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A Common Fixed Point Theorem in Cone Metric Spaces

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Abstract

In this paper we prove a fixed point theorem in cone metric spaces, which is an extension of metric space into cone metric spaces. Our result generalizes and extends some recent results.

Keywords: Cone Metric Space, Fixed Point, Asymptotically Regular.

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the centre of strong research activity. In 2007 Huang and Zhang [5] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently , Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3,4] and the references mentioned therein). In this paper we extend the fixed point theorem of P.D.Proinov [7] in metric space to cone metric spaces.

Throughout this paper, E is a real Banach space, N = {1, 2, 3,……} the set of all natural numbers. For the mappings f, g: $X\to X$, let C(f, g) denotes set of coincidence points of f , g , that is,
$C(f,g):=\{z\in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties [5].

1.1 Definition

Let E be a real Banach Space and P a subset of E .The set P is
Called a cone if and only if:

(a). P is closed, nonempty and $P\neq\{0\}$;

(b). $a, b\in R$, $a, b\geq 0$, $x, y\in P$ implies $ax + by\in P$;

(c) . $x\in P$ and $-x\in P$ implies $x = 0$.

1.2 Definition

Let P be a cone in a Banach Space E, define partial ordering '$\leq$'on E with respect to P by $x\leq y$ if and only if $y-x \in P$. We shall write x<y to indicate $x \leq y$ but $x \neq y$ while X<<y will stand for $y-x\in \text{Int } P$, where Int P denotes the interior of the set P. This Cone P is called an order cone.

1.3 Definition

Let E be a Banach Space and $P\subset E$ be an order cone. The order cone P is called normal if there exists L>0 such that for all $x, y\in E$,

$$0\leq x\leq y \text{ implies } \|x\|\leq L\|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

1.4 Definition

Let X be a nonempty set of E .Suppose that the map d: $X\times X\to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and
$d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

1.5 Example ([5])

Let $E = R^2$, $P = \{ (x , y) \in E$ such that : $x, y \geq 0\} \subset R^2$, X = R and d: $X \times X \to E$ such that $d(x, y) = (| x - y | , \alpha | x - y |)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

1.6 Definition

Let (X, d) be a cone metric space .We say that $\{x_n\}$ is
(a) a Cauchy sequence if for every c in E with $0 << c$, there is N such that
for all n , m > N, $d(x_n, x_m) << c$;
(b) a convergent sequence if for any $0 << c$,there is N such that for all
$n > N$, $d(x_n, x) << c$, for some fixed $x \in X$.
A Cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

1.7 Lemma ([5])

Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L. Let $\{x_n\}$ be a sequence in X. Then
(i). $\{x_n\}$ converges to x if and only if $d(x_n ,x) \to 0$ $(n \to \infty)$.
(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n , x_m) \to 0$ $(n,m \to \infty)$.

1.8 Definition ([9])

Let f, g: X→X. Then the pair (f, g) is said to be (IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

2. Common Fixed Point Theorem

In this section we obtain a common fixed point theorem in cone metric spaces, which extend metric space into cone metric spaces.

2.1 Notation

Throughout this section we define the function $E: X \to R_+$ by the formula $E(x) = d(x, Tx)$ for $x \in X$.

The following theorem generalizes and extends Theorem 4.1 of [7].

2.2 Theorem

Let T be a continuous and asymptotically regular self-mapping on a complete cone metric space (X, d) and P be an order cone satisfying the following conditions:

(A1): $d(Tx ,Ty) \leq \varphi (D(x ,y))$, for all $x , y \in X$;

Where, $D(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$, $0 \leq \gamma \leq 1$.
Then T has a unique fixed point.

Proof. We shall show that $(T^n x)$ is a Cauchy sequence for each $x \in X$ and put $x_n = T^n x$ and $E_n = E(T^n x)$ for $n \in N$. Without loss of generality we may assume that $\delta < 2\varepsilon$. Since T is asymptotically regular, then $E_n \to 0$. Hence, there exists an integer $N \geq 1$ such that

$$E_n < \frac{\delta - \varepsilon}{1 + 2\gamma} \quad \text{for all } n \geq N. \tag{2.3}$$

By induction we shall show that

$$d(x_n, x_m) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} \quad \text{for all } m,n \in N \text{, with } m, n \geq N. \tag{2.4}$$

Let $n \geq N$ be fixed, obviously, (2.4) holds for $m = n$.
Assuming (2.4) to hold for an integer $m \geq n$, we shall prove it for m+1.
By the triangle inequality, we get that

$$\begin{aligned} d(x_n, x_{m+1}) &= d(T^n x, T^{m+1} x) \\ &\leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{m+1} x) \\ &= E_n + d(Tx_n, Tx_m) \end{aligned}$$

$$d(x_n, x_{m+1}) \leq E_n + d(Tx_n, Tx_m). \tag{2.5}$$

We claim that

$$d(Tx_n, Tx_m) \leq \varepsilon. \tag{2.6}$$

If $d(Tx_n, Tx_m)$ not less than or equal to ε, then

$$\varepsilon < d(Tx_n, Tx_m) \leq \varphi(D(x_n, x_m)) < \delta.$$

$\Rightarrow$ $\varepsilon < D(x_n, x_m) \leq \varepsilon$, which is a contradiction.

Therefore, $d(Tx_n, Tx_m) \leq \varepsilon$.
Hence, the claim.
From (2.5), (2.6) and (2.4), it follows that

$$d(x_n, x_{m+1}) \leq E_n + \varepsilon < \frac{\delta - \varepsilon}{1 + 2\gamma} + \varepsilon = \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} \quad \text{for all } m, n \geq N.$$

$$d(x_n, x_{m+1}) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} \quad \text{for all } m, n \geq N.$$

Therefore, (2.4) is proved.
Since $\delta < 2\varepsilon$, then (2.4) implies that $d(x_n, x_m) < 2\varepsilon$ for all integers m and n with $m \geq n \geq N$.
Therefore, $\{x_n\}$ is a Cauchy sequence.
Since X is a complete cone metric space, then $\{x_n\}$ converges to a point $z \in X$. If T is continuous, then z is a fixed

point of T.
Uniqueness, let w be another fixed point of T then, (A1) it follows that

$$d(z,w) = d(Tz, Tw) \leq \varphi(D(z,w))$$

$$= \varphi(d(z,w)+ \gamma[d(z,Tz)+d(w,Tw)])$$

$$= \varphi(d(z,w)+ \gamma[d(z,z)+d(w,w)])$$

$$\leq \varphi(d(z,w)) < d(z,w), \text{ a contradiction .}$$

Therefore, T has a unique fixed point.

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References

[1] M.Abbas and G.Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces. *J. Math. Anal. Appl*. **341**(2008) 416-420.
[2] M.Abbas and B.E.Rhoades, Fixed and periodic point results in cone metric Spaces. *Appl.Math . Lett*. **22(**2009), 511-515.
[3] V.Berinde, A common fixed point theorem for compatible quasi contractive self mappings in metric spaces ,*Appl.Math .Comput*.,**213**(2009),348-354.
[4] B.Fisher, Four mappings with a common fixed point *J.Univ .Kuwait Sci*., (1981), 131-139.
[5] L.G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J.Math.Anal.Appl*.**332(**2)(2007)1468-1476.
[6] G.Jungck, Common fixed points for non continuous non self maps on non-metric spaces, *Far East J. Math. Sci.* (FJMS) **4** (1996) 199-215.
[7] P.D.Proinov , Fixed point theorems in metric spaces, *Nonlinear Analysis* **64** (2006), 546-557.
[8] S.Rezapour and Halbarani, Some notes on the paper "cone metric spaces and fixed point theorem of contractive mappings " ,*J.Math. Anal. Appl*., **345** (2008), 719-724.
[9] S.L.Singh, Apichai Hematulin and R.P.Pant, New coincidence and common fixed point theorem, *Applied General Topology* **10** (2009), no.1, 121-130.