

Subclasses of Analytic Functions Associated With a Family of Multiplier Transformations

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Abstract

In the present paper, we introduce and investigate some new subclasses of analytic functions associated with a family of Multiplier transformations . Such results as subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties are proved. Several sandwich-type results are also derived.
Keywords: Analytic functions, Hadamard product (or convolution), subordination and superordination between analytic functions, Multiplier transformations.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} \dots\}.$$

Let $f, g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad .$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For two functions f and g , analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

Indeed , it is known that

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore , if the function g is univalent in $\mathbb{U}$, then we have the following equivalence :

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For any real numbers s , Kwon and Cho [4] defined the multiplier transformations I_λ^s of functions $f \in \mathcal{A}$ by

$$I_\lambda^s f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^s a_k z^k \quad (\lambda > -1).$$

Obviously , we observe that

$$I_\lambda^s \left(I_\lambda^t f(z) \right) = I_\lambda^{s+t} f(z)$$

for all real numbers s and t.

For $\lambda = 1$ and any integer s , the operator I_λ^s was studied by Uralegaddi and Somanathe [14] . Also , for $s = -1,$ the operator I_λ^s is the integral operator studied by Owa and Srivastava [10] . Moreover , the operator I_λ^s is closely related to the multiplier transformation studied by Jung et al. [3] (also see [2]) , and the differential operator defined by Salagean [11] .

Let

$$f_\lambda^s(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^s z^k \; (s \in \mathbb{R}\,; \lambda > -1)$$

and let $f_{\lambda,\mu}^s$ be defined such that

$$f_\lambda^s(z) * f_{\lambda,\mu}^s(z) = \frac{z}{(1-z)^\mu} \; (\mu > 0\,; z \in \mathbb{U}), \tag{1.2}$$

then , motivated essentially by the Choi - Saigo - Srivastava operator [1] (see also [5] , [8] and [9]) , Kwon and Cho [4] introduced and investigated the operator

$$I_{\lambda,\mu}^s : \mathcal{A} \longrightarrow \mathcal{A},$$

which are defined here by

$$I_{\lambda,\mu}^s f(z) = \left(f_{\lambda,\mu}^s * f\right)(z) \quad (f \in \mathcal{A}\,; s \in \mathbb{R}\,;\ \lambda > -1\,;\ \mu > 0), \tag{1.3}$$

In particular , we note that $I_{0,2}^0 f(z) = zf'(z)$ and $I_{0,2}^1 f(z) = f(z).$
It is easily verified from (1.3) that

$$z(I_{\lambda,\mu}^s f(z))' = \mu\, I_{\lambda,\mu+1}^s f(z) - (\mu - 1)\, I_{\lambda,\mu}^s f(z), \tag{1.4}$$

and

$$z(I_{\lambda,\mu}^{s+1} f(z))' = (\lambda + 1)\, I_{\lambda,\mu}^s f(z) - \lambda\, I_{\lambda,\mu}^{s+1} f(z), \tag{1.5}$$

By making use of the subordination between analytic functions and the operator $I_{\lambda,\mu}^s$, we now introduce the following subclasses of analytic functions .

Definition 1.1 . A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}_{\lambda,\mu}^s(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{I_{\lambda,\mu}^s f(z)}{z} + \alpha\frac{I_{\lambda,\mu+1}^s f(z)}{z} \prec \phi(z) \;\; (z \in \mathbb{U};\ \alpha \in \mathbb{C}\,;\ \phi \in \mathbb{P}) \tag{1.6}$$

Definition 1.2 . A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_{\lambda,\mu}^s(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s+1} f(z)}{z}+\alpha\frac{I_{\lambda,\mu}^{s} f(z)}{z} \prec \phi(z) \ \ (z \in \mathbb{U}\,;\ \alpha \in \mathbb{C}\,;\ \phi \in \mathbb{P}) \qquad (1.7)$$

In the present paper, we aim at proving some subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties associated with the operator $I_{\lambda,\mu}^{s}$. Several sandwich-type results involving this operator are also derived.

2. Preliminary results

In order to prove our main results , we need the following lemmas .

Lemma 2.1. ([6]) Let the function Ω be analytic and convex (univalent) in $\mathbb{U}$ with $\Omega(0) = 1$. Suppose also that the function θ given by

$$\theta(z) = 1 + c_n\, z^n + c_{n+1}\, z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$. If

$$\theta(z) + \frac{z\,\theta'(z)}{\xi} \prec \Omega(z) \ \ (Re(\xi) > 0\,;\ \xi \neq 0\,; z \in \mathbb{U}), \qquad (2.1)$$

then

$$\theta(z) \prec \chi(z) = \frac{\xi}{n} z^{-\frac{\xi}{n}} \int_0^z t^{\frac{\xi}{n}-1} \ \ h(t)dt \prec \Omega(z) \ \ (z \in \mathbb{U}),$$

and χ the best dominant of (2.1) .

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \{\varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\epsilon} f(z) = \infty\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 2.2 .([7]) Let q be convex univalent in $\mathbb{U}$ and $k \in \mathbb{C}$. Further assume that $Re(\bar{k}) > 0\,.If$

$$p \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p + k\,z\,p'$ is univalent in $\mathbb{U}$, then

$$q(z) + k\,z\,q'(z) \prec p(z) + k\,z\,p'(z)$$

implies $q \prec p$, and q is the best subdominant .

Lemma 2.3 .([12]) Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma, \eta \in \mathbb{C}$ with

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > max\left\{0\,,-Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\sigma\, p(z)+\eta z\, p'(z) \prec \sigma q(z)+\eta z\, p'(z),$$

then $p \prec q$, and q is the best dominant .

Lemma 2.4 .([13]) Let the function γ be analytic in $\mathbb{U}$ with

$$\gamma(0)=1 \quad \text{and} \quad Re\big(\gamma(z)\big) > \tfrac{1}{2} \qquad (z \in \mathbb{U}).$$

Then , for any function ψ analytic in $\mathbb{U}$, $(\gamma * \psi)(\mathbb{U})$ is contained in the convex hull of $\psi(\mathbb{U})$.

3. Properties of the function class $\mathcal{F}^{S}_{\lambda,\mu}(\alpha\,;\,\phi)$

We begin by proving our first subordination property given by Theorem 3.1 below .

Theorem 3.1 . Let $f \in \mathcal{F}^{S}_{\lambda,\mu}(\alpha\,;\,\phi)$ with $Re(\alpha) > 0$. Then

$$\frac{I^{S}_{\lambda,\mu} f(z)}{z} \prec \frac{\mu}{\alpha}\, z^{-\frac{\mu}{\alpha}} \int_0^z t^{\frac{\mu}{\alpha}-1}\, \phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}) \;. \tag{3.1}$$

Proof. Let $f \in \mathcal{F}^{S}_{\lambda,\mu}(\alpha\,;\,\emptyset)$ and suppose that

$$h(z)=\frac{I^{S}_{\lambda,\mu} f(z)}{z} \qquad (z \in \mathbb{U}). \tag{3.2}$$

Then h is analytic in $\mathbb{U}$. Combining (1.4) , (1.6) and (3.2) , we easily find that

$$h(z)+\frac{\alpha}{\mu} zh'(z) = (1-\alpha)\frac{I^{S}_{\lambda,\mu} f(z)}{z} + \alpha\frac{I^{S}_{\lambda,\mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U})\,. \tag{3.3}$$

Therefore , an application of Lemma 2.1 for $n=1$ to (3.3) yields the assertion of Theorem 3.1 .

By virtue of Theorem 3.1 , we easily get the following inclusion relationship .

Corollary 3.1 . Let $Re(\alpha) > 0$. Then $\mathcal{F}^{S}_{\lambda,\mu}(\alpha\,;\,\phi) \subset \mathcal{F}^{S}_{\lambda,\mu}(0\,;\,\phi)$.

Theorem 3.2 . Let $\alpha_2 > \alpha_1 \geq 0$. Then $\mathcal{F}^{S}_{\lambda,\mu}(\alpha_2\,;\,\phi) \subset \mathcal{F}^{S}_{\lambda,\mu}(\alpha_1\,;\,\phi)$.

Proof . Suppose that $f \in \mathcal{F}^{S}_{\lambda,\mu}(\alpha_2\,;\,\phi)$. It follows that

$$(1-\alpha_2)\frac{I^{S}_{\lambda,\mu}}{z} + \alpha_2\,\frac{I^{S}_{\lambda,\mu+1}}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.4}$$

Since

$$0 \leq \frac{\alpha_1}{\alpha_2} < 1$$

and the function ϕ is convex and univalent in $\mathbb{U}$, we deduce from (3.1) and (3.4) that

$$(1-\alpha_1)\frac{I^{S}_{\lambda,\mu}}{z} + \alpha_1\,\frac{I^{S}_{\lambda,\mu+1} f(z)}{z}$$

$$= \frac{\alpha_1}{\alpha_2}\left[(1-\alpha_1)\frac{I_{\lambda,\mu}^{s} f(z)}{z} + \alpha_1 \frac{I_{\lambda,\mu+1}^{s} f(z)}{z}\right] + \left(1-\frac{\alpha_1}{\alpha_2}\right)\frac{I_{\lambda,\mu}^{s} f(z)}{z}$$
$$\prec \phi(z) \quad (z \in \mathbb{U}),$$

which implies that $f \in \mathcal{F}_{\lambda,\mu}^{s}(\alpha_1 ; \phi)$. The proof of Theorem 3.3 is evidently completed .

Theorem 3.3 . Let $f \in \mathcal{F}_{\lambda,\mu}^{s}(\alpha ; \phi)$. If the integral operator F is defined by

$$F(z) = \frac{\nu+1}{z^{\nu}} \int_0^z t^{\nu-1} f(t)dt \quad (z \in \mathbb{U} ; \nu > -1), \tag{3.5}$$

then

$$\frac{I_{\lambda,\mu}^{s} F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.6}$$

Proof. Let $f \in \mathcal{F}_{\lambda,\mu}^{s}(\alpha ; \phi)$. Suppose also that

$$G(z) = \frac{I_{\lambda,\mu}^{s} F(z)}{z} \quad (z \in \mathbb{U}). \tag{3.7}$$

From (3.5) , we deduce that

$$z\,(I_{\lambda,\mu}^{s} F(z))' + \nu\, I_{\lambda,\mu}^{s} F(z) = (\nu+1) I_{\lambda,\mu}^{s} f(z)\,. \tag{3.8}$$

Combining (3.1) , (3.7) and (3.8) , we easily gat

$$G(z) + \frac{1}{(1+\nu)}\, zG'(z) = \frac{I_{\lambda,\mu}^{s} f(z)}{z} \prec \phi(z) \;\; (z \in \mathbb{U})\ . \tag{3.9}$$

Thus , by Lemma 2.1 and (3.9), we conclude that the assertion (3.6) of Theorem 3.3 holds .

Theorem 3.4 . Let $F_{\lambda,\mu}^{s}(\xi ; \phi)$ and $g \in \mathcal{A}$ with $Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Suppose also that

$$H(z) = (1-\alpha)\frac{I_{\lambda,\mu}^{s} f(z)}{z} + \alpha \frac{I_{\lambda,\mu+1}^{s} f(z)}{z} \prec \phi(z) \;\; (z \in \mathbb{U}) \quad . \tag{3.10}$$

It follows from (3.10) that

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s}(f*g)(z)}{z} + \frac{I_{\lambda,\mu+1}^{s}(f*g)(z)}{z} = H(z) * \frac{g(z)}{z} \quad (z \in \mathbb{U})\ . \tag{3.11}$$

Since the function ϕ is convex and univalent in $\mathbb{U}$, by virtue of (3.10) , (3.11) and Lemma 2.2 , we conclude that

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s}(f*g)(z)}{z} + \alpha\frac{I_{\lambda,\mu+1}^{s}(f*g)(z)}{z} \prec \phi(z) \;\; (z \in \mathbb{U}), \tag{3.12}$$

which implies that the assertion of Theorem 3.5 holds .

Theorem 3.5 . Let q_1 be univalent in $\mathbb{U}$ and $Re(\alpha) > 0$. Suppose also that q_1 satisfies

$$Re\left(1 + \frac{zq_1''(z)}{q_1'(z)}\right) > \max\left\{0\,, -Re\,\left(\frac{\mu}{\alpha}\right)\right\}. \tag{3.13}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s}(f*g)(z)}{z} + \alpha\frac{I_{\lambda,\mu+1}^{s}(f*g)(z)}{z} \prec q_1(z) + \frac{\alpha}{\mu}\, zq_1'(z), \tag{3.14}$$

then

$$\frac{I_{\lambda,\mu}^{s} f(z)}{z} \prec q_1'(z),$$

and q_1 is the best dominant .

Proof . Let the function h be defined by (3.2). We know that (3.3) holds. Combining (3.3) and (3.14) , we find that

$$h(z)+\frac{\alpha}{\mu}zh'(z)\prec q_1(z)+\frac{\mu}{\alpha}zq_1'(z)\ . \tag{3.15}$$

By Lemma 2.3 and (3.15) , we readily get the assertion of Theorem 3.5 .

If f is subordinate to $\mathcal{F}$, then $\mathcal{F}$ is superordinate to f . We now derive the following superordination result for the class $\mathcal{F}_{\lambda,\mu}^{s}\ (\alpha\ ;\ \phi)$.

Theorem 3.6 . Let q_2 be convex univalent in $\mathbb{U}$, $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$. also let

$$\frac{I_{\lambda,\mu}^{s}\,f(z)}{z}\ \in\ \mathcal{H}[q_2(0),1]\cap Q$$

and

$$(1-\alpha)\ \frac{I_{\lambda,\mu}^{s}\,f(z)}{z}+\alpha\ \frac{I_{\lambda,\mu+1}^{s}\,f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_2(z)+\frac{\alpha}{\mu}z\,q_2'(z)\prec(1-\alpha)\frac{I_{\lambda,\mu}^{s}\,f(z)}{z}+\alpha\frac{I_{\lambda,\mu+1}^{s}\,f(z)}{z},$$

then

$$q_2(z)\prec\frac{I_{\lambda,\mu}^{s}\,f(z)}{z},$$

and q_2 is the best subdominant .

Proof . Let the function h be defined by (3.2) . Then

$$q_2(z)+\frac{\alpha}{\mu}z\,q_2'\,(z)\prec(1-\alpha)\ \frac{I_{\lambda,\mu}^{s}\,f(z)}{z}+\alpha\frac{I_{\lambda,\mu+1}^{s}\,f(z)}{z}=h(z)+\frac{\alpha}{\mu}\ z\,h'\,(z)$$

An application of Lemma 2.4 yields the desired assertion of Theorem 3.6 .

Combining the above results of subordination and superordination , we easily get the following " Sandwich – type result " .

Theorem 3.7 . Let q_3 be convex univalent and q_4 be univalent in $\mathbb{U}$, $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$. Suppose also that q_4 satisfies

$$Re\left(1+\frac{z\,q_4''(z)}{q_4'(z)}\right)>\max\left\{0\,,-Re\left(\frac{\mu}{\alpha}\right)\right\}.$$

If

$$0\neq\frac{I_{\lambda,\mu}^{s}\,f(z)}{z}\in\mathcal{H}\ [q_3(0),1\ ]\cap Q,$$

and

$$(1-\alpha)\ \frac{I_{\lambda,\mu}^{s}\,f(z)}{z}+\alpha\ \frac{I_{\lambda,\mu+1}^{s}\,f(z)}{z}$$

is univalent in $\mathbb{U}$, also

$$q_3(z)+\frac{\alpha}{\mu}z\,q_3'\,(z)\prec(1-\alpha)\ \frac{I_{\lambda,\mu}^{s}\,f(z)}{z}+\alpha\frac{I_{\lambda,\mu+1}^{s}\,f(z)}{z}\prec q_4(z)+\frac{\alpha}{\mu}z\,q_4'\,(z),$$

Then

$$q_3(z)\prec\frac{I_{\lambda,\mu}^{s}\,f(z)}{z}\prec q_4(z)\,,$$

and q_3 and q_4 are , respectively , the best subordinant and the best dominant .

4. Properties of the function class $\mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi)$

By means of (1.5) , and by similarly applying the methods used in the proofs Theorem 3.1 – 3.7 , respectively , we easily get the following properties for the function class $\mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi)$ .Here we choose to omit the details involved .

Corollary 4.1 . Let $f \in \mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi)$ with $R(\alpha) > 0$. Then

$$\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} \prec \frac{\lambda+1}{\alpha}\, z^{-\frac{\lambda+1}{\alpha}} \int_0^z t^{\frac{\lambda+1}{\alpha}-1}\, \phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.2 . Let $\alpha_2 > \alpha_1 \geq 0$. Then $\mathcal{H}_{\lambda,\mu}^{s}(\alpha_2\,;\,\phi) \subset \mathcal{H}_{\lambda,\mu}^{s}(\alpha_1\,;\,\phi)$.

Corollary 4.3 . Let $f \in \mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi)$. If the integral operator F is defined by (3.5) , then

$$\frac{I_{\lambda,\mu}^{s+1} F(z)}{z} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Corollary 4.4 . Let $f \in \mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi)$. And $g \in \mathcal{A}$ with $Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Then

$$(f * g)(z) \in \mathcal{H}_{\lambda,\mu}^{s}(\alpha\,;\,\phi).$$

Corollary 4.5 . Let q_5 be univalent in U and $Re(\alpha) > 0$. Suppose also that q_5 satisfies

$$Re\left(1+\frac{zq_5''(z)}{q_5'}\right) > max\left\{0\,,-Re\left(\frac{\lambda+1}{\alpha}\right)\right\},$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} + \alpha\frac{I_{\lambda,\mu}^{s} f(z)}{z} \prec q_5(z) + \frac{\alpha}{\lambda+1}\, zq_5'(z),$$

then

$$\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} \prec q_5'(z)\,,$$

and q_5 is the best dominant .

Corollary 4.6 . Let q_6 be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Also Let

$$\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} \in \mathcal{H}[q_6(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} + \alpha\frac{I_{\lambda,\mu}^{s} f(z)}{z}$$

be univalent in $\mathbb{U}$ .If

$$q_6(z) + \frac{\alpha}{\lambda+1} zq_6'(z) \prec (1-\alpha)\frac{I_{\lambda,\mu}^{s+1} f(z)}{z} + \alpha\frac{I_{\lambda,\mu}^{s} f(z)}{z}$$

then

$$q_6(z) \prec \frac{I_{\lambda,\mu}^{s+1} f(z)}{z}\,,$$

and q_6 is the best subdominant .

Corollary 4.7 . Let q_7 be convex univalent and q_8 be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Suppose also that q_8 satisfies

$$Re\left(1+\frac{z\,q_8''(z)}{q_8'z}\right) > max\left\{0\,,-Re\left(\frac{\lambda+1}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{I_{\lambda,\mu}^{s+1}\,f(z)}{z} \in \mathcal{H}[q_7(0),1] \cap Q.$$

and

$$(1-\alpha)\frac{I_{\lambda,\mu}^{s+1}\,f(z)}{z} + \alpha\frac{I_{\lambda,\mu}^{s}\,f(z)}{z}$$

is univalent in $\mathbb{U}$, also

$$q_7(z) + \frac{\alpha}{\lambda+1}zq_7'(z) \prec (1-\alpha)\frac{I_{\lambda,\mu}^{s+1}\,f(z)}{z} + \alpha\frac{I_{\lambda,\mu}^{s}\,f(z)}{z} \prec q_8(z) + \frac{\alpha}{\lambda+1}zq_8'(z).$$

then

$$q_7(z) \prec \frac{I_{\lambda,\mu}^{s+1}\,f(z)}{z} \prec q_8(z)\,,$$

and q_7 and q_8 are , respectively , the best subordinant and the best dominant .

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