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Stability Results of Nonlinear Integro-differential Equations

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Abstract
In this paper, the stability of a class of nonlinear integro-differential equation is investigated and analyzed. By defining a suitable Lyapunov functional we establish necessary and sufficient condition -for the stability of the zero solution. Our results extends known results in the literature.
Keywords: Integro-differential equation, Lyapunov functional, Nonlinear, Stability.

1. Introduction

Volterra integro-differential equations have wide applications in biology, ecology, medicine, physics and other scientific areas and thus has been extensively studied. The equilibrium or the steady state of a linear or nonlinear equation can either be stable or unstable. The steady state is called a stable system if after been disturbed by some physical phenomenon returns to its uniform state of rest or its normal position. When a system tends to a new position after a slight displacement, such equilibrium is called unstable equilibrium.

The origin of stability in science and engineering can be track down to the work of Aristotle and Archimedes (Magnus 1959). Alexander Lyapunov was the first to define the notion of stable system in Mathematical form in 1892, in his book on “the general problem of stability”.

The stability theorem for motion studied by A.M Lyapunov has proven to be highly useful and applicable in the field of science and engineering. The notion of stability is studied in the literature under three classes, namely; Bounded input and bounded output (BIBO), Zero-input stability and Input-state Stability. Over the years, Lyapunov method for the stability of integro-differential equation have been proposed by different researcher (Stamove and Stomov (2001, 2013), Tunc (2016), Tunc and Sizar (2017) Vanulailai and Nakagiri (2003), Carabollo et al. (2007), Segeev (2007)).

In particular Stamova and Stomov (2001) worked on the stability of the zero state solution of impulsive function differential equation. They applied the Lyapunov-Razumikhin method and Piecewise continuous function to check the behavioral solution of equation. Vanualailai and Nakagiri (2003) established stability of systems of Volterra integro-differential equation. They used a known form of Lyapunov functional to establish the stability condition for the system. Carabollo et al. (2007) worked on construction of lyapunov functionals to check and investigate the stability for hereditary system. Cemil (2016) studied certain nonlinear Volterra integro differential equations with delay. He established stability and boundedness condition of the solution by defining a suitable Lyapunov functional used to prove the result.

Sergeev (2007) establishes the stability of the solutions of a class of integro-differential equations of Volterra type whose nonlinear term is assumed to be holomorphic function of variables and possible some integral form in a small neighborhood of zero. Stability in Lyapunov's sense of single zero root and of pair of pure imaginary roots for the unperturbed equation is analyzed by relying on functional in integral form represented by Frechet series.

2. Preliminaries

Our aim in this paper is to use a suitable Lyapunov functional and determine necessary and sufficient condition for the stability of the zero solution of the nonlinear integro – differential equation of Volterra type defined by

$$y'(t) = B(t)g\big(y(t)\big) + \int_0^t G\big(t,s,y(s)\big) \qquad (2.1)$$

Where $y \in R$, the functions G is continuous in $(t,s,y)\ for\ 0 \le s \le t < \infty$, B(t) continuous for $0 \le t < \infty$, $g(y(t))\ is\ continuous\ on\ (-\infty,\infty)$ and

$$\int_0^t G\big(t,s,y(s)\big)\,ds < \infty, \quad \int_0^t tG\big(t,s,y(s)\big)\,ds < \infty \qquad (2.2)$$

We use the following notation and basic information throughout this paper. For any $t_0 \ge 0$ and initial function $\phi \in [t_0,t]$, let $y(t) = y(t,t_0,\phi)$ denote the solution of eq. (2.1) on $[t_0,t]$ such that $y(t) = \phi(t)$. Let $C([t_0,t_i])$ and $C([t_0,\infty])$ denote the continuous of real valued functions on $[t_0,t_1]$ and $[t_0,\infty]$ respectively. For $\phi \in C[t_0,0]$, $|\phi| = \sup\{|y(t)| : 0 \le t \le t_0\}$.

Definition 2.1: The zero solution of eq. (2.1) is stable if for each $\varepsilon > 0$ and each $t_0 \ge 0$, there exist $\delta(\varepsilon) > 0$ such that $\|\phi\|_{t_0} < \delta(\varepsilon)$ which implies that $\|y(t,t_0)\| < \varepsilon, for\ t \ge 0$ where $y(t,\phi)$ is a solution of eq. (2.1) which is defined for $t \ge t_0$.

Definition 2.2: The zero solution of eq. (2.1) is uniformly stable if for each $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that $\phi \in [0,t_0]$ with $\|\phi\| < \delta$ (any $t_0 \ge 0$) implies that $\|y(t,\phi)\| < \varepsilon$ for all $t \ge t_0$

Definition 2.3: The function $f(t,y): R^+ \times X \to X$ is called Lipchitz in y if $\forall\ n > 0,\ \exists\ L \ge 0$ such that $\|f(t,y_1) - f(t,y_2)\| \le \llcorner\|y_1 - y_2\|$, for all $y_{1,}\ y_2 \in B_n,\ t \ge 0\ where\ \llcorner$ is called Lipschitz constant and B_n is a close ball with radius n.

Definition 2.4: The zero solution of eq. (2.1) is said to be asymptotically stable if it is stable and there is a number $\delta > 0$ such that any solution $y(t)$ with $\|\phi\| < \delta$ satisfies $\lim_{t\to\infty}\|y(t)\| = 0$.

Definition 2.5: Suppose $\phi \in R^n$ for each solution $y(t,\phi)$ and if $\exists$ $B(y_0)$ such that $|y(t,y_0)| \leq B(y_0)$ $for\ t \geq 0$. Then the solution of eq. (2.1) is bounded.

The following theorem is essential for stability result and is a basic tool for our results.

Theorem 2.1 [Driver (1962)]. If there exists a functional $V(t,\phi(.))$, defined whenever $t \geq t_0 \geq 0$ and $\phi \in C([0,t],R^n)$ such that

i. $V(t,0) \equiv o, V$ is continuous in t and locally Lipschitz in ϕ

ii. $V(t,\phi(t)) \geq W(|\phi(t)|)$, $W:[0,\infty) \to [0,\infty)$ is a continuous function with $W(0)=0, W(r)>0,\ ifr>0$ and W is strictly increasing (positive definiteness), and

iii. $V'(t,\phi(.)) \leq 0$

then the zero solution of eq. (2.1) is stable and

$$V(t,\phi(.)) = V(t,\phi(s)) : 0 \leq s \leq t$$

Is called a Lyapunov function of eq. (2.1)

3. Main result

Theorem 3.1 $If\ B(t) < 0\ , G(t,s,y(s)) > 0\ and$

$$B(t)g(y(t)) + \int_0^t G(t,s,y(s))\,ds \neq 0 \qquad (3.1)$$

Then the statements below are equivalent.

i. The solution of eq. (2.1) tends to zero.

ii. $B(t)g(y(t)) + \int_0^t G(t,s,y(s))\,ds < -\xi,\ \xi > 0$

iii. Every solution of (2.1) is a Lebesgue integrable function with respect to the vector space R^n.

Proof

We shall adopt the method of Lakshmikantan (1995) to show that (iii) ⇒ (i), (i) ⇒ (ii) and (ii) ⇒ (iii).

Given that $q \in L'(R_+)$, the zero solution $y = 0$ of eq. (2.1) is uniformly stable if and only if the two positive functions $m(t) and\ n(t)$ are uniformly bounded on R_+, it is uniformly asymptotically stable if and only if it is uniformly stable and both $m(t) and\ n(t)$ tend to zero as $t \to \infty$.

We are going to show that (i) ⇒ (ii) and (ii) ⇒ (iii) and we are done.

Let $y(t,t_{0,}1) > 1$ be any solution of (2.1) with initial function $\theta(t) = 0$ on the interval $[0,t_0]$. We claim that $y(t) = y(t,t_0,1) > 0$ on $[0,t_0]$, if not, there exists a $t_1 > t_0$ with $y(t_1) = 1$. Hence

$y'(t_1) < 0$, thus it follows from (i) that

$$y'(t) = B(t)g(y(t)) + \int_0^{t_1} G(t,s,y(s))\,ds > B(t)g(y(t))$$

$$+ \int_0^{t_0} G(t,s,y(s))\,ds \ \geq 0 \qquad (3.2)$$

This is contradiction. Thus, (i) ⇒ (ii), then we are left to prove that (ii) ⇒ (iii).

Choosing a functional candidates

$$V\left(t,s,g\big(y(t)\big)\right) = y(t) + \int_0^t \int_0^\infty G\big(\beta,s,y(s)\big)\ d\beta\ g(y(s))\ ds \qquad (3.3)$$

Then for all $y \neq 0$, assuming $y(t)$ is a solution of (2.1), differentiating (3.3) along the solution of (2.1), we have

$$V'\left(t,s,g\big(y(t)\big)\right) \leq B(t)g\big(y(t)\big) + \int_0^t G\big(t,s,y(s)\big)g\big(y(s)\big)\ ds$$

$$+\int_0^\infty G\big(\beta,s,y(s)\big)\ d\beta\ g\big(y(t)\big) - \int_0^t G\big(t,s,y(s)\big)g\big(y(s)\big)\ ds$$

$$= B(t)g\big(y(t)\big) + \int_0^\infty G\big(\beta,s,y(s)\big)\ d\beta\ g\big(y(t)\big)$$

$$= g\big(y(t)\big)\big[B(t) + \int_0^\infty G\big(\beta,s,y(s)\big)\ d\beta\big]$$

$$V'\left(t,s,g\big(y(t)\big)\right) = -g\big(y(t)\big)\xi$$

Where,

$$\xi > 0, g(y(t)) > 0, then \int_0^\infty g(y(s))\ ds\ < \infty\ and \int_0^\infty G\big(\beta,s,y(s)\big)\,d\beta < \infty$$

Thus the solution of eq. (2.1) is Lebesgue integrable having satisfy the conditions of a Lebesgue integral.

Theorem 3.2

Suppose

i) $\emptyset :\ R_+ \to R$ is continuous and $G(t,s,y)$ is continuous for $0 \leq s \leq t \leq \infty$ and satisfy the Lipchitz condition.

$$\big\|B(t)g\big(y_1(s)\big) - B(t)g\big(y_2(s)\big)\big\| + \left\|\int_0^t G(t,s,y_1)ds - \int_0^t G(t,s,y_2)ds\right\|$$

$$\leq L_1\|y_1 - y_2\| + \emptyset L_2\|y_1 - y_2\|$$

For all $\|t - t_0\| \leq \propto$, $\|y_1\| < \infty$, $\|y_2\| < \infty\ and\ \emptyset > 0$. Then eq. (2.1) is unique.

ii) If the integral $\int_0^t \big|G\big(\beta,s,y(s)\big)\big|\ ds$ is defined and continuous for $0 \leq s \leq t \leq \infty$ and given positive number ξ such that

$$2|B(t)| + \int_0^t G(t,s,y_1)ds - \int_0^t G(t,s,y_2)ds \leq -\xi$$

Then the zero solution is stable if and only if $G(t) < 0$.

Proof:

Taking $y(t_0) = y_0$, and integrating eq. (2.1), we have

$$y = y_0 + \int_0^t B(t)g(y(t))dt + \int_0^t \left(\int_0^t G(t,s,y(s))ds\right)\ dt$$

where y_0 is constant.

$$y = y_0 + \int_0^t \left(B(t)g(y(t))dt + \int_0^t G(t,s,y(s))ds\right)\ dt$$

$$y = y_0 + \int_0^t F(t,s,y)\ dt\ where\ F(t,s,y) = B(t)g(y(t))dt + \int_0^t G(t,s,y(s))ds.$$

We observe that y_0 is contact and always continuous. $F(t,s,y)$ is continuous for $0 \leq s \leq t < \infty$ and B is continuous in $0 \leq t \leq \infty$. Again we are to show that the function satisfies the libschitz condition.

$$\|F(t,s,y_1)-F(t,s,y_2)\|$$
$$=\left\|B(t)g(y_1)+\int_0^t G\big(t,s,y_1(s)\big)ds-\ B(t)g(y_2(t))+\int_0^t G(t,s,y_2(s))ds\right\|$$
$$\leq \left\|B(t)g\big(y_1(t)\big)-B(t)g\big(y_2(t)\big)\right\|+\left\|\int_0^t G\big(t,s,y_1(s)\big)ds-\int_0^t G(t,s,y_2(s))ds\right\|$$
$$\leq L_1\|y_1-y_2\|+\emptyset L_2\|y_1-y_2\|$$
$$\leq \|y_1-y_2\|(L_1+\emptyset L_2)$$

Here F satisfies the Lipchitz condition, B and G are continuous and eq. (2.1) has a unique solution.

Assuming $G(t)<0$, choosing a lyapunov functional candidate.

$$V\left(t,s,g\big(y(t)\big)\right)=g(y^2(t))+\int_0^t\int_0^\infty\left|G\big(\beta,s,y(s)\big)ds\ g(y^2(s))\right|ds \qquad (3.4)$$

Differentiating the Lyapunov functional along the solution of equation (2.1) with respect to time, we have

$$V'\left(t,s,g\big(y(t)\big)\right)\leq 2B(t)g(y^2(t))+2\int_0^t\left|G\big(t,s,y(s)\big)\right|\left|g\big(y(s)\big)\right|ds$$
$$+\int_0^\infty G\big(\beta,t,y(s)\big)d\beta\ g(y^2(s))\ ds-\int_0^t\left|G\big(t,s,y(s)\big)\right|g(y^2(s))\ ds$$

Since

$$2\left|g\big(y(s)\big)\right|\leq g(y^2(s))+g(y^2),$$

it then follow that

$$V'\leq 2B(t)g(y^2(t))+\int_0^t\left|G\big(t,s,y(s)\big)\right|[g(y^2(s))+g(y^2)]$$
$$+\int_t^\infty\left|G\big(\beta,t,y(s)\big)\right|d\beta\ g(y^2(s))\ ds-\int_0^t\left|G\big(t,s,y(s)\big)\right|g(y^2)\ ds$$
$$\leq 2B(t)g(y^2(t))+\int_0^t\left|G\big(t,s,y(s)\big)\right|g(y^2(s))ds+\int_t^\infty\left|G\big(t,s,y(s)\big)\right|g(y^2)ds$$
$$+\int_t^\infty\left|G\big(\beta,t,y(s)\big)\right|d\beta\ g(y^2(s))\ ds-\int_0^t\left|G\big(t,s,y(s)\big)\right|g(y^2)\ ds$$
$$\leq 2B(t)g(y^2(t))+\int_0^t\left|G\big(t,s,y(s)\big)\right|g(y^2(s))ds+\int_t^\infty\left|G\big(\beta,t,y(s)\big)\right|g(y^2(s))d\beta$$
$$=\left[2B(t)+\int_0^t\left|G\big(t,s,y(s)\big)\right|ds+\int_t^\infty\left|G\big(\beta,t,y(s)\big)\right|d\beta\right]g(y^2(s))$$
$$\leq -\xi g\big(y^2(s)\big) \qquad (3.5)$$

Hence,

$$V'\left(t,s,g\big(y(t)\big)\right)\leq -\xi\ g\big(y^2(s)\big)\text{ where }\ \xi>0.$$

Thus, the zero solution of (2.1) is stable if V is positive definite and $V'\left(t,s,g\big(y(t)\big)\right)\leq 0$. Now suppose that $G(t)>0$ and define the lyapunov functional

$$V\left(t,s,g\big(y(t)\big)\right)=g(y^2(t))+\int_0^t\int_0^\infty\left|G\big(\beta,s,y(s)\big)\right|d\beta\ g(y^2(s))\ ds \qquad (3.6)$$

$$V'\left(t,s,g\big(y(t)\big)\right)\geq 2B(t)g\big(y^2(t)\big)-2\int_0^t\left|G\big(t,s,y(s)\big)\right|\left|g\big(y(s)\big)\right|ds+$$

$$\int_0^\infty G(\beta,t,y(s))d\beta\; g(y^2(s))\,ds + \int_0^t |G(t,s,y(s))| g(y^2(s))\,ds$$
$$= \left[2B(t) - \int_0^t |G(t,s,y(s))|ds - \int_t^\infty |G(\beta,t,y(s))|d\beta\right] g(y^2(s))$$

hence,

$$V'\left(t,s,g(y(t))\right) \geq \xi\, g(y^2(s)) \tag{3.7}$$

Now given any $t_0 \geq 0$ and $\delta \geq 0$, $g(y(t)) : [0,t_0] \to R$ is continuous with

$$\|g(y(t))\| < \delta \text{ and } \mathrm{V}(t_o, s_0, \mathrm{g}(\mathrm{y}(\mathrm{t})) > 0$$

such that if

$$y(t) = y(t_0, t, g(y(t)))$$

is a solution of (2.1), then we obtain (3.6) and (3.7) such that

$$g(y(t)) \geq V\left(t,s,g(y(t))\right) \geq \mathrm{V}(t_o,\mathrm{s},\mathrm{g}(\mathrm{y}(\mathrm{t})) + \xi \int_0^t g(y(s))$$
$$= \mathrm{V}(t_o,\mathrm{s},\mathrm{g}(\mathrm{y}(\mathrm{t})) + \xi \mathrm{V}(t_o,\mathrm{s},\mathrm{g}(\mathrm{y}(\mathrm{t}))(t-t_0)$$

Hence

$$|g(y(t))| \to \infty \; as \; t \to \infty$$

and the proof is complete.

Conclusion

The behavior of integro-differential equation is frequently described by the construction of lyapunov functional. The method of lyapunov functional construction has a wide range of application in investigating the stability of functional differential equation, difference equation with continous or discrete time etc. In this paper by constructing a suitable Lyapunov function we proved necessary and sufficient condition for the stability of the zero solution of a class of nonlinear integro-differential equation.

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