightarrow a' * y \in f(A) \;\; \forall a' \in f(A)$ and $y \in Y$
   $\Rightarrow f(A) * Y \subseteq f(A)$

Then, f(A) is ($*$ -closed ideal).

**Proposition 3.12**

Let X is BCK-algebra and let f: X → X' is BCK-algebra homomorphism of X if B is ($*$ -closed ideal) of X',

then f -1(B) = {$a \in X$: $f(a) \in B$} is ($*$ -closed ideal) of X.

**Proof**

Let f: X → X' is BCK-algebra homomorphism of X if B is ($*$ -closed ideal) of X', then:

1) Let $a, b \in$ f -1(B) $\Rightarrow f(a), f(b) \in B$
   Since B is ($*$ -closed ideal) then:
   $f(a) * f(b) = f(a * b) \in B$; [since B is ($*$ -closed ideal)]
   $\Rightarrow a * b \in$ f -1(B)
2) Let $a \in$ f -1(B) and $x \in X$ so $f(x) \in X' \Rightarrow f(a) \in B$ and $f(x) \in X'$
   $\Rightarrow f(a) * f(x) = f(a * x) \in B$; [since B is ($*$ -closed ideal)]
   $\Rightarrow a * x \in$ f -1(B) $\forall a \in$ f -1(B) and $x \in X$
   $\Rightarrow$ f -1(B) $* X \subseteq$ f -1(B) $\Rightarrow$ f -1(B) is ($*$ -closed ideal).

**Proposition 3.13**

Let X is BCK-algebra and let I , J be ($*$ -closed ideal) of X. Then $I \times J$ is ($*$ -closed ideal) of $X \times X$.

**Proof**

Let X is BCK-algebra, and let I, J be ($*$ -closed ideal) of X

1) Let $x = (a, a') \in I \times J$ and $y = (b, b') \in I \times J$
   $\Rightarrow x * y = (a, a') * (b, b') = (a * b, a' * b')$

then $a * b \in I$ and $a' * b' \in J$; [since I, J are ($*$ -closed ideal)]
$\Rightarrow (a * b, a' * b') \in I \times J$ so $x * y \in I \times J$

2) Let $(x1, x2) \in X \times X$ and $(a1, a2) \in I \times J$
$\Rightarrow a1 * x1 \in I$, $a2 * x2 \in J$ because I and J are ($*$ -closed ideal)

Then $(a1, a2) * (x1, x2) = (a1 * x1, a2 * x2) \in I \times J$ Then $I \times J$ is ($*$ -closed ideal)

**Proposition 3.14**

Let X is BCK-algebra and let $I' = \{(a , 0) / a \in X\}$ and $J' = \{(0 , b) / b \in X\}$.

Then I' and J' are ($*$ -closed ideal) of $X \times X$.

**Proof**

Let X is BCK-algebra to prove that I' is ($*$ -closed ideal).

1) Let $x, y \in I' \Rightarrow x = (a, 0), y = (b, 0)$
$\Rightarrow x * y = (a , 0) * (b , 0) = (a * b , 0) \in I'$; [since $a * b \in X$]
$\Rightarrow x * y \in I'$

2) Let $x = (a, 0) \in I'$ and $t = (r, s) \in X \times X$
$\Rightarrow x * t = (a , 0) * (r , s) = (a * r, 0 * s) = (a * r , 0)$; [by 5 of definition 2.1]
$\Rightarrow x * t = (a * r , 0) \in I'$; [ since $a * r \in X$ ]
$\Rightarrow I' * X \times X \subseteq I'$ then I' is ($*$ -closed ideal) of $X \times X$.

In a similar way, we can prove that J' is ($*$ -closed ideal) of $X \times X$.

**Remark 3.15**

Let X is BCK-algebra and let I' and J' be defined as in the above proposition.

Then $I' \cap J' = (0, 0)$.

**Proof**

Let X is BCK-algebra and let I' and J' is ($*$ -closed ideal) and

let $x \in I' \cap J' \Rightarrow x \in I'$ and $x \in J'$ then $x = (a , 0)$ and

$x = (0, b)$ where $a \in X$ and $b \in X \Rightarrow (a, 0) = (0, b) \Rightarrow a = 0, b = 0$

$\Rightarrow x = (0, 0) \Rightarrow I' \cap J' = (0, 0)$.

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