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Solution Non Linear Equation Singularly Perturbed Of First Order Ode On Argument In Critical Case

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Summary

In this paper we have a first order ODE that contain a small argument in front of the derivative was the same as power argument in both of right – left hand equation, and we define asymptotic solution using this argument of the power 0 and 1. Also we prove that there exists a unique solution of the main equation.

Key words : ordinary differential equation, singularly perturbed problems, asymptotic solution method.

1- Introduction

(Lin Wuzhong, 1995) was proved, utilizing subjective investigation and asymptotic techniques for integrals, that for $\varepsilon \to 0$ the constraining arrangement of the independently irritated quasilinear BVP with defining moments $\varepsilon x'' + b(z)x' = 0, x(0) = x_0,\ x(1) = x_1$, is an expanding piecewise steady capacity. A gauge of the seasons of every monotone change layer is additionally given.

(Ni Min Kan, A. B. Vasilieva, and M. G. Dmitriev,2006) they were built up the equality of two arrangements of progress guides comparing toward arrangements of singularly perturbed BVP with inside limit layers. The first set shows up in the formalism for building the asymptotics of the arrangement of a BVP and the second, in the immediate plan formalism for developing the asymptotics of the arrangement of a variational issue.

(Wen-Zhi Zhang, Pei-Yan Huang ,2013) they introduced a high request duplication bother strategy for uniquely annoyed two-point limit esteem issues with the limit layer toward one side. Additionally he changed the non-homogeneous standard differential conditions (ODEs) into the homogeneous ODEs.

(V. F. Butuzov, 2017) a BVP for a uniquely irritated differential condition of second request is considered in two cases, when one base of the decline condition is two-tuple. It is demonstrated that in the primary case the issue has an answer with the progress from the two-tuple foundation of the deteriorate condition to one-tuple root in the little neighborhood of an inward purpose of the interim, and in the second case the issue has an answer which has the spike in the inside layer. Such arrangements are named, correspondingly, a difference structure of step-sort and a complexity structure of spike-sort. For each situation the asymptotic development of the differentiation structure is built. It recognizes from the known development for the situation, when every one of the underlying foundations of the decline condition are one-tuple, specifically, the inside layer is multizonal.

(Pontryagin L.S., 1970) examined the conduct of arrangement of first order ODE with boundary value, and this solution of first order has proven a existence and uniqueness theorem for this equation.

(T. Valanarasu, N. Ramanujam, 2003) displayed a limited distinction technique for second request of singular perturbation problems. It depends upon a work determination technique inferred by utilizing adequate conditions which guarantee the well molding of tridiagonal frameworks. Specifically the usage parts of the technique are examined. Numerical tests are accounted for to prove the adequacy of this technique and its aggressiveness as for known solvers for BVPs.

(Vasil'eva, A. D., V. F. Butuzov ,1973) The asymptotic expansion of the solutions of the initial or BVP in the case $p = 0$ was constructed by the scheme described. Also he introduced the curve L_0 in the space (x, t), consisting of three parts. The asymptotics of the initial problem in the case $p \neq 0$ and $m = 0$ was constructed by(Vasil'eva, A. D., V. F. Butuzov, 1970).

(F. Mazzia D. Trigiante 1993) displayed a limited distinction technique for Second Order Singular Perturbation Problems ODE. It depends on a work determination technique inferred by utilizing adequate conditions which guarantee the well molding of tridiagonal frameworks. Specifically the usage parts of the technique are examined. Numerical tests are accounted for to prove the adequacy of this technique and its aggressiveness as for known solvers for BVPs.

(A. B. Vasilieva, 2009) studied a reduced equation singularly perturbed problem depending on small parameter ε, with Multiple Roots. additionally built an asymptotic portrayal of $y(x, \varepsilon)$ as for ε with an $O(\varepsilon^n)$ leftover portion.

(Sibuya Yasutaka,1965) improved some results of matrices whose elements A (x) have continuous derivatives of all orders with respect to several number of real variables or complex variable x . Also he proven the six theorems of A(x).

2- Main problem of ODE

$$\mu \frac{dx}{dt} - A(t)x = f_0(x, t, \mu) + \mu f_1(x, t, \mu) \quad (2\text{-}1)$$

where $\mu > 0$ is small parameter, and x, f_0, f_1 are $M \times M$-dimensional vector functions, $A(t)$ is an $(m \times m)$ matrix and $0 \leq t \leq 1$.

Requirement 1 : Let the eigenvalues $\lambda_i(t)$ ($i = 1,2,3, \dots, M$) of the matrix A (t) satisfy the conditions

$\lambda_i(t) = 0$, $(i = 1,2,3, \dots, p)$,

$\mathrm{Re}\, \lambda_i(t) < 0$, $(i = p + 1, p + 2, \dots, p + k)$,

$\mathrm{Re}\, \lambda_i(t) > 0$, $(i = p + k + 1, p + k + 2, \dots, p + k + m = M)$,

The asymptotic expansion of the solutions of the IVP or BVP in the case $p = 0$ is constructed by the scheme described by (A. B. Vasil'eva, V. F. Butuzov, 1973). The asymptotics of the initial problem in the case $p \neq 0$ and $m = 0$ was constructed by (A. B. Vasil'eva, V. F. Butuzov, 1970). In this paper, under certain conditions, the asymptotics of solve equation (2-1) with conditions of boundary

$$ax(0, \mu) = ax^0, bx(1, \mu) = bx^0, a = \begin{pmatrix} E_{k+p_1} & 0 \\ 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 & 0 \\ 0 & E_{m+p_2} \end{pmatrix}, \quad (2\text{-}2)$$

Where $p_1 + p_2 = p$, $E_i - (i \times i)$ - unit matrix . In the case $p \neq 0$ degenerate system $A(t)\bar{x}(t) = 0$ haven't isolated root. It, rather, has a group of solvable relied upon one or a few parameters. In this paper we construct an asymptotic solution with respect to μ of the solution of the boundary value problem (2-1), (2-2) and we show that this solution for $\mu \to 0$ tends to some solution of the degenerate system that is found by a specific rule. haven't

Requirement 2 : Let the eigenvalue $\lambda_i(t) = 0$ of multiplicity p corresponds to each $t \in [0,1]$ exactly p linearly independent eigenvectors $e_i(t)(i = 1,2,3, \dots, p)$ of the Matrix $A(t)$.also we use the : degenerate system of $f_0(x, t, 0) = f_1(x, t, 0) = 0$ to help us found the solution of this system.

Requirement 3 : Let $A(t), f_0(x, t, \mu), f_1(x, t, \mu)$ are sufficiently smooth for $0 \leq t \leq 1,\ x \in D \subset R_M, 0 < \mu \leq d$ (d is constant). The degree of smoothness and the domain D was refined in the built of asymptotics. The asymptotic expansion of solve equations (2-1), (2-2) like :

$$x(t,\mu)=\sum_{k=0}^{1}\mu^k\left(\bar{x}_k(t,\mu)+\prod_k x(\tau,\mu)+Q_k x(s,\mu)\right) \quad (2\text{-}3)$$

We will extension the parts of above series in powers of μ as the following

$$\bar{x}(t,\mu)=\bar{x}_0(t)+\mu\bar{x}_1(t)+\mu^2\bar{x}_2(t)+\cdots+\mu^k\bar{x}_k(t)+\cdots,$$

$$\prod(\tau,\mu)=\prod_0 x(\tau)+\mu\prod_1 x(\tau)+\mu^2\prod_2 x(\tau)+\cdots+\mu^k\prod_k x(t)+\cdots,$$

$$Q_x(s,\mu)=Q_0x(s)+\mu Q_1x(s)+\mu^2Q_2x(s)+\cdots+\mu^kQ_kx(s)+\cdots.$$

Where $\tau=\frac{t}{\mu}$, $s=\frac{t-1}{\mu}$.

For the coefficients of this expansion, we can obtain the equations by the rule described in section 14 by (A. B. Vasil'eva, V. F. Butuzov, 1973). The main terms of the expansion are related to this systems . also we will take only the case when the power of μ in zero and one.

Now we will substitution the (2-3) in (2-1) and (2-2) to have For $\bar{x}_0(t),\prod_0 x(\tau)$ and $Q_0x(s)$ with requirement 2 we get

$$A(t)\bar{x}_0(t)=0, \quad (2\text{-}4)$$

$$\frac{d\prod_0 x(\tau)}{d\tau}=A(0)\prod_0 x(\tau), \qquad \frac{dQ_0x(s)}{ds}=A(1)Q_0x(s) \quad (2\text{-}5)$$

With boundary conditions

$$a[\bar{x}_0(0)+\prod_0 x(0)\,]=ax^0,\ \ b[\bar{x}_0(1)+Q_0x(0)]=bx^0 \quad (2\text{-}6)$$

The general solution of equation (2-4),and by irtue of requirement 2, can be written in the form

$$\bar{x}_0(t)=c_{10}(t)e_1(t)+c_{20}(t)e_2(t)+c_{30}(t)e_3(t)+\cdots+c_{p0}(t)e_p(t)$$

$$=\sum_{i=1}^{p}c_{i0}(t)e_i(t), \quad (2\text{-}7)$$

Where the $c_{i0}(t)$ are arbitrary scalar functions, $e_i(t)$ $(i=1,2,3,..,p)$ linearly independent eigenvectors corresponding to the zero eigenvalue of the matrix A (t). It follows from the work by (Sibuya Yasutaka 1965) that one can construct $e_i(t)$, having the same degree of smoothness as A (t). These are our own vectors and we will use.

We generalize the solutions of systems (2-5) in the form by (Pontryagin L.S. 1970)

For $\prod_0 x(\tau)$, we obtain the problem

$\frac{d\prod_0 x(\tau)}{d\tau}=A(0)\prod_0 x(\tau)$ and by the (Pontryagin L.S. 1970), we have general solution of $\prod_0 x(\tau)$ that is

$$\begin{aligned}\prod_0 x(\tau)=\ &\alpha_{10}e_1(0)+\alpha_{20}e_2(0)+\alpha_{30}e_3(0)+\cdots+\alpha_{i0}e_i(0)+\beta_{10}e^{\bar{\lambda}_1(0)\tau}w_1(\tau)+\beta_{20}e^{\bar{\lambda}_2(0)\tau}w_2(\tau)\\&+\beta_{30}e^{\bar{\lambda}_3(0)\tau}w_3(\tau)+\cdots+\beta_{i0}e^{\bar{\lambda}_i(0)\tau}w_i(\tau)+\bar{\beta}_{10}e^{\bar{\lambda}_1(0)\tau}\bar{w}_1(\tau)+\bar{\beta}_{20}e^{\bar{\lambda}_2(0)\tau}\bar{w}_2(\tau)\\&+\bar{\beta}_{30}e^{\bar{\lambda}_3(0)\tau}\bar{w}_3(\tau)+\cdots+\bar{\beta}_{i0}e^{\bar{\lambda}_i(0)\tau}\bar{w}_i(\tau)\end{aligned}$$

$$=\sum_{i=1}^{p}\alpha_{i0}e_i(0)+\sum_{i=1}^{k}\beta_{i0}e^{\bar{\lambda}_i(0)\tau}w_i(\tau)+\sum_{i=1}^{m}\bar{\beta}_{i0}e^{\bar{\lambda}_i(0)\tau}\bar{w}_i(\tau) \quad (2\text{-}8)$$

For $Q_0x(s)$, we obtain the problem

$\frac{dQ_0 x(s)}{ds} = A(1)Q_0 x(s)$ and by the (Pontryagin L.S. 1970), we have general solution of $Q_0 x(s)$ that is

$$Q_0 x(s) = \overline{\alpha}_{10}e_1(1) + \overline{\alpha}_{20}e_2(1) + \overline{\alpha}_{30}e_3(1) + \cdots + \overline{\alpha}_{i0}e_i(1) + \overline{\gamma}_{10}e^{\lambda_1(1)s}\overline{z}_1(\tau) + \overline{\gamma}_{20}e^{\lambda_2(1)s}\overline{z}_2(s) + \overline{\gamma}_{30}e^{\lambda_3(1)s}\overline{z}_3(s) + \cdots + \overline{\gamma}_{i0}e^{\lambda_i(1)s}\overline{z}_i(s) + \gamma_{10}e^{\overline{\lambda}_1(0)\tau}z_1(s) + \gamma_2 e^{\overline{\lambda}_2(0)\tau}z_2(s) + \gamma_{30}e^{\overline{\lambda}_3(0)\tau}z_3(s) + \cdots + \gamma_{i0}e^{\overline{\lambda}_i(1)\tau}z_i(s)$$

$$= \sum_{i=1}^{p}\overline{\alpha}_{i0}e_i(1) + \sum_{i=1}^{k}\overline{\gamma}_{i0}e^{\lambda_i(1)s}\,\overline{z}_i(s) + \sum_{i=1}^{m}\gamma_{i0}\,e^{\overline{\lambda}_i(1)s}\,z_i(s) \quad (2\text{-}9)$$

Where $\lambda_i(t)$ ($i = 1,2,3,\dots,k$) have a $Re\,\lambda_i(t) < 0$, $\overline{\lambda}_i(t)$ ($i = 1,2,3,\dots,m$) have $Re\,\lambda_i(t) > 0; w_i(\tau)$ and $\overline{w}_i$ [$\overline{z}_i(s)$ and $z_i(s)$] are Vector of a function whose components are polynomials with respect to $\tau[s]$; α_{i0}, $\overline{\alpha}_{i0}$, β_{i0}, $\overline{\beta}_{i0}$, γ_{i0}, $\overline{\gamma}_{i0}$ are Some constants. Let.us.require.that.the.entire.boundary function $\prod_0 x(\tau)$ and $Q_0 x(s)$ approach as

$$\prod{}_0 x(\tau) \to 0 \text{ at } \tau \to \infty \text{ and } Q_0 x(s) \to 0 \text{ at } \tau \to -\infty, \quad (2\text{-}10)$$

Then in the representations (2-8), (2-9) all α_{i0}, $\overline{\alpha}_{i0}$, β_{i0}, $\overline{\beta}_{i0}$,$\overline{\gamma}_{i0}$ must be set equal to zero. Substituting (2-8) and (2-9) into the boundary conditions (2-6) and taking (2-10) into account, we have:

$$\left.\begin{aligned} &\sum_{i=1}^{p} c_{i0}(0)ae_i(0) + \sum_{i=1}^{p}\beta_{i0}aw_i(0) = ax^0 \\ &\sum_{i=1}^{p} c_{i0}(1)be_i(1) + \sum_{i=1}^{p}\gamma_{i0}bz_i(0) = bx^0. \end{aligned}\right\} \quad (2\text{-}11)$$

To find $c_{i0}(0)$ we use the condition for the solvability of the equation for the first approximation

$$A(t)\overline{x}_1(t) = \frac{d\overline{x}_0(t)}{dt} - s(\overline{x}_0(t), t, 0) + 0 \times s(\overline{x}_0(t), t, 0).$$

$$A(t)\overline{x}_1(t) = \frac{d\overline{x}_0(t)}{dt} - s(\overline{x}_0(t), t, 0) \quad (2\text{-}12)$$

Since $detA(t) \equiv 0$ for $0 \le t \le 1$, then for solvability of (2-12) is necessary and sufficient , so that

$$\sum_{i=1}^{p}\langle g_i(t), e_i(t)\rangle c'_{i0}(0) = \langle g_i(t), f_0\left(\sum_{i=1}^{p} c_{i0}e_i, t, 0\right) + f_1\left(\sum_{i=1}^{p} c_{i0}e_i, t, 0\right) - \sum_{i=1}^{p} c_{i0}e'_i\rangle,$$
(2-13)

where <.., ...> denotes the scalar product, $g_i(t)$ are the eigenvectors of the matrix $A(t)$ conjugate to $A^*(t)$ corresponding to the zero eigenvalue.

You can choose $g_i(t)$ of the same degree of smoothness as $A(t)$. By virtue of requirement 2 , $det\,\|< g_i(t), e_i\| >\neq 0$ for $t \in [0,1]$ and the system (2-13) is solvable with respect to $c'_{i0}(t)$ $(i = 1,2,\dots,p)$:

$$c'_{i0}(t) = G_{i0}(c_{10}(t), c_{20}(t), c_{30}(t), \dots, c_{p0}(t), t). \quad (2\text{-}14)$$

Requirement 4 Let the system (2-14) have a solution

$$c_{i0}(t) = \varphi_{i0}(\alpha_{10}, \alpha_{20}, \alpha_{30}, \dots, \alpha_{p0}, t) \quad (i = 1,2,\dots,p). \quad (2\text{-}15)$$

where α_{10}, α_{20}, $\alpha_{30}, \dots, \alpha_{p0}$, are constants integration and $\alpha_0 = (\alpha_{10}, \alpha_{20}, \alpha_{30}, \dots, \alpha_{p0})$ Belongs to some set Λ.

Substituting (2-15) into (2-11), we have

$$\sum_{i=1}^{p}\varphi_{i0}(\alpha_{10}, \alpha_{20}, \alpha_{30}, \dots, \alpha_{p0}, 0)ae_i(0) + \sum_{i=1}^{p}\beta_{i0}aw_i(0) = ax^0$$

$\sum_{i=1}^{p}\varphi_{i0}(\alpha_{10},\ \alpha_{20},\ \alpha_{30},\dots,\alpha_{p0},1)be_i(1)+\ \sum_{i=1}^{p}\gamma_{i0}bz_i(0)=bx^0.$ (2-16)

Requirement 5. Let the system (2-16) be solvable with respect to the coefficients $\alpha_{10},\ \alpha_{20},\ \alpha_{30},\dots,\alpha_{p0}$, $\beta_{10},\beta_{20},\beta_{30},\dots,\beta_{k0},\ \gamma_{10},\gamma_{20},\gamma_{30},\dots,\gamma_{m0},\ \alpha_0\in\ \Lambda$

And the functional determinant (2-16) for the coefficients is different from zero in some neighborhood of this solution.

Through the coefficients obtained, the zero approximation is determined, and there exist constants $c>0$ and $\delta>0$, such that

$\|\Pi_0x(\tau)\|\le c\ exp(-\delta\tau)\ at\ \tau\ge 0,\ \|Q_0x(s)\|\le c\ exp(\delta s)\quad at\ s\le 0.$

We introduce the curve L_0 in the space (x,t), as in section 14 by (A. B. Vasil'eva, V. F. Butuzov, 1973), consisting of three Parts:

$L_{01}=\{(x,t):x=\ \overline{x}_0(0)+\prod_0x(\tau),\tau\ge 0,t=0\},$

$L_{02}=\{(x,t):x=\ \overline{x}_0(t)\},0\le t\le 1,$

$L_{01}=\{(x,t):x=\ \overline{x}_0(1)+\mathrm{Q}_0x(s),s\le 0,t=1\}$.

Requirement 6. Let $A\,(t),f_0\ (x,t,\mu)$ and $f_1\ (x,t,\mu)$ have continuous derivatives with respect to all arguments order $(n+2)$ inclusive in the η –tube of the curve L_0 and for $0<\mu\ \le d$.

As by (A. B. Vasil'eva, V. F. Butuzov, 1970), the first approximation $\overline{x}_1(t)$ is determined from (2-12)

$\overline{\mathrm{x}}_1(\mathrm{t})=\sum_{i=1}^{p}c_{i1}(t)\ e_i(t)+\tilde{x}_1(t),$ (2-17)

Where $c_{i1}(t)$ are arbitrary functions, $\tilde{x}_1(t)$ is a partial solution of (2-12).

For $\prod_1x(\tau)$ and $\mathrm{Q}_1x(s)$we obtain linear inhomogeneous differential systems whose general solutions of which are written similarly of equation (2-8) and (2-9) with the addition of particular solutions$\prod_1x(\tau)$ and $\mathrm{Q}_1x(s)$, which can be chosen so that $\|\tilde{\Pi}_1x(\tau)\|\le c\ \exp(-\delta\tau)$ at $\tau\ge 0$, $\|\tilde{Q}_1x(s)\|\le c\ \exp(\delta s)$ at $s\le 0$. Requiring that $\prod_1x(\tau)$ and $\mathrm{Q}_1x(s)$ be boundary functions, we obtain from (2-2) in the first approximation

$$\left.\begin{array}{l}\sum_{i=1}^{p}c_{i1}(0)ae_i(0)+\ \sum_{i=1}^{k}\beta_{i1}aw_i(0)=-a\left(\tilde{x}_1(0)+\tilde{\Pi}_1x(0)\right),\\ \sum_{i=1}^{p}c_{i1}(1)be_i(1)+\ \sum_{i=1}^{m}\gamma_{i1}bz_i(0)=-b\left(\tilde{x}_1(1)+\tilde{Q}_1x(0)\right).\end{array}\right\}\quad (2\text{-}18)$$

The functions $c_{i1}(1)$ are determined from the condition for the solvability of a linear algebraic system for $\overline{\mathrm{x}}_2(\mathrm{t})$ in the same way as $c_{i0}(1)$, and for the definition of$c_{i1}(1)$ $(i=1,2,3,\dots,p)$ we obtain a system of p linear differential equations that has a solution $c_{i1}(t)=\varphi_{i1}(\alpha_{11},\ \alpha_{21},\ \alpha_{31},\dots,\alpha_{p1},t)$ $(i=1,2,\dots,p)$. Substituting this into the equation (2-18), We obtain a linear algebraic system with respect to $\alpha_{11},\ \alpha_{21},\ \alpha_{31},\dots,\alpha_{p1}$, $\beta_{11},\beta_{21},\beta_{31},\dots,\beta_{k1},\ \gamma_{11},\gamma_{21},\gamma_{31},\dots,\gamma_{m1},$ The determinant of which coincides with the functional determinant (2-16), Since systems for the determination of derivatives $\frac{\partial\varphi_{i0}}{\partial\alpha_{i0}}$ and $\frac{\partial\varphi_{i1}}{\partial\alpha_{i1}}$ was same. In this way we determine the zero approximation, where

$\|\prod_1\mathrm{x}(\tau)\|\le \mathrm{c}\ \exp(-\delta\tau)$ at $\tau\ge 0$,and $\|\mathrm{Q}_1\mathrm{x}(\mathrm{s})\|\le \mathrm{c}\ \exp(\delta\mathrm{s})$ at $\mathrm{s}\le 0$.

The coefficients of the following approximations are defined similarly.

Theorem under the requirement from 1-5 there are constants $0 < \mu_0 \le d, \eta > 0$, such that for $0 < \mu \le \mu_0$,in the η -tube of L_0 , there exists a unique solution $x\,(t,\mu)$ of problem (2-1) and (2-2) and following inequality holds :

$$\|x\,(t,\mu) - X_2\,(t,\mu)\| \le c\mu^2 \text{ at } 0 \le t \le 1. \qquad (2\text{-}19)$$

Comment. The results can be generalized to the system

$$\mu\frac{dx}{dt} - A(t)x = h(t,\mu) + f_0(x,t,\mu) + \mu f_1(x,t,\mu)$$

provided that $h(t,0)$ is orthogonal to all $g_j(t)$ $(j = 1,2,3,\ldots,p)$, since the degenerate system has $A\,(t)\,\overline{x}_0(t) = -h(t,0)$.

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