Fourier Cosine and Sine Transform with Product of Polynomial Function

Getachew Abiye Salilew

Department of Mathematics, College of Natural and Computational Science
Madda Walabu University, Bale Robe, Ethiopia
e-mail: getachewabye@gmail.com

Abstract

The purpose of this paper is to provide presumably new properties of Fourier cosine and sine transform of a function with product of a polynomial function. The author presented very short form of general properties of Fourier cosine and sine transform with product of a polynomial function having coefficients which are numbers.

Key words: Fourier transform, Fourier cosine transform, Fourier sine transform

1. Introduction and preliminaries

The theory of Fourier series and the Fourier transform is concerned with dividing a function into a superposition of sine and cosine, its components of various frequencies. It is a crucial tool for understanding waves, including water waves, sound waves and light waves. Fourier analysis is a mathematical technique which enables us to decompose an arbitrary function into a superposition of oscillations which can be resolved into a sum of sine and cosine. The theory of Fourier series can be used to analyze the flow of heat in a bar and the motion of a vibrating string. Joseph Fourier a 21 years old mathematician and engineer announced a thesis which began a new chapter in the history of mathematics. Fourier's original investigations which led to the theory of Fourier series were motivated by an attempt to understand heat flow [1-3]. Nowadays, the notion of dividing a function into its components with respect to an appropriate "orthonormal basis of functions" is one of the key ideas of applied mathematics, useful not only as a tool for solving partial differential equations but also for many other purposes as well. In this paper we presented provide presumably new general properties regarding to the Fourier cosine and sine transform with product of a polynomial function having coefficients which are numbers, by using very elementary ways.

Definition

The Fourier cosine transform (FCT) of the function $f(x)$ is given by

$$\mathcal{F}_c(f(x)) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty f(x)\cos(x\omega)\,dx.$$

The Fourier sine transform (FST) of the function $f(x)$ is given by

$$\mathcal{F}_s(f(x)) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty f(x)\sin(x\omega)\,dx.$$

The Fourier transform (FT) of the function $f(x)$ is given by

$$\mathcal{F}_s(f(x)) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{-ix\omega}\,dx.$$

Where c and s represent cosine and sine respectively [4-7].

Lemma: **(**Properties of Fourier Cosine and Sine Transform)

Let $f(x)$ has both Fourier cosine and sine transform. For an even natural number n, we have:

$$\mathcal{F}_c(x^n f(x)) = (-1)^{\frac{n}{2}} \frac{d^n}{d\omega^n}[\mathcal{F}_c(f(x))].$$

And for an odd natural number n, we have:

$$\mathcal{F}_c(x^n f(x)) = (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^n}{d\omega^n}[\mathcal{F}_s(f(x))].$$

Proof. We can deduce the properties here by using the very recent work [8]. We omit the details.

Here, we aim to present to equivalent identities for this lemma. That is, for an even natural number n, we have:

$$\mathcal{F}_s(x^n f(x)) = (-1)^{\frac{n}{2}} \frac{d^n}{d\omega^n}[\mathcal{F}_s(f(x))].$$

And for an odd natural number n, we have:

$$\mathcal{F}_s(x^n f(x)) = (-1)^{\left(\frac{n+1}{2}\right)} \frac{d^n}{d\omega^n}[\mathcal{F}_c(f(x))].$$

Since the proof is similar to the above lemma we omit the details.

2. Main Theorem

Theorem: Consider a polynomial function, $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, with coefficients are numbers and assume that $f(x)$ has both Fourier cosine and sine transform. Then the Fourier cosine transform of a new function, $P_n(x)f(x)$, given by:

For an even natural number n, we have:

$$\mathcal{F}_c\big(P_n(x)f(x)\big) = \sum_{k=0}^{n} a_k(-1)^{\frac{k}{2}} \frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big] + \sum_{l=1}^{n-1} a_l(-1)^{\left(\frac{l+3}{2}\right)} \frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big]$$

Where $l = 1, 3, 5, \ldots, n-1$; $k = 0, 2, 4, 6, \ldots, n$.

For an odd natural number n, we have:

$$\mathcal{F}_c\big(P_n(x)f(x)\big) = \sum_{l=1}^{n} a_l(-1)^{\left(\frac{l+3}{2}\right)} \frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big] + \sum_{k=0}^{n-1} a_k(-1)^{\frac{k}{2}} \frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big]$$

Where $l = 1, 3, 5, \ldots, n$; $k = 0, 2, 4, 6, \ldots, n-1$.

Proof. Let $f(x)$ has Fourier cosine and sine transform and for an even natural number n, we have:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

$$P_n(x) = (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0) + (a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_3x^3 + a_1x).$$

$$P_n(x) = h(x) + g(x).$$

Where $h(x) = a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0$ implies all powers of variable x are even and

$g(x) = a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_3x^3 + a_1x$ with all powers of variable x are odd. The Fourier cosine transform of a new function, $h(x)f(x)$, given by

$$\mathcal{F}_c\big(h(x)f(x)\big) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty h(x)f(x)\cos(x\omega)\,dx$$

$$\mathcal{F}_c\big(h(x)f(x)\big) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \{a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0\} f(x)\cos(x\omega)\, dx.$$

By using lemma and arranging the powers of suitable terms, after simplification, we obtain

$$\mathcal{F}_c\big(h(x)f(x)\big) = a_0\mathcal{F}_c\big(f(x)\big) - a_2\frac{d^2}{d\omega^2}\big[\mathcal{F}_c\big(f(x)\big)\big] + \cdots + a_n(-1)^{\frac{n}{2}}\frac{d^n}{d\omega^n}[\mathcal{F}_c(f(x))]$$

$$\mathcal{F}_c\big(h(x)f(x)\big) = \sum_{k=0}^{n} a_k(-1)^{\frac{k}{2}}\frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big]. \qquad (1)$$

Where $k = 0, 2, 4, 6, \ldots, n$.

Similarly the Fourier cosine transform of a new function, $g(x)f(x)$, given by

$$\mathcal{F}_c\big(g(x)f(x)\big) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} g(x)f(x)\cos(x\omega)\, dx$$

$$\mathcal{F}_c\big(g(x)f(x)\big) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \{a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_3x^3 + a_1x\} f(x)\cos(x\omega)\, dx.$$

Using lemma and arranging the powers of suitable terms, after little algebra, we obtain

$$\mathcal{F}_c\big(g(x)f(x)\big) = a_1\frac{d}{d\omega}\big[\mathcal{F}_s\big(f(x)\big)\big] - a_3\frac{d^3}{d\omega^3}\big[\mathcal{F}_s\big(f(x)\big)\big] + \cdots + a_{n-1}(-1)^{\left(\frac{n+2}{2}\right)}\frac{d^{n-1}}{d\omega^{n-1}}[\mathcal{F}_s(f(x))]$$

$$\mathcal{F}_c\big(g(x)f(x)\big) = \sum_{l=1}^{n-1} a_l(-1)^{\left(\frac{l+3}{2}\right)}\frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big]. \qquad (2)$$

Where $l = 1, 3, 5, \ldots, n-1$.

Combining (1) and (2), the Fourier cosine transform of a new function, $P_n(x)f(x)$, obtained as:

$$\mathcal{F}_c\big(P_n(x)f(x)\big) = \sum_{k=0}^{n} a_k(-1)^{\frac{k}{2}}\frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big] + \sum_{l=1}^{n-1} a_l(-1)^{\left(\frac{l+3}{2}\right)}\frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big].$$

Similarly for an odd natural number n, we have:

$$P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

$$P_n(x) = (a_nx^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x) + (a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_2x^2 + a_0).$$

$$P_n(x) = q(x) + r(x).$$

Where $q(x) = a_nx^n + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_1x$ implies all powers of variable x are odd and

$r(x) = a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_2x^2 + a_0$ with all powers of variable x are even. Then using the previous suitable properties with simplification of little algebra the Fourier cosine transform of a new functions, $q(x)f(x)$ and $r(x)f(x)$, are given by

$$\mathcal{F}_c\big(q(x)f(x)\big) = \sum_{l=1}^{n} a_l(-1)^{\left(\frac{l+3}{2}\right)} \frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big] \;\; and \;\; \mathcal{F}_c\big(r(x)f(x)\big) = \sum_{k=0}^{n-1} a_k(-1)^{\frac{k}{2}} \frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big]$$

respectively.

Thus the Fourier cosine transform of a new functions, $P_n(x)f(x)$, are given by

$$\mathcal{F}_c\big(P_n(x)f(x)\big) = \mathcal{F}_c\big([q(x)+r(x)]f(x)\big) = \sum_{l=1}^{n} a_l(-1)^{\left(\frac{l+3}{2}\right)} \frac{d^l}{d\omega^l}\big[\mathcal{F}_s\big(f(x)\big)\big] + \sum_{k=0}^{n-1} a_k(-1)^{\frac{k}{2}} \frac{d^k}{d\omega^k}\big[\mathcal{F}_c\big(f(x)\big)\big].$$

Where $l = 1, 3, 5, \dots, n$; $k = 0, 2, 4, 6, \dots, n-1$.

3. Conclusions

The objective of this paper is to provide a brief representation of any function in the integral form. Fourier cosine and sine transform of a function after multiplying the given function by a polynomial function with coefficient are numbers provide the relationship between Fourier cosine and sine transform. Significance of this study will help to represent the solutions of ODEs, PDEs, and integral equations that involving polynomial function terms in the integral form of simper functions of cosine and sine.

References

1. Stephen L. Campbell and Richard Haberman, *Solutions Manual to Introduction to Differential Equations with Dynamical Systems*, Princeton University Press, 2008.
2. John Douglas Moore, *Introduction to Partial Differential Equations*, 2003.
3. James S. Walker, University of Wisconsin-Eau clarie.
4. Linear partial differential equations for scientists and engineers, Tyn Myint-U Lokenath Debnath, 2007.
5. Peter J. Collins, Differential equation and integral equation, 2006.
6. James R. Brannan, William E. Boyce, Differential equations, 2nd edition.
7. Peter V. Oneil, Advanced engineering mathematics, 7th edition.
8. Shiferaw Geremew Kebede, *Properties of Fourier cosine and sine transforms*, IJSBAR, 35(3) (2017) 184-193.