Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.1, 2013 www.iiste.org

# A New Class of A-stable Implicit Schemes for Treatment of Stiff System of Ordinary Differential Equations

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**Abstract**

In this paper, a new class of A-Stable Implicit Rational Runge-Kutta schemes were developed, analyzed and computerized to solve stiff system of ordinary differential equations. The method is motivated by the Implicit Conventional Runge – Kutta Schemes and Rational function approximation. While its development and analyses make use of Taylor series expansion (Taylor and Binomial) and Pade's approximation techniques respectively. The schemes are convergent and A-stable.

**Keywords:** Rational, Runge-Kutta, Consistent, effective, Error bound, Implementation, Convergent, A-stable, A (α) stable

## 1. Introduction

An n$^{th}$ order ordinary differential equation is of the general form

$$y' = f(x, y),\ y(x_o) = y_o \qquad (1.1)$$

where

$y_o = (y_o, y_{o2}, y_{o3} \ldots y_{on})$

A differential equations (1) whose Jacobian possesses eigen values

$$\lambda_j = U_j + iV_j, \quad j = 1(1)n \qquad (1.2)$$

where $i = \sqrt{-1}$ , satisfying the following conditions.

(a) $U_i << 0,\ j = 1(1)n$

(b) $\text{Max}|U_j(x)| >> \min|U_j(x)|$

is Stiff. In this case condition (a) show that the system is stable while (b) indicates that the system possesses some components decay very rapidly.

The problems associated with numerical solution of stiff ODEs were first recognized by Curtis and Hirschfelder (1952). Other requirement include the necessity for the numerical scheme to be either A-stable, Stiffly stable, A (α)-stable and A(o) –stable. These stability criteria require that the numerical schemes must be implicit Dahlquist (1963). In the present of all these problems, Hong Yuanfu (1982) proposed a more general form of this scheme called Explicit Rational R-K scheme. The general form of the scheme is given by

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \qquad (3)$$

where, $K_1 = hf(x_n, y_n)$

$$K_i = hf\left( x_n + c_i h, y_n + \sum_{i=1}^{S} a_{ij} k_j \right), i = 1(1)R$$

$H_1 = hg\,(x_n, z_n)$

$$H_i = hg\left(x_n + d_i h, z_n + \sum_{i=1}^{S} b_{ij} H_j\right) \quad (4)$$

with $$g(x_n, z_n) = -Z_n^2 f(x_{n,}\ y_n) \quad (5)$$

and $$Z_n = \frac{1}{y_n} \quad (6)$$

In his development, $a_{ij} = 0$, $b_{ij} = 0$ for $j > i$. He developed families of methods of orders two and three of these schemes. During analysis, he discover that the schemes are A-stable. This property prompted Okunbor (1985) to develop the order four of this methods. From Okunbor's work, it is observed that the higher the stage of the method, the poorer's the stability. Their performance on stiff oscillatory problem is nothing to write home about.

However, experience with the conventional R-K have shown that Implicit R – K scheme have better resolution properties than Explicit ones. This expectation is the chief mover of the present consideration by Babatola (1999).

## 2. The Development of the Proposed Schemes

An R-stage Implicit Rational R-K scheme is of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \quad (2.1)$$

where,

$$K_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i} a_{ij} k_j\right)$$
$$H_i = hg\left(x_n + d_i h, z_n + \sum_{j=1}^{i} b_{ij} H_j\right) \quad (2.2)$$

and $$g(x_n, z_n) = -Z_n^2 f(x_{n,} y_n) = \frac{1}{y_n^{\ 2}} f(x_n, y_n)$$

with the constraints

$$c_i = \sum_{j=1}^{R} a_{ij}, \qquad d_i = \sum_{jj=1}^{iR} b_{ij} \quad (2.3)$$

The parameters $V_i$, $W_i$, $C_i$, $d_i$, $a_{ij}$ and $b_{ij}$ are to be determined from the system of non-linear equation generated by adopting the following step;

(i) Obtained the Taylor series expansion of $y_{n+1}$, Ki's and Hi's about point ($x_{n,,}$ $y_n$) for i =1(1)R.

(ii) Insert the series expansion into (7).

(iii) Compare the final expansion with Taylor series expansion of $y_{n+1}$ about ($x_n$, $y_n$) in the power of h.

The number of parameters normally exceeds the number of equations, but these parameters are choosen to ensure that (one or more of the following conditions are satisfied).

1. Adequate order of accuracy of the scheme (King 1966).
2. Minimum bound of local truncation error (Gill, 1951).
3. The method has maximum interval of Absolute stability (Blum 1952).
4. Minimize computer storage facilities.

*2.1 One Stage Scheme*

The general one-stage Implicit Rational R-K scheme is of the form

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \quad (10)$$

where,

$$K_1 = hf(x_n + c_1 h,\ y_n + a_{11}K_1)$$
$$H_1 = hg(x_n + d_1 h,\ z_n + b_{11}H_1) \quad (11)$$
$$g(x_n\ z_n) = -Z_n^2 f(x_n, y_n) \quad (12)$$

with the constraints

$$c_1 = a_{11}$$
$$d_1 = b_{11} \quad (13)$$

Adopting binomial expansion theorem on the RHS of equation (10) and ignoring higher order terms, yields

$$y_{n+1} = y_n + W_1 K_1 - y_n^2 V_1 H_1 + (\text{higher order terms}) \quad (14)$$

The Taylor series expansion of $y_{n+1}$ gives

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}Df_n + \frac{h^3}{3!}\left(D^2 f_n + fyDf_n\right) + \frac{h^4}{4!}\left(D^3 f_n + f_y D^2 f_n) - 3Df_n Df_y + f_y^2 Df_n\right) + 0h^5$$

(15)

where,

$$Df_n = f_x + f_n f_y$$
$$D^2 f_n = f_{xx} + 2f_n f_{xy} + 2f_n f_{xy} + f_n^2 f_{yy}$$
$$D^3 f_n = f_{xxx} + 3f_n f_{xxy} + 3 f_n^2 f_{xyy} + f_n^3 f_{yyy}$$

Similarly expand $K_1$ about $(x_n, y_n)$ we have,

$$K_1 = hA_1 + h^2 B_1 + h^3 D_1 + 0h^4 \quad (16)$$

where,

$$A_1 = f_n \qquad B_1 = C_1 Df_n$$
$$D_1 = C_1^2 \ (Df_n f_y + ½ D^2 f_n) \quad (17)$$

In a similar manner, expansion of $H_1$ about $(x_n, y_n)$ yields

$$H_1 = hN_1 + h^2 M1 + h^3 R_1 + 0h^4 \quad (18)$$

where,

$$N_1 = \frac{-f_n}{y_n^2},\ M_1 = \frac{-d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right)$$

$$R_1 = \frac{-d_1^2}{y_n^2}\left[\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{f_n^2}{y_n}\right)\right] + ½\left(D^2 f_n - \frac{2f_n}{y_n}\left(f_{nn}^2 + f_x\right)\right] \quad (19)$$

Adopting (16) and (18) in (14), we obtained

$$y_{n+1} = y_n + W_1\left(hA_1 + h^2 B_1 + h^3 D_1 + 0h^4\right) - y_n^2\left(hN_1 + h^2 M_1 + h^3 R_1 + 0h^4\right)$$
$$= y_n\left(W_1 A_1 - y_n^2 V_1 N_1\right)h + \left(W_1 B_1 - y_n^2 V_1 M_1\right)h^2 + \left(W_1 D_1 - y_n^2 V_1 R_1\right)h^3 + 0(h^4) \quad (20)$$

Comparing the coefficient of the powers of h and $h^{2/n}$ equations (15) and (20) and substitute (17) and (19) to get

$W_1 + V_1 = 1$ (21)

$W_1 C_1 + V_1 d_1 = ½$

With the constraints (13), we obtained family of one stage scheme of order two.

(i) $W_1 = 0$, $V_1 = 1$, $c_1 = d_1 = ½$ , $a_{11} = b_{11} = ½$

scheme (10) yields

$$y_{n+1} = \frac{y_n}{1 + y_n H_1} \tag{22}$$

where $H_1 = hg\ (x_n + ½\ h,\ Z_n + ½\ H_1)$.

Also with

(ii) $V_1 = W_1 = ½$ , $c_1 = a_{11} = ¾$, $d_1 = b_{11} = ¼$ .

The scheme (10) result into

$$y_{n+1} = \frac{y_n + \frac{1}{2}K_1}{1 + \frac{y_n}{2}H_1} \tag{23}$$

where

$K_1 = hf\ (x_n + ¾\ h,\ y_n + ¾\ K_1)$

$H_1 = hf(x_n + ¼\ h,\ z_n + ¼\ H_1)$

Also with

(iii) $W_1 =1$, $V_1 = 0$, $c_1 =d_1 = ½$ , $a_{11} = b_{11} = -\ ½$ .

Scheme (10) result into

$y_{n+1} = y_n + K_1$

where,

$K_1 = hf(x_n + ½h,\ y_n + ½\ K_1)$

Which coincide with Implicit Euler's Scheme of order 2.

## *2.2 Two Stage Schemes*

The general two-stage implicit of Rational Runge-Kutta scheme is of the form

$$y_{n+1} = \frac{y_n + W_1 K_1 + W_2 K_{21}}{1 + y_n (V_1 H_1 + V_2 H_2)} \tag{25}$$

where

$$K_i = hf\left( x_n + c_i h, y_n + \sum_{j=1}^{2} a_{ij} k_j \right), i = 1(1)2$$

$$H_i = hg\left( x_n + d_i h, z_n + \sum_{j=1}^{2} b_{ij} H_j \right), i = 1(1)2 \tag{26}$$

Adopting the same procedure as in one-stage scheme, we obtained the following system of equation for family of two-stage schemes of order three.

$W_1 + W_2 + V_1 + V_2 = 1$

$W_1C_1 + W_2C_2 + V_1d_1 + V_2d_2 = ½$

$W_1 (a_{11}C_1 + a_{12}c_2) + W_2 (a_{21}c_1 + a_{22}c_2) + V_1 (b_{11}d_1 + b_{12}d_2) + V_2(b_{21}d_1 + b_{22}d_2) = 1/6$

$$W_1C_1^2 + W_2C_2^2 + V_1d_1^2 + V_2d_2^2 = 1/3 \qquad (27)$$

with the constraints

$$a_{11} + a_{12} = c_1$$
$$a_{21} + a_{22} = c_2$$
$$b_{11} + b_{12} = d_1$$
$$b_{21} + b_{22} = d_2 \qquad (28)$$

Solving these equations (27 & 28) we obtained family of two stage implicit rational R-K schemes of order three.

(1) $W_1 = W_2 = 0$, $V_1 = ¼$, $V_2 = ¾$ , $c_1 = d_1 = a_{12} = b_{12} = 1$

$a_{11} = b_{11} = a_{21} = b_{21} = 0$

$c_2 = d_2 = a_{22} = b_{22} = 1/3$ (29)

$$y_{n+1} = \frac{y_n}{1 + y_n/4\,(H_1 + 3H_2))}$$

where,

$H_1 = hg(x_n + h, \ z_n + H_2)$

$H_2 = hg(x_n + 1/3h, \ z_n + 1/3H_2)$

(2) Also by setting the values of the parameters we obtain

$V_1 = W_1 = 0$, $V_2 = W_2 = ½$ , $c_1 = d_1 = 0$, $c_2 = ½ + \sqrt{3/6}$

$$d_2 = 1/2 - \frac{v^3}{6}, a_{22} = b_{22} = 1/3, \ a_{21} = \frac{1+\sqrt{3}}{6}$$

$$b_{11} = a_{11} = 1/3, b_{12} = a_{12} = -1/3$$

equation (25) yields

$$y_{n+1} = \frac{yn + 1/2\,K_2}{1 + y_n/2\,H_2} \qquad (30)$$

where

$K_1 = hf(x_n, y_n + 1/3\, K_1 - 1/3\, K_2)$

$$K_2 = hf\left(x_n + (1/2 + \sqrt{\frac{3}{6}}\right)h, y_n + \left(\frac{1+\sqrt{3}}{6})K_1 + 1/3\,K_2\right)$$

$H_1 = hg\ (x_n, \ z_n + 1/3\ H_1 = 1/3\ H_2)$

$$H_2 = hg\left(x_n + (½ + \sqrt{\frac{3}{6}}\right)h,\ Z_n + \frac{\left(1+\sqrt{3}\right)}{6}K_1 + ⅓H_2) \quad (31)$$

Imposing condition $T_{n+1} = 0(h^5)$

We obtain the following equations of two stage family of order four.

$V_1+V_2+W_1+W_2 = 1$

$W_1c_1+W_2c_2+V_1d_1+V_2d_2 = ½$

$W_1c_1^2 + W_2c_2^2 + V_1d_1^2 + V_2d_2^2 = ⅓$

$W_1c_1^3 + W_2c_2^3 + V_1d_1^3 + V_2d_2^3 = ¼$

$W_1 (a_{11}c_1+a_{12}c_2) + W_2 (a_{21}c_1+a_{22}c_2) + V_1 (b_{11}d_1+b_{12}d_2) + V_2(b_{21}d_1 +b_{22}d_2) = ⅙$

$W_1c_1 (a_{11}c_1+a_{12}c_2) + W_2c_2 (a_{21}c_1+a_{22}c_2)+ V_1d_1(b_{11}d_1+b_{12}d_2) +V_2d_2(b_{21}d_1 +b_{22}d_2)= ¼$

$W_1(a_{11}c_1^2 + a_{12}c_2^2) + W_2(a_{21}c_1^2 + a_{22}c_2^2) + V_1(b_{11}d_1^2 + b_{12}d_2^2) + V_2(b_{21}d_1^2 + b_{22}d_2^2) = ½$

$$\begin{aligned} W_1 &= \left[a_{11}\left(a_{11}c_1 + a_{12}c_2\right) + a_{12}\left(a_{21}c_1 + a_{22}c_2\right) + W_2\left[a_{21}\left(a_{11}c_1 + a_{12}c_2\right) + a_{21}c_1 + a_{22}c_2\right)\right] + \\ & V_1\left(b_{11}\left(b_{11}d_1 + b_{12}d_2\right)b_{12}\left(b_{21}d_1 + b_{22}d_2\right) + V_2\left(b_{21}\left(b_{11}d_1 + b_{12}d_2\right) + b_{22}\left(b_{21}d_1 + b_{22}d_2\right)\right] = \tfrac{1}{24} \end{aligned} \quad (32)$$

With the equations (28)and (32). Possible family of two-stage schemes of order four are obtained by setting

(1) $V_1 = V_2 = 0$, $W_1 = W_2 = ½$ , $d_1=c_1 = ½ + \sqrt{3/6}$

$$d_2 = c_2 = ½ - \frac{\sqrt{3}}{6},\ a_{22} = b_{22} = b_{11} = a_{11} = ¼$$

$$a\,a_{12} = b_{12} = ¼ + \frac{\sqrt{3}}{6},\ a_{21} = b_{21} = ¼ - \frac{\sqrt{3}}{6}$$

These into equation (25) yields

$y_{n+1} = y_n + ½\ (K_1 + K_2)$

where $K_1 = hf\,(x_n + \left(½ + \frac{\sqrt{3}}{6}\right)h,\ y_n + ¼K_1 + \left(¼ + \frac{\sqrt{3}}{6}\right)K_2$

$$K_2 = hf\,(x_n + \left(½ + \frac{\sqrt{3}}{6}\right)h + y_n + (¼ - K_1 - \frac{\sqrt{3}}{6}\Big)K_1 + ¼K_2) \quad (33)$$

Which incidentally coincide with 2-stage Implicit R-K scheme of order four. Proposed by Harmmer and Holling Worth (1955).

(ii) $W_1 = W_2 = 0$, $V_1 = V_2 = ½$, $c_2 = d_2 = ½ - \sqrt{3/6}$

$a_{11} = b_{11} = a_{22} = b_{22} = ¼$ , $a_{12} = b_{12} = ¼$

Equation (25) yields

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{2}\left(H_1 + H_2\right)} \quad (34)$$

$$H_1 = hg\,(x_n + \left(½ + \frac{\sqrt{3}}{6}\right), z_n + ¼H_1 + \left(¼ + \frac{\sqrt{3}}{6}\right)H_2)$$

$$H_3 = hg\ (x_n + \left(\tfrac{1}{2} - \frac{\sqrt{3}}{6}\right),\ z_n + \left(\tfrac{1}{4} - \frac{\sqrt{3}}{6}\right)H_1 + \tfrac{1}{4}H_2)$$

Next section analyses the error, consistency, convergence and stability property of these schemes.

## 3. Error, Convergence and Stability Properties

In this section, we shall consider the error, convergence, consistency and stability properties of these schemes.

### *3.1 Error Analysis*

Error of numerical approximation techniques for Stiff ODEs arise from different causes that can be majorly classified into discretization, truncation, and round –off error respectively.

Round-off error is an error introduced as a results of the computing device. Mathematically it can be expressed as

$$Y_{n+1} = y_{n+1} - P_{n+1} \qquad (35)$$

where $y_{n+1}$ is the expected solution of the difference equation (10), while $P_{n+1}$ is the computer output at $(n+1)^{th}$ iteration.

Truncation error on the other hand is the error introduced as a result of ignoring some of the higher terms of the power series (Taylor and Binomial series expansion) during the development of the new schemes.

Discretization error $e_{n+1}$ associated with the formular (10) is the difference between the exact solution $y(x_{n+1})$ and the numerical solution $y_{n+1}$ generated by (10) at point $x_{n+1}$. That is

$$e_{n+1} = y_{n+1} - y(x_{n+1}) \qquad (36)$$

### *3.2 Consistency Property*

The one-step scheme is said to be consistent if

$$\lim_{h\to o}\frac{y_{n+1} - y_n}{h} = f(x_n, y_n) \qquad (37)$$

To show the consistency, we recall that

$$y_{n+1} = y_n - y_n^2\sum_{i=1}^{R} V_i H_i + \sum_{i=1}^{R} W_i K_i + (\text{Higher order terms}) \qquad (38)$$

Subtract $y_n$ from both sides and ignoring higher order terms

$$y_{n+1} - y_n = \sum_{i=1}^{R} W_i K_i - y_n^2\sum_{i=1}^{R} V_i H_i \qquad (39)$$

Substituting the expression for $H_i$ and $K_i$ in equation (8)

$$y_{n+1} - y_n = \sum_{i=1}^{R} W_i\ hf\left(x_n + c_1 h_1 y_n + \sum_{i=1}^{j} a_{ij} K_j\right) + -y_n^2\sum V_i hg\left(x_n + d_i h, z_n + \sum_{j=1}^{i} b_{ij} H_j\right) \qquad (40)$$

Dividing by h and taking limit as $h \to o$

$$\lim_{h\to o}\frac{y_{n+1} - y_n}{h} = \sum_{i=1}^{R} W_1 f(x_{n,}, y_n) - y_n^2\sum V_i g(x_n, z_n) \qquad (41)$$

$$\lim_{h\to o}\frac{y_{n+1} - y_n}{h} = \sum_{i=1}^{R} (W_1 + V_1) f(x_n, y_n) \qquad (42)$$

 

$$\therefore \lim_{h\to o}\frac{y_{n+1}-y_n}{h} = f(x_n, y_n) \quad (43)$$

This shows that Implicit Rational R-K scheme is consistent.

According to Lambert (1963), a consistent one-step method is convergent. Hence the new scheme is convergent.

*3.3. Stability Property*

To show the stability of the scheme, we apply (10) to Dahlquist (1963) stability scalar test initial value problem.

$$y' = \lambda y, \, y(x_o) = y_o \quad (44)$$

For example, the stability scheme (34) with

$V_1 = V_2 = ½, \; W_1 = W_2 = 0, \; c_1 = d_1 = ½ - \sqrt{3/6}$ ,
$c_2 = d_2 = ½ + \sqrt{3/6}$ , $b_{11} = b_{22} = a_{11} = a_{22} = ¼$ .
$b_{12} = a_{12} = ¼ + \sqrt{3/6}$ , $a_{21} = b_{21} = ¼ - \sqrt{3/6}$

is

$$\mu(Z) = \frac{1 + \frac{1}{2}Z + \frac{1}{2}Z^2}{1 - \frac{1}{2}Z - \frac{5}{12}Z^2} \quad (45)$$

This scheme is A-stable with ( -∞, 0) as interval of Absolute stability. Since

$$\lim_{Z\to\infty} it\left|\mu(Z)\right| < 1 \quad (46)$$

*3.4 Numerical Computations and Results*

In order to access the performance of the schemes the following sample problem were solved.

**Problem 1:**

Consider the Stiff systems of ODEs

$$Y' = AY \quad (47)$$

Where A

$$= \begin{bmatrix} 1.0 & -4.99 & 0 \\ 0 & -5.0 & 0 \\ 0 & 2.0 & -12 \end{bmatrix} \quad (48)$$

with initial condition $y(o) = (2, 1, 2), 0 \le x \le 1$

Using step size h = 0.01, the method is implemented and the results are shown in Table (1).

**Problem 2:**

The second sample problem considered is the Stiff system of initial values problem in ODEs.

$$y' = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 \\ 0 & 0 & -9.0 & 0 \\ 0 & 0 & 0 & -10.0 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \quad (49)$$

With initial condition $y(o) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$,

The results are shown in Table 2.

## 4. Conclusion

Implicit Rational Runge-Kutta method for the integration of Stiff system of ODEs has been proposed. Theoretically it has been showed that the scheme is consistent, convergent and A – stable. Numerical results showed that the scheme is accurate and effective. Also from the above results the error is very minimal and this implies that the scheme is very accurate.

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TABLE 1: NUMERICAL RESULT OF A - STABLE IMPLICIT RATIONAL RUNGE-KUTTA SCHEMES FOR SOLVING STIFF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

| | | Y1 | Y2 | Y3 |
|---|---|---|---|---|
| **Xn** | **CONTROL STEP SIZE h** | **E1** | **E2** | **E3** |
| | | .1980099667D+01 | .9706425830D+00 | .8869204674D+00 |
| .3000000000D – 01 | .3000000000D – 01 | .8291942688D-09 | .3281419103D-07 | .8161313500D-05 |
| | | .1885147337D+01 | .8379203859D+00 | .4917945068D+00 |
| .1774236000D+00 | .1771470000D-01 | .9577894033D-01 | .3422855333D-08 | .5357828618D-06 |
| | | .1791235536D+01 | .7191953586D+00 | .2663621637D+00 |
| .3307246652D+00 | .1046033532D-01 | .11050933794D-10 | .35587255336D-09 | .3474808041D-07 |
| | | .1694213422D+01 | 6088845946D+00 | .1365392880D+00 |
| .4977858155D+00 | .6176733963D-02 | .1269873096D-11 | .3655098446D-10 | .2146555961D-08 |
| | | .1556933815D+01 | .4729421983D+00 | .4953161076D-01 |
| .7512863895D+00 | .3647299638D-01 | .1425978891D-08 | .3505060447D-07 | .1010194837D-05 |
| | | .1435390902D+01 | .3709037123D+00 | .1867601194D-01 |
| .9951298893D+00 | .2153693963D-01 | .1594313570D-09 | .3316564301D-08 | .4481540687D-07 |

TABLE 2: NUMERICAL RESULT OF A-STABLE IMPLICIT RATIONAL RUNGE-KUTTA SCHEMES FOR SOLVING STIFF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

| | | Y1 | Y2 | Y3 | Y4 |
|---|---|---|---|---|---|
| **Xn** | **CONTROL STEP SIZE h** | **E1** | **E2** | **E3** | **E4** |
| | | .9950124792D+00 | .9900498337D+00 | .9139311928+00 | .9048374306D+00 |
| .3000000000D – 01 | .3000000000D – 01 | .2597677629D-10 | .4145971344D-09 | .2617874150D-05 | .3971726602D-05 |
| | | .9708623323D+00 | .9425736684D+00 | .5872698932D+00 | .5535451450D+00 |
| .1774236000D+00 | .1771470000D-01 | .3078315380D-11 | .4788947017D-10 | .2005591107D-06 | .2890213078D-06 |
| | | .9402798026D+00 | .8841261072D+00 | .3300866691D+00 | .2918382654D+00 |
| .3694667141D+00 | .1046033532D-01 | .3621547506D-12 | .5454525720D-11 | .1355160001D-07 | .1829417523D-07 |
| | | .9144602205D+00 | .8362374949D+00 | .1999708940D+00 | .1672231757D+00 |
| .5365278644D+00 | .6176733963D-02 | .4285460875D+13 | .6268319197D-12 | .9915873955D-09 | .1265158728D-08 |
| | | .8693495443D+00 | .7557686301D+00 | .8044517344D-01 | .6079796167D-01 |
| .8400599835D+00 | .3647299638D-01 | .4961209221D-10 | .6922001861D-09 | .5087490103D-06 | .5899525189D-06 |