Using An Accelerating Method With The Trapezoidal And Mid-Point Rules To Evaluate The Double Integrals With Continuous Integrands Numerically

Azal Taha Abdul Wahab[1*] Rusul Hassan Naser[2]

1. Faculty of Education for girls\University of Kufa, Najaf, IRAQ.
2. Faculty of Education for girls\University of Kufa, Najaf, IRAQ.

* E-mail of the corresponding author: Azalt.almussawy@uokufa.edu.iq

Abstract

In this research we have used a compound method which is composed of Trapezoidal and Mid-Point Rules to evaluate the approximate values of the double Integrals with Continuous Integrands because of the resultant approximate values are fast when approximating from the true values of integrals if compared with other Newton -Cotes formulas. [S.S. Sastry, 2008". The symbol of this rule is *TM*, in this method 2n is, the number of subintervals [a,b], equivalent to 2m (the number of subintervals of [c,d] and (h=ħ), we accelerate the resultant approximate to the true values of integrals through applying Aitken's accelerate on the compound rule *TM* to procure a new method we called it *A(TM)* where the symbol A indicates to the Aitken's acceleration method and the symbol *TM* (*y* indicates to the Trapezoidal rule on the external and *x* indicates to the Mid-point rule on the internal dimensions).

Keywords: double integrals, Trapezoidal and Mid-Point Rules and Newton-Cotes

Introduction

Numerical analysis topic is characterized by innovating new several various methods to reach approximate solutions of certain mathematics problems effectively, since the method efficiency depends on smoothness when applying it, therefor the selection of the appropriate method for approximating problem solution gets effected by change that occurs in computer technology.

The importance of double Integrals in finding surface area mid-point positions and moment of inertia of flat surface and to find a volume laying under a double Integrals surface, all that urge many researchers to work in double Integrals field, among those who shed light on evaluating Integrals with Continuous Integrands of the formulas $f(x, y)= f_1(x)f_2(y)$ are Hans Schjar and Jacobsen in 1973[Hans Schjar and Jacobsen 1973], some studied Integrals with improper Integrands but they neglected impropriety as Phillip J. Davis and Phillip Rabinowitz in 1975[3][Phillip J. Davis and Phillip Rabinowitz, 1975].

In 2010, Akar [Master thesis submitted by Batool Hatam Akar, 2010] used a new method where she introduced a numerical method to evaluate double Integrals values through using Romberg's method on the resultant values of Mid-point Rules in two dimensions *x* and *y* when number of subintervals of the internal integrals are equivalent to number of subintervals of the external integrals, where ($h=\hbar$) we call it *RMM* where (*MM*) symbolizes to Mid-point rule method that is applied on both dimensions and *R* indicates to acceleration method and good results were obtained with a few number of used subinterval.

In 2012, the researcher, Nada Ahmed Muhammed [Master thesis submitted by Muhammed, Nada Ahmed, 2012], presented three new theorems by using the Trapezoidal and Mid-Point Rules with derivation of error formula to each method she called it *TM*, *TT and MT*, and accelerated reaching to the results by using Richard's accelerate. Applying Richard accelerate with the three rules have given good results regarding accuracy when applied on continuous double integrands. In this research we apply Aitkin accelerate on three values resulting from application of *TM* to calculate the approximate values of double integrals with continuous integrals when *2n* the symbol of the number of subintervals [a, b], is equivalent to *2m* equal to number of subintervals of [*c*, *a*] and we have obtained a new method and called it *ATM*, this rule gives faster results in reaching to analytical analysis than the rule *TM* when applying on the continuous integrands.

2- *TM* Method

To derive a method to evaluate the double Integrals with Continuous Integrands Numerically, we assume the double Integrals *I* it is defined as following

$$I = \int_c^d \int_a^b f(x, y)dxdy \qquad \text{.................... (1)}$$

Where *f(x, y)* is a Continuous Integrands in each point of Integrals area [*a*, *b*] ×[*c*, *d*]

In general, the integral can be written as following:

$$I = \int_c^d \int_a^b f(x, y)dxdy = GG(h) \qquad \text{.................. (2)}$$

Where *GG(h)* represents Integrals value numerically by using of the formulas TM and

$$h = \frac{(b-a)}{n} = \frac{(d-c)}{m}$$

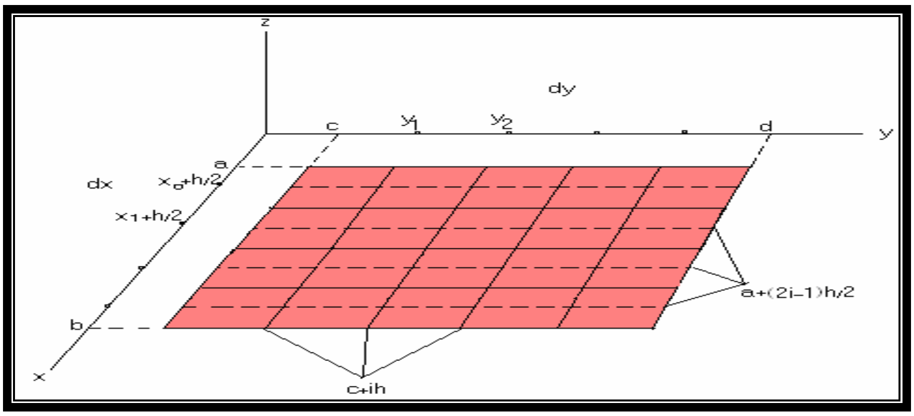
(We will choose $m=n$ because of attaining better and faster results when reaching to the true values. Akkar [Master thesis submitted by Batool Hatam Akar, 2010] and Nassir [Master thesis submitted by Nassir, Rusul Hassan, 2011])

The formulas *TM* can be obtained through application Mid-Point rule on the internal dimension x and the Trapezoidal rule on the external dimension when $m=n$ (n represents the number of subintervals [*a*, *b*] and m represents number of subintervals of [*c*, *d*] which means ($h=\hbar$), where

$$, \quad \bar{h} = \frac{d-c}{m} \quad , \quad h = \frac{b-a}{n}$$

Nada Ahmed Muhammed [Master thesis submitted by Muhammed, Nada Ahmed, 2012]

$$3- \quad \int_c^d \int_a^b f(x, y) = \frac{h^2}{2} \sum_{i=0}^{n-1} \left(f(x_i + \tfrac{h}{2}, c) + f(x_i + \tfrac{h}{2}, d) + 2\sum_{j=1}^{n-1} f(x_i + \tfrac{h}{2}, y_j) \right) \qquad \text{... (3)}$$



Geometric representation illustrates integral area and two -dimension integral subinterval according to the rule.

When we start with placing *n=m=2* into the formula above, that means we evaluate the approximate values of the double Integrals *I* with *TM* method and write that value in the table when *n=m=2*, then we place *n=m=4* and evaluate *TM*, also we write that value in the table as it is the approximate values of the double Integrals, by using the same method to evaluate the approximate values of the double Integrals when *n=m=8*. Thus, it is reachable to attain a better value through using each three values from *TM* rule by applying Aitken's acceleration method to procure better value for integral.

3- *TM* Method with Aitken's acceleration method.

Aitken's acceleration method is a process to accelerate the convergence of the approximate values to the true values of integrals. To apply *TM* method with Aitken's acceleration method, we use the following rule:

$$A(TM) \cong \frac{(\mathrm{TM(h)} \times (\mathrm{TM}(\frac{h}{4})) - (TM(\frac{h}{2}))^2}{(TM(\frac{h}{4})) - 2(TM(\frac{h}{2})) + (TM(\mathrm{h}))} \qquad ...(4)$$

Where *A(TM)* is a value in the new column and each of

$$TM(h), \mathrm{TM}(\frac{h}{2}), \mathrm{TM}(\frac{h}{4})$$

are values in the column that lays before the first one, and we will apply them in the mentioned rule to accelerate obtaining better values of integrals instead of continuity in increasing the subintervals, so if it was for instance, *n* represents a value in the *TM* rule we will get *n-2* value when applying Aitken's accelerate on it through use of the equation (6), then we also use Aiken's on *n-2* value that leads to get *n-4* value. Thus, we continue conducting that till we obtain the desirable accuracy. Nassir [Master thesis submitted by Nassir, Rusul Hassan, 2011]

Significant notes concerning the selected examples:

1- In *TM* rule we used the values *m= 1, 2, 4, 8, 16, …. n= 1, 2, 4, 8, 16, ….* Nassir [Master thesis submitted by Nassir, Rusul Hassan, 2011]
2- Numerical values of integrals are evaluated through MATLAB programming language, which is considered one of the most significant programs that presents solutions in mathematic field and what emanates of it from engineering specialties that depend basically upon mathematics.

3. **Examples:-** The integrand of integral $I = \int_1^2 \int_1^2 ln(x+y)dxdy$

defined of each $(x, y) \in [1,2]\,[1,2]$ and it's analytical value is 1.0891386520 which is approximated to ten decimal places

$A(TM)$	$A(TM)$	TM	$m=n$
		1.0879317225	2
		1.0888332015	4
	1.0891399121	1.0890620506	8
	1.0891387320	1.0891194867	16
1.0891386520	1.0891386571	1.0891338598	32

Table (1) double integral evaluation $I = \int_1^2 \int_1^2 \ln(x+y)dxdy$ **by using** $A(TM)$ **method**

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We conclude from the table (1) the following: when $n = m = 32$, the value in *TM* rule is true only for the five decimal places. When using Aitken's acceleration method with this rule we have obtained a value that is identical with analytical value which is true for ten decimal places with (210 subinterval).

For evaluating the double integral numerically

$$I = \int_0^1 \int_0^1 xe^{-(x+y)} dxdy \quad ,$$

it is obvious that integrand is defined to each $(x, y) \in [0,1]\ [0,1]$ and it's analytical value is 0.1670322429 which is approximated to ten decimal places

$A(TM)$	$A(TM)$	TM	$m = n$
		0.1771090298	2
		0.1695491974	4
	0.1670329578	0.1676613238	8
	0.1670322950	0.1671895030	16
0.1670322425	0.1670322463	0.1670715573	32
0.1670322429	0.1670322432	0.1670420715	64

Table (2) double integral calculation $I = \int_0^1 \int_0^1 xe^{-(x+y)} dxdy$ **by using** $A(TM)$ **method**

We notice in this table, when $n=m=64$, the value in *TM* rule is true only for four decimal places. When using Aitken's acceleration method with this rule we have attained a value that is identical with analytical value, which is true for ten decimal places, when $n=m=64$ (2^{12} subinterval)

Also the integrand of integral

$$I = \int_0^1 \int_0^1 \sin(\frac{\pi}{2}(x+y)) dxdy$$

is defined of each $(x, y) \in [0,1]\ [0,1]$ and it's analytical value is 0.810569469 which is approximated to nine decimal places

$A(TM)$	$A(TM)$	TM	$m = n$
		0.788580507	2
		0.805290378	4
	0.810501997	0.809262986	8
	0.810565122	0.810243674	16
0.810569476	0.810569195	0.810488072	32
0.810569469	0.810569452	0.810549123	64

Table (3) double integral calculation $I=\int_0^1\int_0^1 \sin(\frac{\pi}{2}(x+y))dxdy$ **by using** $A(TM)$ **method.**

We notice in this table when *n=m=64*, the value in *TM* rule is true only for four decimal places. When using Aitken's acceleration method with this rule we have obtained a value that is identical with analytical value, which is true for nine decimal places, with (2^{12} subinterval)
Also the integrand of integral

$$I=\int_1^2\int_0^1 \frac{\ln(x+y)}{x+y}dxdy$$

defined of each (*x*, *y*) ∊ [0,1] [0,1] and it's analytical value 0.3262692696 which is approximated to ten decimal places. The table (4) illustrates the results.

$A(TM)$	$A(TM)$	TM	$m=n$
		0.3233689196	2
		0.3254535167	4
	0.3263063117	0.3260587250	8
	0.3262715900	0.3262162019	16
0.3262692701	0.3262694154	0.3262559754	32
0.3262692696	0.3262692787	0.3262659444	64

Table (4) illustrates that the numerical value of integral $I=\int_1^2\int_0^1 \frac{\ln(x+y)}{x+y}dxdy$ **by using *A(TM)* Method**

It is obvious from this table that the numerical value

$$I=\int_1^2\int_0^1 \frac{\ln(x+y)}{x+y}dxdy$$

When *n=m=64* rule is true only for five decimal places by using *A(TM)* method and by applying Aitken's acceleration method on the resulted values, we have obtained a value that is identical with (the analytical value), which is true for ten decimal places, when *n=m=64*, that means (2^{12} subinterval)

5- Discussion

It is obvious from what have come in all tables mentioned above that the application of *TM* rule alone for Continuous Integrands will give true approximate values for four of five decimal places when $m=n$ (n represents the number of subintervals [a,b] and m represents the number of subintervals [a,b] [c,d] that means (h=ħ), we could accelerate the results through implementing Aitken's acceleration method on the resultant values and obtaining numerical values identical to true analytical values for ten decimal places in some examples, nine decimal places in other examples by using relatively a few subintervals and in a short time where the time was calculated through MATLAB program language.

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