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The Correspondence between Composite Functions in Modern Algebra and Calculus

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Abstract

The study of maps which act on a finite set of objects is of special importance in modern algebra. The main objective of this paper is to compare between the structures of groups that yield from the definition of composite functions in group theory (permutation groups) and those in calculus.

In calculus there no detailed study for groups, but we took the definition of composite functions as a binary operation to define a group. This study can be considered as an essential model for isomorphic groups.

1. Introduction

Definition 1:

To define maps of Sets: Let f: $G \rightarrow G$ be a map of G into itself such that G: ξ, η, ζ, δ, … be a finite or infinite set of objects, then f is a rule whereby to each $\xi \in G$ there is assigned a unique object $\eta \in G$, called the image of ξ under f. We can write $\eta = f(\xi)$. Two maps f and g are equal if and only if $f(\xi) = g(\xi)$ $\forall \xi \in G$. The composite of f and g is the map $f_{\bullet}g$ defined by $f_{\bullet}g(\xi) = g(f(\xi))$ which means that $f_{\bullet}g$ is obtained by letting f be followed by g. Thus if $f(\xi) = \eta$, then $f_{\bullet}g(\xi) = g(\eta)$. But the definition in calculus is: $f_o g(\xi) = f(g(\xi))$ [5], [6].

Now let f, g and h be three maps of G into itself. We can show that the composition of these maps always obeys the associative law. Let $\xi \in G$ and put $f(\xi) = \eta$, $g(\eta) = \zeta$, $h(\zeta) = \tau$.

Then

$f_{\bullet}(g_{\bullet}h)(\xi) = g_{\bullet}h(f(\xi)) = g_{\bullet}h(\eta) = h(g(\eta)) = h(\zeta) = \tau$

and $(f_{\bullet}g)_{\bullet}h(\xi) = (g(f(\xi))_{\bullet}h = (g(\eta))_{\bullet}h = h(g(\eta)) = h(\zeta) = \tau$.

Since ξ was an arbitrary element of G, it follows that

$$f_{\bullet}(g_{\bullet}h) = (f_{\bullet}g)_{\bullet}h \quad (1)$$

Similarly: $f_o(g_o h) = (f_o g)_o h$ (2)

The study of maps which act on a finite set *G* of objects is of especial importance. For simplicity the objects of *G* are often denoted by the integers 1, 2, 3,…, *n* [1], [4].

Definition 2: A map of G (of order *n*) onto itself is called a permutation of degree *n*.

It is explicitly described by the symbol:

$$\pi = \begin{pmatrix} 1 & 2 \ldots j \ldots n \\ a_1 & a_2 \ldots a_j \ldots a_n \end{pmatrix} \quad (3)$$

Where $a_j = \pi j$ is the image of *j* under π. Thus the second row in (3) is a rearrangement of the integers 1, 2, …, *n*. From elementary algebra it is known that there are *n*! such rearrangements. Hence there *n*! permutations of degree *n*. The complete set of permutations will be denoted by s_n. In fact the different rearrangements of the columns in (3) give the same permutations for example;

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \Lambda \ .$$

If we have another permutation *p* such that

$$p = \begin{pmatrix} 1 & 2 \ \ldots \ n \\ b_1 & b_2 \ \ldots b_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \ldots a_n \\ c_1 & c_2 \ldots c_n \end{pmatrix} \quad (4)$$

Then from (3) and (4) , $\pi p = \begin{pmatrix} 1 & 2 \ldots n \\ c_1 & c_2 \ldots c_n \end{pmatrix}$ [2], [7].

Definition3: Let G and $G^{/}$ be two (finite or infinite) groups, where the identity elements of G and $G^{/}$ are denoted by e and $e^{/}$ respectively. Suppose there exists a one-to-one correspondenceθ: G→ $G^{/}$ between the elements of G and $G^{/}$, that is to each *x* in G there is a unique image *y* in $G^{/}$ and to each image *y* in $G^{/}$ there is a unique element *x* in G. In other words the elements G and $G^{/}$ have been paired off in such a way that each element of G and $G^{/}$ occurs in precisely one pair. Then we say G and $G^{/}$ are isomorphic [3].

Example1: Let z range over the extended z-plane, that is over all complex numbers and the point at infinity. The following six maps transform the extended z-plane into itself and therefore constitute an associative system under composition.

$$f_1 : z \to z \ , \ f_2 : z \to \frac{1}{1-z} \ , f_3 : z \to \frac{z-1}{z} \ , \ f_4 : z \to \frac{1}{z} \ , f_5 : z \to 1-z \ ,$$

$$f_6 : z \to \frac{z}{z-1}, e.g.\ z(f_4 \circ f_3) = f_3(f_4) = f_3(\frac{1}{z}) = \frac{\frac{1}{z}-1}{\frac{1}{z}} = 1-z = f_5.$$

The complete multiplication table of the system is as follows:

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_3	f_1	f_5	f_6	f_4
f_3	f_3	f_1	f_2	f_6	f_4	f_5
f_4	f_4	f_6	f_5	f_1	f_3	f_2
f_5	f_5	f_4	f_6	f_2	f_1	f_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1

Table 1: Multiplication of Composite Functions in z-plane.

If $\Sigma = \{f_1, f_2, f_3, f_4, f_5, f_6\}$, then the system (Σ , $_\circ$) satisfied all the group postulates:

(I) Closed (II) Associative (III) There exists an identity element (f_1)

(IV) The inverse element exists for every element as:

f	f_1	f_2	f_3	f_4	f_5	f_6
f^{1}	f_1	f_3	f_2	f_4	f_5	f_6

We can note that, table 1 is not symmetric about the main diagonal, then the system (Σ , $_\circ$) represents (non Abelian) group [3].

2. Results and Discussion

Now the following examples for investigating the composition of permutation groups according to the definitions in group theory (modern algebra) and the definitions in calculus correspondingly.

Example 2: Let S = {1, 2} and S_2 = { λ_0 , λ_1} such that: $\lambda_0 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $\lambda_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then multiplication tables are:

$\bullet$	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	λ_0

(i) Table 2: Multiplication Table of (S_2 , $\bullet$) in group theory.

The system (S_2 , $\bullet$) satisfied all the group postulates, then represents permutation Abelian group.

$\circ$	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	λ_0

(ii) Table 3: Multiplication Table of (S_2 , o) in calculus.

The system (S_2 , o) satisfied all the group postulates , then represents the same permutation Abelian group.

Example 3: Let S = {1, 2, 3} and S_3 = {λ_0, λ_1, λ_2, λ_3, λ_4, λ_5} such that:

$$\lambda_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \lambda_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Then multiplication tables are:

$\bullet$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_0	λ_3	λ_2	λ_5	λ_4
λ_2	λ_2	λ_4	λ_0	λ_5	λ_1	λ_3
λ_3	λ_3	λ_5	λ_1	λ_4	λ_0	λ_2
λ_4	λ_4	λ_2	λ_5	λ_0	λ_3	λ_1
λ_5	λ_5	λ_3	λ_4	λ_1	λ_2	λ_0

(i) Table 3: Multiplication Table of (S_3 , $\bullet$) in group theory.

The system (S_3 , $\bullet$) satisfied all the group postulates, then represents (non Abelian) group of center λ_0 (commutative elements).

$\circ$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_0	λ_4	λ_5	λ_2	λ_3
λ_2	λ_2	λ_3	λ_0	λ_1	λ_5	λ_4
λ_3	λ_3	λ_2	λ_5	λ_4	λ_0	λ_1
λ_4	λ_4	λ_5	λ_1	λ_0	λ_3	λ_2
λ_5	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0

(ii) Table 4: Multiplication Table of (S_3 , o) by the definition in calculus.

The system (S_3 , o) satisfied all the group postulates, then represents different (non Abelian) group of center λ_0.

Example 4: Let S = {1, 2, 3, 4} and S_4 = {λ_0, λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8, λ_9, λ_{10}, λ_{11}, λ_{12}, λ_{13}, λ_{14}, λ_{15}, λ_{16}, λ_{17}, λ_{18}, λ_{19}, λ_{20}, λ_{21}, λ_{22}, λ_{23}} such that:

$$\lambda_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \lambda_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\lambda_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \lambda_9 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \lambda_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \lambda_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

$$\lambda_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \lambda_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \lambda_{14} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \lambda_{15} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$\lambda_{16} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \lambda_{17} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \lambda_{18} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \lambda_{19} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$\lambda_{20} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}, \lambda_{21} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \lambda_{22} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \lambda_{23} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

For example (by definition in group theory) $\lambda_5 \bullet \lambda_{17} = \lambda_{12}$, but $\lambda_5 \circ \lambda_{17} = \lambda_{15}$ (by definition in calculus). Obviously the systems (S_4 , $\bullet$) and (S_4 , o) are also represent two different (non Abelian) groups of centers λ_0 with respect to group theory and calculus definitions.

3. Conclusion

We can conclude our discussion in three cases:

(a) For S_1 there is no rearrangement.

(b) For S_2 the both definitions of composition in group theory and in calculus yielded the same Abelian groups of order two.

(c) Lastly, S_n where $n > 2$, the definitions of composition in group theory and in calculus, yielded different non Abelian groups but are isomorphic groups.

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