Hybrid Linear Multistep Methods with Nested Hybrid Predictors for Solving Linear and Non-linear Initial Value Problems in Ordinary Differential Equations

Esuabana, Ita Micah [1], Ekoro Samson Evale [2]

1. Department of Mathematics, University of Calabar, Calabar, Cross River State, Nigeria.
E-mail: esuabanaita@gmail.com
2. Department of Mathematics, University of Calabar, Calabar, Cross River State, Nigeria.
E-mail: ekorosam@yahoo.com

Abstract:

In this paper, we present new class of hybrid linear multistep methods with nested hybrid predictors for the numerical integration of initial value problems in ordinary differential equations. The derivation of the method is based on interpolation and collocation procedures. The region of absolute stability of the new scheme is investigated using the boundary locus method. The method is demonstrated on some linear and non-linear problems; numerical results are tabulated and are compared with some existing methods.

Key words: Interpolation, collocation, nesting, hybrid, predictors, linear multistep methods.

1.Introduction

Most mathematical formulation of practical problems in Science and Engineering often lead to solving initial value problems in ordinary differential equations of the form

$$y' = f(x, y), \; y(x_0) = y_0 \in \Re \qquad (1.1)$$

$$f : \Re^{m+1} \to \Re^m$$

The solution to this class of problems is the major interest in this paper. Some of these problems do not have analytic solutions. It is of this need that many researchers have developed numerical methods to finding the approximate solutions to the problems. The Authors (Butcher, 2005), (Butcher and Hojjati, 2005), (Burrage and Tian, 2001), (Enright, 1974,2000), (Okuonghae, 2010), (Okuonghae and Ikhile, 2011) among others have developed methods for solving first order initial value problems in Ordinary Differential equations. This is still an active part of research.

Linear multistep methods are popular methods for solving initial value problem in ordinary differential equations. The general k-step linear multistep methods is of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.2)$$

where $\propto_j, \beta_j$ are uniquely determined and $\propto_j + \beta_j \neq 0$ and $\propto_k$is taken as I. The method in (1.2) has order and stability barrier, (Dalquist, 1963). These constraint has motivated many researchers (Ikhile, 2007), (Okuonghae and Ikhile 2013) among others to derived hybrid linear multistep method with the aim to circumvent the constraint.

Okuonghae and Ikhile (2012), proposed a class of A($\propto$)- stable hybrid linear multistep method for stiff, IVPs in ODEs. They modified the second donatives linear multistep method (SGLMM) proposed by (Enright, 1974) of the form

$$y_{n+v_l} = y_{n+k} + \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \lambda_k f_{n+k} \qquad (1.3)$$

by adding an extra hybrid term hf_{n+v} to the right hand side of (1.3) and replaced the term f'_{n+k} by f'_{n+v} to obtain the continuous algorithms of the form

$$y\left(x_n+h(t+1)\right)=\alpha_{k-1}(t)\,y_{n+k-1}+h\sum_{j=0}^{k}\beta_j(t)f_{n+j}+h_v\,f_{n+v}+h^2\Upsilon_v(t)\,f'_{n+v}$$

$t=k-1$, $v=k-\frac{1}{2}$ with the hybrid predictor

$y\left(x_n+vh\right)=\sum_{j=0}^{k}\alpha_j(t)y_{n+j}+h\varphi(t)\,f_{n+k}+h^2\lambda_k(t)\,f'_{n+k}$, $k=t+1$ and $t=k-\frac{1}{2}$, $k=1,2...$

This method contains more stable members than the SDLMM proposed by (Enright, 1974). Following similar ideas of (Enright,1974) and (Cash, 1981), we derived a method of the form

$$y_{n+k}=\sum_{j=0}^{k-1}\alpha_j^{(m)}y_{n+j}+h\left(\sum_{j=0}^{k}\beta_j^{(m)}f_{n+j}+\beta_v^{(m)}f_{n+v_m}\right)+h^2\lambda_k f_{n+k} \quad (1.4)$$

of order $p=2k+2$

with one off-step point and nested hybrid predictors of the form

$$y_{n+v_{l+1}}=\sum_{j=0}^{k}\alpha_j^{(l)}y_{n+j}+h\left(\sum_{j=0}^{k}\beta_j^{(l)}f_{n+j}+\beta_{v_l}^{(l)}f_{n+v_l}\right)+h^2\lambda_k^{(l)}f'_{n+k} \quad (1.5)$$

of order $p^*=2k+3$ and

$$y_{n+v_0}=\sum_{j=0}^{k}\alpha_j^{(-l)}y_{n+j}+h\sum_{j=0}^{k}\beta_j^{(-l)}f_{n+j}+h^2\lambda_k^{(-l)}f'_{n+k} \quad (1.6)$$

of order $p^{**}=2k+2$ where $\left\{\alpha_j^{(m)}\right\}_{j=0}^{k-1}$, $\left\{\beta_j^{(m)}\right\}_{j=0}^{k}$, $\beta_v^{(m)}$, $\left\{\alpha_j^{(-l)}\right\}_{j=0}^{k}$ $\left\{\beta_j^{(-l)}\right\}_{j=0}^{k}$, $\lambda_k^{(-l)}$ are constant parameters to be determined. These constant parameters are chosen in order to make the order as high as possible as well as given small error constants and minimum number of functions evaluation (Okuonghae and Ikhile, 2011). The α_k in (1.4) are normalized i.e $\alpha_k=1$ and $h=x_{n+1}-x_n$ is a fixed mesh size.

The hybrid parameters V_m, V_l *and* V_{l+1} are incorporated to provide collocation points. Where $y_{n+v_m}, y_{n+vl}, y_{n+v_{l+1}}$ and y_{n+v_0} are the hybrid solution at the respective grid points, $y_{n+k,}$ is the ultimate solution. The v_m and v_l are off-steps and are chosen as $v_m=k-\frac{1}{2}, v_l=\frac{v_{l+1}+k}{2}$, $l=0(1)m-1$, $v_l\in(0,k), v_l\neq j,\ j=0(1)k$ where k is the step-number and taken as $k=1,2,3,...$ The hybrid predictor generates other hybrids and are nested into the method to improve stability of the method.

2. Derivation of the hybrid methods

We considered the solution to the IVP in (1.1) in the form

$$y(x)=\sum_{j=0}^{N}a_jx^j \quad (2.1)$$

with $N=2k+2$ as the order of the method, $\left\{x^j\right\}_{j=0}^{N}$ is polynomial basis function and α_j are real parameter constants to be determined. Differentiating (2.1) twice we obtain

$$y'(x)=\sum_{j=1}^{N}ja_jx^{j-1} \quad (2.2)$$

$$y''(x)=\sum_{j=2}^{N} j(j-1)a_j x^{j-2} \quad (2.3)$$

Interpolating (2.1) and (2.3) at $x=x_{n+k}$ and collocating (2.2) at $x=x_{n+j}$, $j=1,2,3,...,k-2$ to obtain the system of equations

$$\begin{pmatrix} 1 & x_n & x_n^2 & . & . & . & x_n^N \\ 1 & x_{n+1} & x_{n+1}^2 & . & . & . & x_{n+1}^N \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & . & . & . & x_{n+k-1}^N \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_N \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+k-1} \end{pmatrix} \quad (2.4)$$

Solving equation (2.4) with mathematica 10.0 software, we obtain the scheme in y_{n+j}, f_{n+j}, f_{n+v_m}, f'_{n+k}, $j=0(1)k$, $k=1,2,...$

3. Derivation of the hybrid predictors

The corresponding numerical solution of the nested hybrid predictor in (1.5) is assume in the form of the polynomial interpolant

$$y\left(x_{n+v_{l+1}}\right)=\sum_{j=0}^{2k+3} c_j x^j \quad (3.1)$$

c_j, $j=0(1)2k+3$ are real constants to be determined following similar procedures as in section 2 and appeared in nested form.

4. The Order of the Hybrid Method

The difference operator L associated with the hybrid linear multistep method version of (1.4) is defined as

$$L\{y(x_n);h\}=\sum_{j=0}^{k}\alpha_j^{(m)}y(x_n+jh)-h\sum_{j=0}^{k}\beta_j^{(m)}y'(x_n+jh)+\beta_v^{(m)}y'(x_n+v_m h) - h^2\lambda_k^{(m)}y''(x_n+kh) \quad (4.1)$$

where $y(x_n)\in[a,b]$ is an arbitrary function.

Definition1:

The hybrid linear multistep methods (1.4) is said to be of order p if

$c_0=c_1=...=c_p=0$ and $c_{p+1}\neq 0$ c_{p+1} is the error constant.

Proposition

The error constant c_{p+1} associated with the hybrid linear multistep method (1.4) is

$$c_{p+1}=\sum_{j=0}^{k}\frac{j^{p+1}}{(p+1)!}\alpha_j^{(m)}-\frac{j^p}{p!}\beta_j^{(m)}-\frac{v_m^p}{p!}\beta_v^{(m)}-\frac{\lambda_k^{(m)}k^{p-1}}{(p-1)!} \quad (4.2)$$

Proof:

The difference operator $\mathcal{L}$ associated with the hybrid linear multistep method (1.4) is defined as

$$L\{y(x_n);h\} = \sum_{j=0}^{k} \alpha_j^{(m)} y(x_n + jh) - h\sum_{j=0}^{k} \beta_j^{(m)} y'(x_n + jh) + \beta_v^{(m)} y'(x_n + v_m h)$$
$$- h^2 \lambda_k^{(m)} y''(x_n + kh)$$

where $y'(x_n + jh) = f(x_n + jh, \mathrm{y}_n + jh)$, $y'(x_n + v_m h) = f(x_n + v_m h, \mathrm{y}_n + v_m h)$,

$y''(x_n + kh) = f'(x_n + kh, \mathrm{y}_n + kh)$

The Taylor series expansion about x_n gives

$$L\{y(x_n);h\} = \sum_{j=0}^{k} \alpha_j^{(m)} \left[y(x_n) + jhy'(x_n) + \frac{j^2h^2}{2!} y''(x_n) + \ldots \right] -$$

$$h\sum_{j=0}^{k} \beta_j^{(m)} \left[y'(x_n) + jhy''(x_n) + \frac{j^2h^2}{2!} y'''(x_n) + \frac{j^3h^3}{3!} y^{1v}(x_n) + \ldots \right] +$$

$$\left[\beta_v^{(m)} y'(x_n) + hv_m \beta_v^{(m)} y''(x_n) + \frac{\beta_v^{(m)} v_m^2 h^2}{2!} y'''(x_n) + \frac{\beta_v^{(m)} v_m^3 h^3}{3!} y^{(iv)}(x_n) + \ldots \right] -$$

$$h^2 \lambda_k^{(m)} \left[y''(x_n) + khy'''(x_n) + khy'''(x_n) + \frac{k^2h^2}{2!} y^{iv}(x_n) + \ldots \right]$$

Grouping in power of h, we obtain

$$L\{y(x_n);h\} = \sum_{j=0}^{k} \alpha_j^{(m)} \; y(x_n) + h\left(\sum_{j=0}^{k} j\alpha_j^{(m)} - \beta_j^{(m)} - \beta_v^{(m)} \right) y'(x_n) +$$

$$h^2 \left[\sum_{j=0}^{k} \frac{j^2 \alpha_j^{(m)}}{2!} - j\beta_j^{(m)} - \beta_v^{(m)} v_m - \lambda_k^{(m)} \right] y''(x_n) +$$

$$h^3 \left(\sum_{j=0}^{k} \frac{j^3 \alpha_j^{(m)}}{3!} - \frac{j^2 \beta_j^{(m)}}{2!} - \frac{\beta_v^{(m)} v_m^2}{2!} - \lambda_k^{(m)} \right) y'''(x_n) + \ldots +$$

$$h^p \left(\sum_{j=0}^{k} \frac{j^p}{p!} \alpha_j^{(m)} - \frac{j^{p-1}}{(p-1)!} \beta_j^{(m)} - \frac{\beta_v^{(m)} v_m^{p-1}}{(p-1)!} - \frac{\lambda_k^{(m)} k^{p-2}}{(p-2)!} \right) y^p(x_n) +$$

$$h^{p+1} \left(\sum_{j=0}^{k} \frac{j^{p+1}}{(p+1)!} \alpha_j^{(m)} - \frac{j^p \beta_j^{(m)}}{p!} - \frac{v_m^p \beta_v^{(m)}}{p!} - \frac{\lambda_k^{(m)} k^{p-1}}{(p-1)!} \right) y^{p+1}(x_n)$$

$$h^0 : \; c_0 = \sum_{j=0}^{k} \alpha_j^{(m)}$$

$$h^1 : \; c_1 = \sum_{j=0}^{k} j\alpha_j^{(m)} - \beta_j^{(m)} - \beta_v^{(m)}$$

$$h^2:\ c_2=\sum_{j=0}^{k}\frac{j^2}{2!}\alpha_j^{(m)}-j\beta_j^{(m)}-v_m\beta_v^{(m)}-\lambda_k^{(m)}$$

$$h^3:\ c_3=\sum_{j=0}^{k}\frac{j^3}{3!}\alpha_j^{(m)}-\frac{j^2\beta_j^{(m)}}{2!}-\frac{\beta_v^{(m)}v_m^2}{2!}-\lambda_k^{(m)}k$$

$$.h^p:\ c_p=\sum_{j=0}^{k}\frac{j^p}{p!}\alpha_j^{(m)}-\frac{j^{p-1}\beta_j^{(m)}}{(p-1)!}-\frac{\beta_v^{(m)}v_m^{p-1}}{(p-1)!}-\frac{\lambda_k^{(m)}k^{p-2}}{(p-2)!}$$

$$h^{p+1}:\ c_{p+1}=\sum_{j=0}^{k}\frac{j^{p+1}}{(p+1)!}\alpha_j^{(m)}-\frac{j^p\beta_j^{(m)}}{(p)!}-\frac{\beta_v^{(m)}v_m^p}{(p)!}-\frac{\beta_k^{(m)}v_m^p}{(p)!}-\frac{\lambda_k^{(m)}k^{p-1}}{(p-1)!}$$

The proof is complete!

The local truncation error of the predictors in (1.5) and (1.6) are respectively

$$y_{n+v_{l+1}}-y\left(x_{n+v_{l+1}}\right)=c_{p^*+1}h^{p^*+1}y^{p^*+1}(x_n)+0\left(h^{p^*+2}\right)$$

and

$y_{n+v_0}-y\left(x_{n+v_0}\right)=c_{p^{**}+1}h^{p^{**}+1}y^{p^{**}+1}(x_n)+0\left(h^{p^{**}+2}\right)$ where c_{p^*+1} and $c_{p^{**}+1}$ of the predictors in (1.5) and (1.6) respectively

Definition 2:(cf: Widlund, 1967)

A numerical algorithm is said to be $A-$stable for some $\alpha\in\left[0,\frac{\pi}{2}\right]$

If the wedge, $s_\alpha=\left\{z:|Arg(-z)|<\alpha, z\neq 0\right\}$ is contained in its region of absolute stability. The largest angle α is the angle of absolute stability.

Definition 3:

A numerical algorithm is said to be A-stable if the region of absolute stability of the scheme lie entirely in the left half of the complex plane $\not\subset$

$R_A=\{z\in \mathbb{C}: Re(z)<o\}$ $R_A=\left\{z\in\not\subset: \mathrm{Re}(z)<0\right\}$

TABLE 1

The discrete coefficients of the method in (1.4)

K	**1**	**2**	**3**	**4**	**5**
M	0	1	2	3	4
v_m	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$\alpha_0^{(m)}$	1	$\frac{1}{49}$	$\frac{1}{811}$	$\frac{-85}{117677}$	$\frac{-1201}{1335674}$
$\alpha_1^{(m)}$	1	$\frac{48}{49}$	$\frac{27}{811}$	$\frac{25}{39209}$	$\frac{-11875}{66783}$
$\alpha_2^{(m)}$	0	1	$\frac{782}{811}$	$\frac{1008}{39209}$	$\frac{-3500}{667837}$
$\alpha_3^{(m)}$	0	0	1	$\frac{115456}{117627}$	$\frac{2000}{667837}$
$\alpha_4^{(m)}$	0	0	0	1	$\frac{1390625}{1335674}$
$\alpha_5^{(m)}$	0	0	0	0	1
$\beta_0^{(m)}$	$\frac{1}{6}$	$\frac{2}{441}$	$\frac{1}{4055}$	$\frac{-4}{274463}$	$\frac{-365}{2003511}$
$\beta_1^{(m)}$	$\frac{1}{6}$	$\frac{32}{147}$	$\frac{9}{811}$	$\frac{-2948}{588135}$	$\frac{-31875}{4674559}$
$\beta_2^{(m)}$	0	$\frac{32}{147}$	$\frac{189}{811}$	$\frac{-96}{39209}$	$\frac{-29500}{667837}$
$\beta_3^{(m)}$	0	0	$\frac{181}{811}$	$\frac{8704}{39209}$	$\frac{-47500}{667837}$
$\beta_4^{(m)}$	0	0	0	$\frac{26056}{117627}$	$\frac{125625}{667837}$
$\beta_5^{(m)}$	0	0	0	0	1
$\beta_v^{(m)}$	$\frac{3}{2}$	$\frac{256}{441}$	$\frac{2304}{4055}$	$\frac{786432}{1372315}$	$\frac{819200}{14024577}$
$\lambda_k^{(m)}$	0	$\frac{-2}{147}$	$\frac{-12}{811}$	$\frac{-568}{39209}$	$\frac{145765}{667837}$

Examples of the Methods in (1-4 – 1.6) and error constants.

For $k=1,\ p=5, m=0,\ v_0=\frac{1}{2}$

$$y_{n+1} = h\left(\frac{f_n}{6} + \frac{2}{3} f_{n+\frac{1}{2}} + \frac{f_{n+1}}{6}\right) + y_n, \ c_5 = \frac{-1}{2880}$$

with the hybrid

$$y_{n+\frac{1}{2}} = h\left(\frac{f_n}{16} - \frac{f_{n+1}}{4}\right) + \frac{5y_n}{16} + \frac{11y_{n+1}}{16} + \frac{1}{32}h^2 f'_{n+1}, \ c_4 = \frac{-1}{3840}$$

For $k = 2, p = 6, m = 1, \mathrm{v}_1 = \frac{3}{2}$

$$y_{n+2} = h\left(\frac{2f_n}{441} + \frac{32f_{n+1}}{147} + \frac{256}{441} f_{n+\frac{3}{2}} + \frac{32f_{n+2}}{147}\right) + \frac{y_n}{49} + \frac{48y_{n+1}}{49} - \frac{2}{147}h^2 f'_{n+2}, \ c_7 = \frac{-1}{185220}$$

with hybrids

$$v_1 = \frac{3}{2}, v_0 = \frac{7}{4}$$

$$y_{n+\frac{3}{2}} = h\left(\frac{75f_n}{154112} + \frac{87f_{n+1}}{2752} - \frac{96}{301} f_{n+\frac{7}{4}} - \frac{675f_{n+2}}{11008}\right) + \frac{103y_n}{44032} + \frac{405y_{n+1}}{2752} + \frac{37449y_{n+2}}{44032} - \frac{9h^2 f'_{n+2}}{22016}, \mathrm{c}_8 = \frac{11}{4931584}$$

$$y_{n+\frac{7}{4}} = h\left(\frac{63f_n}{32768} + \frac{147f_{n+1}}{4096} - \frac{308}{16384} f_{n+2}\right) + \frac{513y_n}{65536} + \frac{343y_{n+1}}{4096} + \frac{59535y_{n+2}}{65536} + \frac{441h^2 f'_{n+2}}{32768},$$

$$c_7 = \frac{-7}{1310720}$$

For $k = 3, p = 8, m = 2, \mathrm{v}_2 = \frac{5}{2}$

$$y_{n+3} = h\left(\frac{f_n}{4055} + \frac{9f_{n+1}}{811} + \frac{189f_{n+2}}{811} + \frac{2304}{4055} f_{n+\frac{5}{2}} + \frac{181f_{n+3}}{811}\right) + \frac{y_n}{811} + \frac{27y_{n+1}}{811} + \frac{783y_{n+2}}{811} - \frac{12h^2}{811} f'_{n+3}$$

$$c_9 = \frac{1}{10899840}$$

With hybrids

$$v_2 = \frac{5}{2}, \mathrm{v}_1 = \frac{11}{4}, \mathrm{v}_0 = \frac{23}{8}$$

$$y_{n+\frac{5}{2}} = h\left(\frac{1105f_n}{14564352} + \frac{17475f_{n+1}}{6178816} + \frac{20775f_{n+2}}{441344} - \frac{9600}{33187} f_{n+\frac{11}{4}} - \frac{91775f_{n+3}}{1324032}\right) + \frac{241y_n}{662016} + \frac{14075y_{n+1}}{1765376} +$$

$$\frac{38475y_{n+2}}{220672} + \frac{4328575y_{n+3}}{5296128} + \frac{775h^2 f'_{n+3}}{882688}, \mathrm{c}_{10} = \frac{25}{1779499008}$$

$$y_{n+\frac{11}{4}} = h\left(\frac{2001307 f_n}{108943638528} + \frac{9577029 f_{n+1}}{15788933120} + \frac{10044573 f_{n+2}}{1578893312} - \frac{162624}{692645} f_{n+\frac{23}{8}} - \frac{11199881}{4736679936} f_{n+3}\right) +$$
$$\frac{6419 y_n}{74010624} + \frac{10065627 y_{n+1}}{6315573248} + \frac{648539 y_{n+2}}{394217728} + \frac{186044389727 y_{n+3}}{1894671944} - \frac{7073297 h^2 f'_{n+3}}{3157786624},$$
$$c_{10} = \frac{381997}{90944254771200}$$

$$y_{n+\frac{23}{8}} = h\left(\frac{28175 f_n}{201326592} + \frac{388815 f_{n+1}}{134217728} + \frac{833175 f_{n+2}}{67108864} - \frac{22680875 f_{n+3}}{201326592}\right) + \frac{30625 y_n}{50331648} + \frac{1581181 y_{n+1}}{268435456} +$$
$$\frac{119025 y_{n+2}}{8388608} + \frac{788646425 y_{n+3}}{805306368} + \frac{648025 h^2 f'_{n+3}}{134217728}, c_9 = \frac{-18515}{154618822656}$$

5. Stability of the hybrid schemes

The stability of the hybrid schemes are investigated through the boundary locus method. Combing the schemes in (1.4) – (1.6) and applying to the scalar test problem $y' = \lambda y$ yields the stability polynomial of the hybrid scheme as

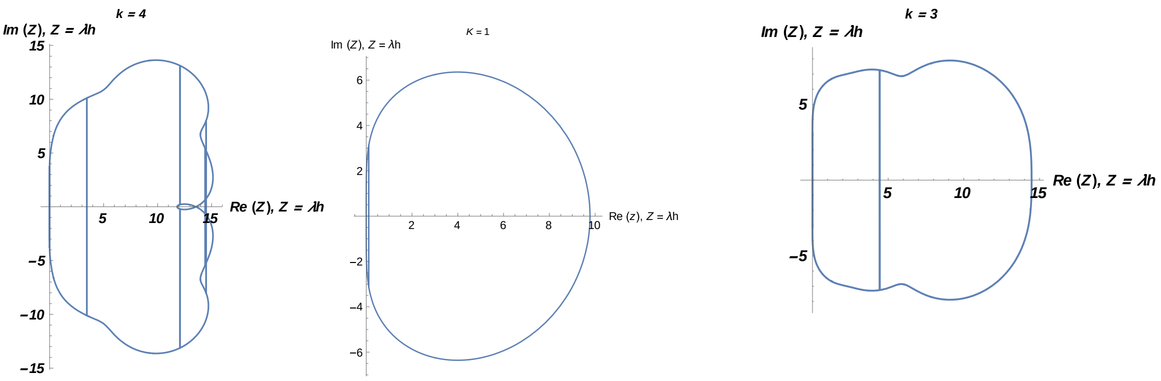
$$\pi(r,z) = r^k - \sum_{j=0}^{k-1} \alpha_j^{(m)} r^j -$$
$$z\left[\left(A_1 + \beta_{v_m}^{(m)}\left(A_2 + z\left(A_3 + \beta_{v_l}^{(l)}\left(\ldots\left(A_4 + zA_5 + z^2 A_6 + z^2 A_7 + z^2 A_8\right)\right)\right)\right)\ldots\right)\right] \quad (5.1)$$

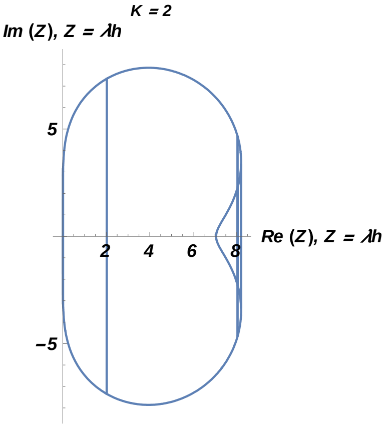
Where $A_1 = \sum_{j=0}^{k} \beta_j^{(m)} r^j$, $A_2 = \sum_{j=0}^{k} \alpha_j^{(l)} r^j$, $A_3 = \sum_{j=0}^{k} \beta_j^{(l)} r^j$, $A_4 = \sum_{j=0}^{k} \alpha_j^{(-l)} r^j$, $A_5 = \sum_{j=0}^{k} \beta_j^{(-l)} r^j$,

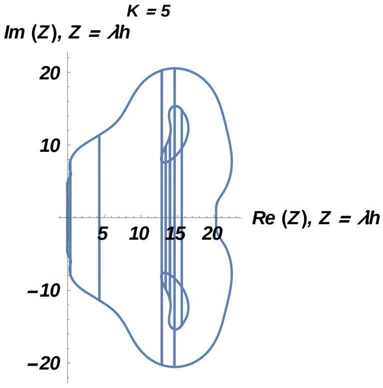
$A_6 = \sum_{j=0}^{k} \lambda_j^{(-l)} r^j$, $A_7 = \lambda_k^{(m)} r^k$, $z = \lambda h$ and $A_8 = \lambda_k^{(l)} r^k$.

To obtain the curves the stability polynomial for each k is plotted and the region of absolute stability of the new scheme reveals that the scheme is zero-stable and $A-$stable for $k = 1, 2, 3, 4, 5$ and $A(\alpha)-$stable for $k = 6, 7$ and instability sets in for $k = 8$. The new scheme contain more stable members than in (Enright, 1974) and (Cash,1981)

Boundary Plots of the scheme







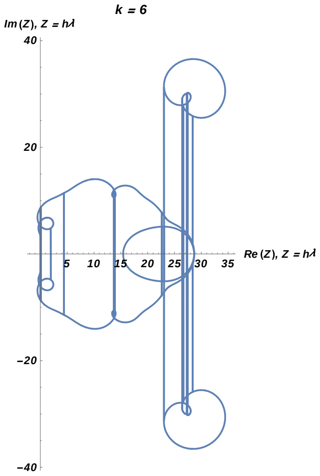


Table 2.

$A(\alpha)$**-stability for $A-EBDF, MEBDF,$ SDLMM, Mehdizadeh, etal and the new method.**

K	1	2	3	4	5	6	7
$A-EBDF\ \alpha\left({}^{\theta}\right)$	90	90	90	88.85	84.2	75	61
$MEBD\ \alpha\left({}^{\theta}\right)$	90	90	90	88.4	82.5	74.5	62
$SDLMM\ \alpha\left({}^{\theta}\right)$	90	90	87.88	82.03	73.10	89.95	37.61
$MEHDZADEH\ etal$ $\alpha\left({}^{\theta}\right)$	90	90	90	90	89.11	73.46	61.05
$NEW\ METHOD$ $\alpha\left({}^{\theta}\right)$	90	90	90	90	90	89.71	80.12

6. Numerical Implementation

Our aim in this section is to solve some existing problems in ordinary differential equations and compare numerical results with other existing methods. The solution components are resolved by applying the new Raphson scheme

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - J\left(y_{n+k}^{[s]}\right)^{-1} F\left(y_{n+k}^{[s]}\right),\ s = 0,1,2,3,\ldots$$

where $J\left(y_{n+k}^{[s]}\right)$ is the Jacobian matrix from the method. The starting value for the Nweton-Raphson scheme is generated from the explicit Trapezoidal rule using fixed step-size h.

We considered the following problems;

Problem 1

Nonlinear problem in (Enright,1974) and (Higham etal, 2000).

$$y'_1 = -0.04y_1 + 10^4 y_2 y_3,\ \ y_1(0) = 1$$

$$y'_2 = 0.04y_1 + 10^4 y_2 y_3 - 3\times 10^7 y_2^2,\ \ y_2(0) = 0$$

$$y'_3 = 3\times 10^7 y_2^2,\ y_3(0) = 0 \quad , \quad x \in [0,3], h = 0.0001$$

Problem 2

Linear problem discussed in Enright,(1974) and Higham etal,(2000).

$$y' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} y,\ y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ y(x) = \begin{pmatrix} e^{-0.1x} \\ e^{-10x} \\ {}^{-100x} \\ e^{-1000x} \end{pmatrix},\ x \in [0,5],\ h = 0.0001$$

Problems 3

Oscillatory problem in Enright etal, (1975).

$$y' = \begin{pmatrix} -10 & \alpha & 0 & 0 & 0 & 0 \\ -\alpha & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{pmatrix} y,\ y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ y(x) = \begin{pmatrix} e^{-10x}(\cos\alpha x + \sin\alpha x) \\ e^{-10x}(\cos\alpha x - \sin\alpha x) \\ e^{-4x} \\ e^{-x} \\ e^{-0.5x} \\ e^{-0.1x} \end{pmatrix}, \alpha = 3$$

Table 3:

Results of Problem 1

X	OKOUNGHAE[2010]	ODEISS	SDLMM[1974]	THE NEW METHOD
0.0000	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
1.0000	3.074626578393852E-005	3.074633616699499E-005	3.814047909744147E-005	3.0745868197E-005
2.0000	2.701783871220487E-005	2.70181637986833E-005	4.008347257110652E-005	2.7017566842E-005
3.0000	2.43833338671147I5E-005	2.48383187366613E-005	4.245457651781839E-005	2.4383138063E-005

Table 4: **Numerical Result of Problem 2**

X	OKOUNGHAE[2010]	ODE15S	SDLMM[1974]	THE NEW METHOD
0.0000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1.0000	0.90437418035960	0.94938740918820	0.904837269447746	0.904028369707018
2.0000	0.819092130934865	0.818730592376307	0.818730592376307	0.818722565811390
3.0000	0.740818220681718	0.741543181821252	0.740818070634993	0.740810812536555
4.0000	0.670320046035639	0.671251130058352	0.670319909449178	0.670313342868700
5.0000	0.606530659712633	0.60736863660	0.60653053598027	0.606524594436367

Table 5: **Numerical Result of Problem 3**

X	EXACT SOLUTION	THE NEW METHOD	ODEI5S	SDLMM[1974]
0.0000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1.0000	-3.8538751357048E-005	-3.8515620444E-005	-3.903944217556801E-005	-3.836188649062032E-005
2.0000	1.40314019820908E-009	1.402503E-009	4.904611584513630E-008	1.403776715337233E-009

Table 6: **ERRORS IN TA BLE 5**

X	NEW METHOD|ERROR|	ODEI5S|ERROR|	SDLMM|ERROR|
0.0000	0.000000000000000	0.000000000000000	0.000000000000000
1.0000	2.3130919039995 27E-008	5.006908185200084E-007	2.31351335723174E-008
2.0000	6.371982090799894E-013	4.764297564692722E-008	6.365171281531113E-013

Summary and conclusion

We have presented new class of hybrid linear multistep method with nested hybrid predictors. The New scheme is found to be A-stable at high order with smaller error constants as seen in table 2. This makes the new scheme suitable for nonlinear and stiff ordinary differential equations. The comparison of the numerical results obtained from the new scheme on some problems with Enright $[1974]$, ODE15s, Cash $[1981]$ among others Shows it reliability for this class of problems and overcome Dahlquist order Barrier.

REFERENCES

Butcher J.C. (2005). High Order A-Stable Numerical Methods for Stiff Problems. *Journal of Science and Computer*, **25**(1), 51 - 66.

Butcher J. C., & Hojjati, G. (2005).Second Derivative methods with RK Stability. *Journal of Numerical Algorithm*, **40**(4), 415 - 429.

Burrage, K. & Tian T. (2001). Stiffly Accurate Runge Kutta Methods for Stiff Stochastic Differential Equations. *Computer Physics and Communication*,**142**, 186 - 190.

Cash, J.R. (1981). Second Derivative Extended Background Differentiation Formula for the Numerical Integration of Stiff Systems. *SIAM, Journal of Numerica Analysis*, **18**, 21 - 36

Cash J. R., Consider, S. (1992). An MEBDF code for Stiff Initial Value Problem. *ACM trans Mathematics Software*,**18,** 29 - 46

Cash, J. R and Gladwell,(1992). *Computational Ordinay Differential Equations* 1 Edition, Oxford: Clarendon Press, 29 - 46.

Dahlquist, G.A. (1963). Special Stability Problems for Linear Multistep Methods, *BIT*, **3**, 27-43.

Enright, W.H.(2000). Continuous Numerical Methods for ODES with defect control. Journal of Computational and Applied Mathematics, **25**, 159 -170.

Enright, W. H., Hull. T. E. and Lindberg, B. (1975). Comparing Numerical Methods for Stiff Systems of ODES. *BIT*, **15**, 1- 48

Enright, W. H. (1974). Second Derivative Multistep Methods for Stiff Ordinary Differential Equations. *Society of applied and industrial Mathematics, Journal of Numerical Analysis*, **11** (2), 321 -331

Fatunla S. O. (1984). Numerical Method for Initial Value Problems in ODES. *Journal of Mathematics Application*, **10**(1), 1-4

Gear, C.W. (1971a). *Numerical Initial Value Problems in Ordinary Differential Equations*. Engle Wood Cliffs, New Jersey: Prentice – Hall.

Gear, C. W. (1968). *The Automatic Integration of Stiff Ordinary Differential Equations*. pp 187 -193, A. J. H Morreli (Ed.) Information Processing 68: Proc. IFIP Congress, Ediburgh ,Nor – Holland, Amsterdam.

Hojjati, G., Rahimi Ardabili M. Y. , & Hosseini, S.M. , (2004). A- EBDF; An Adaptive Method for Numerical Solution Of Stiff Systems of Ordinary Differential Equations. *Journal of Mathematical and computers simulation*, **66**, 33 – 41.

Higham, D.J., Higham, N. J.(2000). – Matlab Guide. *Society of Industrial and Applied Mathematics*, Philadelphia, PA

Khalsaraei, M.M. & Molayi, M.(2015). The New Class of A-stable Hybrid multistep Methods for Numerical Solution of Stiff Initial Value Problems in Ordinary Differential Equations. *Journal of Mathematical Theory and Modelling*, **5**(1), 97-103.

Okuonghae R.I & Ikhile' M. N. O. (2012). A Continuous Formulation of $A(\alpha)$- Stable Linear Multistep Methods for Stiff Initial Value Problems in Ordinary Differential Equations. *Journal of Algorithm & Computer Technology*, **6** (1), 79 – 100

Okuonghae, R. I. (2010). A Class of Continuous hybrid LMM for stiff IVPs in Ordinary Differential Equations. Science of Annals A.I.Cuza University IASi . (Accepted).

Widlund, O. (1967). An Unconditional stable liner multi-step methods. *BIT,* **7**,

Khalsaraei, M.M. & Molayi, M.(2015). The New Class of A-stable Hybrid multistep Methods for Numerical Solution of Stiff Initial Value Problems in Ordinary Differential Equations. *Journal of Mathematical Theory and Modelling*, **5**(1), 97-103.