Some results on the Lomax distribution with three parameters

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Abstract

In this paper, we consider a new way to see the three parameters Lomax distribution and we study the main properties of this distribution, with special emphasis on its expectation, variance, quantiles and some characteristics related to reliability studies. We briefly describe different methods of estimations, namely maximum likelihood estimators, least squares estimators, weighted least squares, maximum product spacing estimates and method of Cramer-von-Misses and and compare them using extensive numerical simulations. Applications reveals that the model proposed can be very useful in fitting real data. Two applications are carried out on real data to show the potentiality of the proposed family.

Keywords: exponential distribution, Lomax distribution, Order Statistics, Maximum Likelihood Estimation, Quantile function, Generating Function, Moments.

1 Introduction

In statistics literature we have many continuous univariate distributions. Some classical distributions have been used for modelling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. The aim of this paper is to consider Lomax distribution with three parameters.

Definition 1. *[11] A continuous random variable X is said to have a Lomax distribution if it has probability density function*

$$p(x,\alpha,\beta)=\frac{\alpha\lambda^{\alpha}}{(x+\beta)^{\alpha+1}},\qquad x>0,\alpha>0,\beta>0. \tag{1}$$

The pdf of the Lomax distribution is a solution to the following differential equation:

$$\left\{\begin{array}{l}(\lambda+x)p'(x)+(\alpha+1)p(x)=0,\\ p(0)=\frac{\alpha}{\lambda}\end{array}\right\}$$

The Lomax distribution is a Pareto Type I distribution shifted so that its support begins at zero. Specifically: If $Y\sim Pareto(x_m=\lambda,\alpha)$, then $Y-x_m\sim Lomax(\lambda,\alpha)$[29].

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The Lomax distribution is a Pareto Type II distribution with $x_m = \lambda$ and $\mu = 0$[29]. If $X \sim Lomax(\lambda, \alpha)$ then $X \sim \mathrm{P(II)}(x_m = \lambda, \alpha, \mu = 0)$. [29]

Recently, Lomax distribution has received much attention. Ghitany et al. (2007) proposed Marshall-Olkin extended Lomax distribution. Lemonte et al. (2013) proposed an extended Lomax distribution. Abdul-Moniem and Abdel-Hameed (2012) generalized the Lomax distribution by powering a positive real number α to the cumulative distribution function (cdf). This new family of distributions called exponentiated Lomax distribution. For any continuous baseline cumulative distribution function (cdf) $G(x)$,the cumulative distribution function of the Lomax generator distribution [16] is defined by:

$$F(x, \alpha, \beta, \boldsymbol{\xi}) = 1 - \left\{ \frac{\beta}{\beta - \ln\left[1 - G(x, \boldsymbol{\xi})\right]} \right\}^{\alpha}. \tag{2}$$

The probability density function (pdf)associated with Equation (2) is given by

$$f(x, \alpha, \beta, \boldsymbol{\xi}) = \alpha\beta^{\alpha} \frac{g(x, \boldsymbol{\xi})}{\left[1 - G(x, \boldsymbol{\xi})\right] \left\{\beta - \ln\left[1 - G(x, \boldsymbol{\xi})\right]\right\}^{\alpha+1}}. \tag{3}$$

Definition 2. *[8] A continuous random variable X is said to have an exponential distribution if it has probability density function*

$$g(x, \lambda) = \lambda e^{-\lambda x}, \qquad x > 0, \lambda > 0. \tag{4}$$

where $\lambda > 0$ is called the rate of the distribution.

The cdf of exponential distribution is given by:

$$G(x, \lambda) = 1 - e^{-\lambda x}, \qquad x > 0, \lambda > 0. \tag{5}$$

In this article we consider a new way of Lomax distribution with three parameters by inserting (5) in (2).

Definition 3. *[16] A random variable X is said to have a Lomax distribution with three parameters if it has the density:*

$$f(x, \alpha, \beta, \lambda) = \frac{\alpha\beta^{\alpha}\lambda}{\left[\beta + \lambda x\right]^{\alpha+1}}, \qquad x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{6}$$

The cumulative distribution function associated with Equation (6) is given by

$$F(x, \alpha, \beta, \lambda) = 1 - \left[\frac{\beta}{\beta + \lambda x}\right]^{\alpha}, \alpha > 0, \beta > 0, \lambda > 0. \tag{7}$$

We can see that (6) is positive and

$$\int_0^\infty f(x, \alpha, \beta, \lambda) dx = \int_0^\infty \frac{\alpha\beta^{\alpha}\lambda}{\left[\beta + \lambda x\right]^{\alpha}} dx$$

$$= |\beta + \lambda x = t, \qquad \lambda dx = dt| = \alpha\beta^{\alpha}\lambda \int_\beta^\infty \frac{\frac{1}{\lambda} dx}{t^{\alpha+1}}$$

$$= \alpha\beta^{\alpha} \frac{t^{-\alpha}}{-\alpha}\Big|_\beta^\infty = 1.$$

2 Reliability Analysis

2.1 Survival function

The reliability function (survival function) of Lomax distribution with three parameters is given by

$$R(x,\alpha,\beta,\lambda) = 1 - F(x,\alpha,\beta,\lambda) = \left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}, \alpha > 0, \beta > 0, \lambda > 0. \quad (8)$$

2.2 Hazard Rate Function

The hazard rate function(failure rate) of a life- time random variable X with Lomax distribution with three parameters is given by

$$h(t,\alpha,\beta,\lambda) = \frac{f(x,\alpha,\beta,\lambda)}{1-F(x,\alpha,\beta,\lambda)} = \frac{\frac{\alpha\beta^{\alpha}\lambda}{[\beta+\lambda x]^{\alpha+1}}}{\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}} = \frac{\alpha\lambda}{\beta+\lambda t}, \alpha > 0, \beta > 0, \lambda > 0. \quad (9)$$

From equation (9) it can be seen that the hazard rate function of Lomax distribution with three parameters is decreasing.

The cumulative hazard rate function of a life- time random variable X with Lomax distribution with three parameters is given by

$$H(t) = \int_0^t h(u)du = -\ln(R(t)) = -\ln\left(\left(\frac{\beta}{\lambda t+\beta}\right)^{\alpha}\right)$$

2.3 Mean Residual Life

The mean residual life(MRL)([23]) function describes the aging process so, it is very important in reliability and survival analysis. The mean residual life(MRL) function of a lifetime random variable x is given by

$$\mu(x) = \frac{1}{R(x)}\int_x^{\infty} tf(t)dt - x, x > 0.$$

where $R(X)$ is given by (8).

Theorem 4. *The MRL function of a lifetime random variable X with Lomax distribution with three parameters is given by*

$$\mu(x) = \frac{\alpha(\beta+\lambda x)}{\lambda(\alpha-1)} - \frac{\beta}{\lambda} - x. \quad (10)$$

Proof.

$$\int_x^{\infty} tf(t)dt = \alpha\beta^{\alpha}\lambda\int_x^{\infty}\frac{tdt}{(\beta+\lambda t)^{\alpha+1}} = |\beta+\lambda t = u|$$

$$= \frac{\alpha\beta^{\alpha}}{\lambda}\int_{\beta+\lambda x}^{\infty}\left[\frac{du}{u^{\alpha}} - \frac{\beta}{u^{\alpha+1}}\right]du = \frac{\alpha\beta^{\alpha}}{\lambda}\left[\frac{1}{(\alpha-1)(\beta+\lambda x)^{\alpha-1}} - \frac{\beta}{\alpha(\beta+\lambda x)^{\alpha}}\right].$$

Now, from definition of MRL, we get

$$\begin{aligned}\mu(x) &= \frac{1}{R(x)}\int_{x}^{\infty} tf(t)dt - x\\ &= \frac{(\beta+\lambda x)^{\alpha}}{\beta^{\alpha}}\cdot\frac{\alpha\beta^{\alpha}}{\lambda}\left[\frac{1}{(\alpha-1)(\beta+\lambda x)^{\alpha-1}} - \frac{\beta}{\alpha(\beta+\lambda x)^{\alpha}}\right] - x\\ &= \frac{\alpha(\beta+\lambda x)}{\lambda(\alpha-1)} - \frac{\beta}{\lambda} - x.\end{aligned}$$

□

2.4 Quantiles

The quantile of any distribution is given by solving the equation

$$F(x_p) = p, \quad 0 < p < 1.$$

Theorem 5. *The quantile of Lomax distribution with three parameters is given by*

$$x_p = \frac{1}{\lambda}\left[\sqrt[\alpha]{(}\frac{\beta^{\alpha}}{1-p}) - \beta\right], \quad 0 < p < 1. \tag{11}$$

Proof.

$$\begin{aligned}&1 - \left[\frac{\beta}{\beta+\lambda x_p}\right]^{\alpha} = p \rightarrowtail 1 - p = \left[\frac{\beta}{\beta+\lambda x_p}\right]^{\alpha}\\ &\rightarrowtail (\beta+\lambda x_p)^{\alpha} = \frac{\beta^{\alpha}}{1-p} \rightarrowtail \beta+\lambda x_p = \sqrt[\alpha]{\frac{\beta^{\alpha}}{1-p}}\\ &\rightarrowtail \lambda x_p = \sqrt[\alpha]{\frac{\beta^{\alpha}}{1-p}} - \beta \rightarrowtail x_p = \frac{1}{\lambda}\left[\sqrt[\alpha]{\frac{\beta^{\alpha}}{1-p}} - \beta\right].\end{aligned}$$

□

Theorem 6. *Let X be a random variable with pdf* (6) *The expectation and variance are given by:*

$$\mathrm{E}(X) = \frac{1}{\lambda}\left[\frac{\alpha\beta}{\alpha-1} - 1\right], \tag{12}$$

and

$$\mathrm{Var}(X) = \frac{1}{\lambda^2(1-\alpha)}\left[2\beta^2 - \frac{[\alpha(\beta-1)+1]^2}{1-\alpha}\right], \tag{13}$$

respectively.

Proof.

$$\mathrm{E}(X) = \int_0^\infty x f(x)dx = \alpha\beta^\alpha\lambda \int_0^\infty \frac{xdx}{(\beta+\lambda x)^{\alpha+1}}dx$$
$$= \alpha\beta^\alpha\lambda \frac{1}{\lambda^2}\int_\beta^\infty \frac{(t-\beta)dt}{t^{\alpha+1}}$$
$$= \frac{1}{\lambda}\left[\frac{\alpha\beta}{\alpha-1} - 1\right]$$

$$\mathrm{E}(X^2) = \alpha\beta^\alpha\lambda \int_0^\infty \frac{x^2dx}{(\beta+\lambda x)^{\alpha+1}}dx$$
$$= \alpha\beta^\alpha\lambda \frac{1}{\lambda^3}\int_\beta^\infty \frac{(t-\beta)^2dt}{t^{\alpha+1}}$$
$$= \frac{2\beta^2}{\lambda^2(1-\lambda)}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \qquad = \frac{1}{\lambda^2(1-\alpha)}\left[2\beta^2 - \frac{[\alpha(\beta-1)+1]^2}{1-\alpha}\right]$$

From equation (6) it can be seen that the probability density function of Lomax distribution with three parameters is decreasing for all $x > 0$.

3 Order Statistics

The kth order statistic of a sample is its kth smallest value. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$range\{X_1, \ldots, X_n\} = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)} \leq \cdots \leq X_{(n)}$ denotes the order statistic of a random sample $X_1, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)\,(F_X(x))^{j-1}\,(1-F_X(x))^{n-j},$$

for $j = 1, \dots, n$. The pdf of the jth order statistic for a Lomax distribution with three parameters is given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{\alpha\beta^{\alpha}\lambda}{[\beta+\lambda x]^{\alpha+1}}\left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}\right]^{j-1}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha(n-j)}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is

$$f_{X_{(n)}}(x) = n\frac{\alpha\beta^{\alpha}\lambda}{[\beta+\lambda x]^{\alpha+1}}\left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n\frac{\alpha\beta^{\alpha}\lambda}{[\beta+\lambda x]^{\alpha+1}}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha(n-1)} = \frac{n\alpha\beta^{\alpha n}\lambda}{(\beta+\lambda x)^{\alpha n+1}}$$

4 Maximum likelihood estimation

The maximum likelihood estimates, MLE's, of the parameters that are inherent within the Lomax distribution with three parameters is obtained as follows: The likelihood function of the observed sample $\underset{\sim}{x} = \{x_1, x_2, \dots, x_n\}$ of size n drawn from the density (6), is defined as

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta, \lambda) = \frac{\alpha^n \beta^{n\alpha}\lambda^n}{\prod\limits_{i=1}^{n}[\beta+\lambda x_i]^{\alpha+1}}$$

The corresponding Log-likelihood function is given by

$$\ell = \ln L = n\ln\alpha + \alpha n\ln\beta + n\ln\lambda - (\alpha+1)\sum_{i=1}^{n}\ln(\beta+\lambda x_i) \qquad (14)$$

Now setting

$$\frac{\partial \ln L}{\partial\alpha} = 0, \quad \frac{\partial \ln L}{\partial\beta} = 0, \quad \textit{and} \quad \frac{\partial \ln L}{\partial\lambda} = 0,$$

we have

$$\frac{\partial \ln L}{\partial\alpha} = \frac{n}{\alpha} + n\,(ln(\beta))\,\beta - \sum_{i=1}^{n}\ln(\lambda x_i+\beta) = 0$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{\alpha\, n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{1}{\lambda x_i+\beta} = 0$$

$$\frac{\partial \ln L}{\partial\lambda} = \frac{n}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{\lambda x_i+\beta} = 0$$

The MLEs $\left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ of (α, β, λ), respectively is obtained by solving this non-linear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the sample likelihood function. Applying the usual large sample approximation, the MLE $\hat{\varphi} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ can be treated as being approximately tri-variate normal with mean $\hat{\varphi}$ and variance-covariance matrix equal to the inverse of the expected information matrix [17], i.e.

$$\sqrt{n}(\hat{\varphi} - \varphi) \to N_3\left(0, nI^{-1}(\varphi)\right)$$

where $I^{-1}(\varphi)$ is the limiting variance-covariance matrix of $\hat{\varphi}$. The elements of the 3×3 matrix $I(\varphi)$ can be estimated by $I_{ij}(\hat{\varphi}) = -\ell_{\varphi_i \varphi_j}|_{\varphi=\hat{\varphi}}$, $i, j \in \{1, 2, 3\}$.

The elements of the Hessian matrix corresponding to the ℓ function in Equation (14) are:

$$I_{11} = \frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$I_{12} = \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{1}{\lambda\, x_i + \beta}$$

$$I_{13} = \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{n} \frac{x_i}{\lambda\, x_i + \beta}$$

$$I_{22} = \frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{\alpha\, n}{\beta^2} + (\alpha + 1) \sum_{i=1}^{n} \frac{1}{(\lambda\, x_i + \beta)^2}$$

$$I_{23} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = (\alpha + 1) \sum_{i=1}^{n} \frac{x_i}{(\lambda\, x_i + \beta)^2}$$

$$I_{33} = \frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{n}{\lambda^2} + (\alpha + 1) \sum_{i=1}^{n} \frac{{x_i}^2}{(\lambda\, x_i + \beta)^2}$$

4.1 Maximum product spacing estimates

The maximum product of spacings (MPS) method for estimating parameters in continuous univariate distributions was proposed by Cheng and Amin [10] and independently by Ranneby [24]. This method is based on an idea that the differences (Spacings) of the consecutive points should be identically distributed. Let $x_1, x_2, \ldots, x_n$ be independent identically distributed (i.i.d) random varibales with cumulative distribution function given by (7) and denote the corresponding order statistics $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Define

$$S_n(\alpha, \beta, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left[F(x_{(i)}, \alpha, \beta, \lambda) - F(x_{(i-1)}, \alpha, \beta, \lambda)\right] \qquad (15)$$

Definition 7. *[26] Any $\bar{\alpha}, \bar{\beta}$ and $\bar{\lambda}$ which maximizes $S_n(\alpha, \beta, \lambda)$ is a maximum product spacings estimator of the unknown true parameters α, β and λ.*

The MPS estimators $\hat{\alpha}_{PS}$, $\hat{\beta}_{PS}$ and $\hat{\lambda}_{PS}$ of α, β and λ can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial S_n(\alpha,\beta,\lambda)}{\partial\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_\alpha(x_{(i)},\alpha,\beta,\lambda) - F'_\alpha(x_{(i-1)},\alpha,\beta,\lambda)}{F(x_{(i)},\alpha,\beta,\lambda) - F(x_{(i-1)},\alpha,\beta,\lambda)}\right] = 0, \tag{16}$$

$$\frac{\partial S_n(\alpha,\beta,\lambda)}{\partial\beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_\beta(x_{(i)},\alpha,\beta,\lambda) - F'_\beta(x_{(i-1)},\alpha,\beta,\lambda)}{F(x_{(i)},\alpha,\beta,\lambda) - F(x_{(i-1)},\alpha,\beta,\lambda)}\right] = 0, \tag{17}$$

and

$$\frac{\partial S_n(\alpha,\beta,\lambda)}{\partial\lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_\lambda(x_{(i)},\alpha,\beta,\lambda) - F'_\lambda(x_{(i-1)},\alpha,\beta,\lambda)}{F(x_{(i)},\alpha,\beta,\lambda) - F(x_{(i-1)},\alpha,\beta,\lambda)}\right] = 0, \tag{18}$$

where,

$$F'_\alpha(x_{(i)},\alpha,\beta,\lambda) = -\left(\frac{\beta}{\lambda\, x_{(i)}+\beta}\right)^{\alpha}\ln\left(\frac{\beta}{\lambda\, x_{(i)}+\beta}\right),$$

$$F'_\beta(x_{(i)},\alpha,\beta,\lambda) = -\frac{\alpha\,\lambda\, x_{(i)}}{\beta\,\left(\lambda\, x_{(i)}+\beta\right)}\left(\frac{\beta}{\lambda\, x_{(i)}+\beta}\right)^{\alpha},$$

and

$$F'_\lambda(x_{(i)},\alpha,\beta,\lambda) = \frac{\alpha\, x_{(i)}}{\lambda\, x_{(i)}+\beta}\left(\frac{\beta}{\lambda\, x_{(i)}+\beta}\right)^{\alpha}.$$

4.2 Least square estimates

The least square estimators and weighted least square estimators were proposed by Swain, Venkataraman and Wilson [27] to estimate the parameters of Beta distribution. The least square estimators of the unknown parameters α, β and λ of Lomax distribution with three parameters by using the same methodology as Swain et al., can be obtained by minimizing

$$\sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2$$

with respect to unknown parameters α, β and λ. Suppose $F(x_{(i)})$ denotes the distribution function of the ordered random variables $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ where $\{x_1, x_2, \cdots, x_n\}$ is a random sample of size n from a distribution function $F(\cdot)$. Therefore, in this case, by using

$$F(x_{(i)}) = (1 - \left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha}$$

the least square estimators of α, β and λ, say $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}$ and $\hat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{i=1}^{n}\left[1 - \left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha} - \frac{i}{n+1}\right]^2$$

with respect to α, β and λ.

The least square estimates (LSEs) $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}$ and $\hat{\lambda}_{LSE}$ of α, β and λ are obtained by minimizing

$$LSE\left(\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[1-\left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha}-\frac{i}{n+1}\right]^{2} \tag{19}$$

Therefore, $\hat{\alpha}_{LS}$, $\hat{\beta}_{LS}$ and $\hat{\lambda}_{LS}$ of α, β and λ can be obtained as the solution of the following system of equations:

$$\begin{aligned}\frac{\partial LSE\left(\alpha,\beta,\lambda\right)}{\partial\alpha}&=\sum_{i=1}^{n}F_{\alpha}'(x_{(i)},\alpha,\beta,\lambda)\left(F\left(x_{(i)},\alpha,\beta,\lambda\right)-\frac{i}{n+1}\right)\\&=\sum_{i=1}^{n}\left[-\left(\frac{\beta}{\lambda x_{(i)}+\beta}\right)^{\alpha}\ln\left(\frac{\beta}{\lambda x_{(i)}+\beta}\right)\right]\\&\times\left[1-\left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha}-\frac{i}{n+1}\right]=0\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial LSE\left(\alpha,\beta,\lambda\right)}{\partial\beta}&=\sum_{i=1}^{n}F_{\beta}'(x_{(i)},\alpha,\beta,\lambda)\left(F\left(x_{(i)},\alpha,\beta,\lambda\right)-\frac{i}{n+1}\right)\\&=-\sum_{i=1}^{n}\frac{\alpha\,\lambda\,x_{(i)}}{\beta\,\left(\lambda x_{(i)}+\beta\right)}\left(\frac{\beta}{\lambda x_{(i)}+\beta}\right)^{\alpha} \qquad (21)\\&\times\left[1-\left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha}-\frac{i}{n+1}\right]=0. \qquad (22)\end{aligned}$$

$$\begin{aligned}\frac{\partial LSE\left(\alpha,\beta,\lambda\right)}{\partial\lambda}&=\sum_{i=1}^{n}F_{\lambda}'(x_{(i)},\alpha,\beta,\lambda)\left(F\left(x_{(i)},\alpha,\beta,\lambda\right)-\frac{i}{n+1}\right)\\&=-\sum_{i=1}^{n}\frac{\alpha\,x_{(i)}}{\lambda x_{(i)}+\beta}\left(\frac{\beta}{\lambda x_{(i)}+\beta}\right)^{\alpha} \qquad (23)\\&\times\left[1-\left[\frac{\beta}{\beta+\lambda x_{(i)}}\right]^{\alpha}-\frac{i}{n+1}\right]=0. \qquad (24)\end{aligned}$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R software namely.

4.3 The weighted least square estimators

The weighted least square estimators [27] of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[F(X_{(j)})-\frac{j}{n+1}\right]^{2}$$

with respect to α, β and λ. The weights w_j are equal to $\frac{1}{V(X_{(j)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$. Therefore, in this case, the weighted least square estimators of α, β and λ, say $\hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE}$ and $\hat{\lambda}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{n-j+1}\left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}-\frac{j}{n+1}\right]^2$$

with respect to α, β and λ.

Therefore, $\hat{\alpha}_{WLSW}, \hat{\beta}_{WLSW}$ and $\hat{\lambda}_{WLSW}$ of α, β and λ can be obtained as the solution of the following system of equations:

$$\frac{\partial WLS\,(\alpha,\beta,\lambda)}{\partial \alpha} = -\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{(n-j+1)}\left(\frac{\beta}{\lambda x+\beta}\right)^{\alpha} \ln\left(\frac{\beta}{\lambda x+\beta}\right) \times \left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}-\frac{i}{n+1}\right]=0$$

$$\frac{\partial WLS\,(\alpha,\beta,\lambda)}{\partial \beta} = -\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{(n-j+1)}\frac{\alpha\,\lambda\,x}{\beta\,(\lambda x+\beta)}\left(\frac{\beta}{\lambda x+\beta}\right)^{\alpha} \times \left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}-\frac{i}{n+1}\right]=0.$$

$$\frac{\partial WLS\,(\alpha,\beta,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{(n-j+1)}\frac{\alpha\,x}{\lambda x+\beta}\left(\frac{\beta}{\lambda x+\beta}\right)^{\alpha} \times \left[1-\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}-\frac{i}{n+1}\right]=0.$$

4.4 Method of Cramér-von-Mises

To motivate our choice of Cramer-von Mises type minimum distance estimators, MacDonald [22] provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, The Cramé-von Mises estimates $\widehat{\alpha}_{CME}, \widehat{\beta}_{CME}$ and $\widehat{\lambda}_{CME}$ of the parameters α, β and λ are obtained by minimizing, with respect to α, β and λ, the function:

$$C(\alpha,\beta,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)} \mid \lambda,\theta\right) - \frac{2i-1}{2n}\right)^2. \tag{25}$$

5 Application

In this section, we will check the performance of the proposed Lomax distribution with three parameters. We use a real data set to show that the Lomax distribution with three parameters is better model than the existing models which is studied before by

Tahir et al.[12], such as Weibull Lomax [12], exponentiated Lomax and Lomax. We use maximum likelihood method estimate the parameters and also their standard errors.

Aircraft Windshield data sets. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. We consider the data on failure and service times for a particular model windshield given in Table 16.11 of Murthy et al. [28]. These data were recently studied by Ramos et al. [25]. The data consist of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h.

5.1 Data set 1: Failure times of 84 Aircraft Windshield

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

5.2 Data set 2: Service times of 63 Aircraft Windshield

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 15 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

In order to compare the distribution models, we consider criteria like -2ℓ, AIC (Akaike information criterion), CAIC (corrected Akaike information criterion) and BIC for the data set. The better distribution corresponds to smaller -2ℓ, AIC, CAIC and BIC values:

- $AIC = -2\log\ell\left(\underset{\sim}{x}, \alpha, \lambda, \boldsymbol{\xi}\right) + 2p,$
- $AICC = AIC + \frac{2p(p+1)}{n-p-1},$
- $BIC = -2\log\ell\left(\underset{\sim}{x}, \alpha, \lambda, \boldsymbol{\xi}\right) + p\log(n)\,,$
  where, p is the number of parameters are to be estimated from the data and n the sample size.

Tables 1 and 2 shows the MLEs under both distributions, Tables 1and 2 shows the values of -2ℓ, AIC, AICC, and BIC values. The values in Tables 1 and 2 indicate that the

Lomax distribution with three parameters leads to a better fit than the Weibull Lomax exponentiated Lomax and Lomax distribution.

Table 1: MLEs and comparison criterium for failure times of 84 Aircraft Windshield data(Data set 1.)

Model	Parameter Estimate	$-\ell$	AIC	CAIC	BIC
Lomax distribution with three parameters	$\hat{\alpha} = 514.2598$	92.4803	190.9607	191.2607	198.2887
	$\hat{\beta} = 130.0092$				
	$\hat{\lambda} = 0.1082$				
Weibull Lomax	$\hat{a} = 0.0128$	127.8652	263.7303	264.2303	273.5009
	$\hat{b} = 0.5969$				
	$\hat{\alpha} = 6.7753$				
	$\hat{\beta} = 1.5324$				
Exponentiated Lomax	$\hat{a} = 3.6261$	141.3997	288.7994	289.0957	296.1273
	$\hat{\alpha} = 20074.5097$				
	$\hat{\beta} = 26257.6808$				
Lomax	$\hat{\alpha} = 51425.3500$	164.9884	333.9767	334.1230	338.8620
	$\hat{\beta} = 131789.7800$				

Table 2: MLEs and comparison criterium for service times of 63 Aircraft Windshield data(Data set 2.)

Model	Parameter Estimate	$-\ell$	AIC	CAIC	BIC
Lomax distribution with three parameters	$\hat{\alpha} = 994.0184$	81.1431	168.2863	168.6931	174.7157
	$\hat{\beta} = 207019.3641$				
	$\hat{\lambda} = 111.7893$				
Weibull Lomax	$\hat{a} = 0.1276$	98.11712	204.2342	204.9239	212.8068
	$\hat{b} = 0.9204$				
	$\hat{\alpha} = 3.9136$				
	$\hat{\beta} = 3.0067$				
Exponentiated Lomax	$\hat{a} = 1.9145$	103.5498	213.0995	213.5063	219.5289
	$\hat{\alpha} = 22971.1536$				
	$\hat{\beta} = 32881.9966$				
Lomax	$\hat{\alpha} = 99269.7800$	109.2988	222.5976	222.7976	226.8839
	$\hat{\beta} = 207019.3700$				

6 Simulation algorithms and study

To generate a random sample of size n from Lomax distribution with three parameters, we follow the following steps:

1. Set n, $\Theta = (\beta)$ and initial value x^0.
2. Generate $U \sim Unifrom\,(0, 1)$.
3. Update x^0 by using the Newton's formula
   $x^\star = x^0 - R(x^0, \Theta)$
   where, $R(x^0, \Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(.)$ and $f_X(.)$ are cdf and pdf of Lomax distribution with three parameters, respectively.
4. If $\mid x^0 - x^\star \mid \leq \epsilon$, (very small, $\epsilon > 0$ tolerance limit), then store $x = x^\star$ as a sample from Lomax distribution wth three parameters.
5. If $\mid x^0 - x^\star \mid > \epsilon$, then, set $x^0 = x^\star$ and go to step 3.
6. Repeat steps 3-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

6.1 Data set 3:Simulation data

R code for simuation random numbers from Lomax distribution with three parameters

```
F=function(x,alpha, beta,lambda)
{ 1-(beta/(lambda*x+beta))^alpha}

f=function(x,alpha, beta,lambda)
{(beta/(lambda*x+beta))^alpha*alpha*lambda/(lambda*x+beta)}
n=500
alpha1=2
beta1=3
lambda1=0.5
u=runif(n)
x=rep(0,n)
for(i in 1:n){
x0=1
xnew=x0-((F(x0,alpha1,beta1,lambda1)-u[i])/f(x0,alpha1,beta1,lambda1))
while(abs(xnew-x0)>0.0001){
x0=xnew
xnew=x0-((F(x0,alpha1,beta1,lambda1)-u[i])/f(x0,alpha1,beta1,lambda1))
}
x[i]=xnew
}
x=sort(x)
```

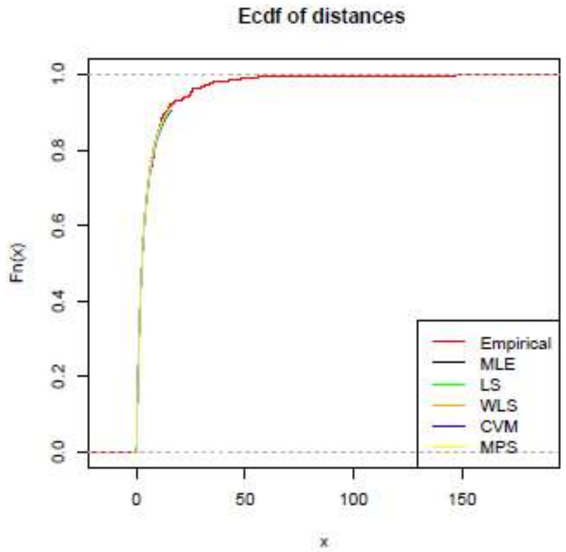

Figure 1: Empirical and fitted Lomax distribution with three parameters for simulated data.

From table 3 Least squares estimator of parameters is superior than the others methods of estimation.

Table 3: Parameter estimation for simulated data and Kolmogorov-Smirnov test

Method	α	β	λ	K-S	Best method
MLE	1.8094188	2.9929555	0.5491893	0.9141862	5
Least Squares	1.5186251	2.9671346	0.6837718	0.9022503	1
Weighted Least Squares	1.7211664	2.9865430	0.5804955	0.910201	3
Cramer Von Mises	1.5263952	2.9686791	0.6780801	0.9023797	2
MPS	1.7365531	2.9867868	0.5790081	0.9117906	4

The fitted cumulative distribution function of the Lomax distribution with three parameters is plotted in Figure 1 for the simulated data.

7 Conclusion

In this paper we study Lomax distribution with three parameters. The CDF, PDF, Hazard function and cumulative hazard function are derived. Additionally, some of the mathematical and statistical properties like quantile function, expectation, variance, and order statistic are also provided. The model parameters for real data sets are estimated by using maximum likelihood estimation. We have considered different methods of estimation of the unknown parameters of the Lomax distribution with three parameters. We briefly describe different methods of estimations, namely maximum likelihood estimators, least squares estimators, weighted least squares, maximum product spacing estimates and method of Cramer-von-Misses and and compare them using extensive numerical simulations.

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