Three Stage Inverse Implicit Runge–Kutta Scheme for Solution of Ordinary Differential Equations

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Abstract

This paper describes the Development, Analysis and Implementation of Three-Stage Inverse Implicit Runge–Kutta Schemes for Treatment Ordinary Differential Equations. Its development adopts Taylor series expansion to generate its parameters. The analysis of its basics properties adopted Dalhquist A–stable model test equation and theoretical results show that the method is A–stable, P-stable, consistent and convergent. Numerical results and comparative analysis with existing method show that the method is accurate and effective.

Keywords: A–stable, P–stable, effective, accurate, inverse, convergent, consistent.

1. Introduction

There are many problems in the area of Engineering, Management and Sciences which can leads to Ordinary Differential Equations (ODEs) of the form:

$$y' = f(x, y), \quad y(x_o) = y_o \tag{1}$$

where y is the dependent variable of x.

In 1982, Hong Yuafu proposed Rational Runge-Kutta method of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{2}$$

where,

$$Ki = hf\left(x_n + c_i h,\ y_n + \sum_{j=1}^{R} a_{ij} K_j\right)$$

$$Hi = hg\left(x_n + d_i h, z_n + \sum_{j=1}^{R} b_{ij} H_j\right) \tag{3}$$

with $$g(x_n, z_n) = -Z_n^2 f(x_n, y_n), \quad z_n = \frac{1}{y_n} \tag{4}$$

In his development, he considered the explicit family of the method, that is the case $a_{ij} = b_{ij} = 0$ for $j \geq i$. During his analysis he discovered that the Explicit family is A-stable and P-stable. Okunbor (1985) extended the scheme to family of order four. Babatola (1999) also considered the Implicit family of the method.

Although the method is suitable, accurate and stable, it is bedeviled by the difficult problem of function evaluation of f and g.

The method which we are considering in this work is of the form:

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{R} V_i H_i} \quad (5)$$

Where

$$H_i = hg\left(x_n + d_i h, z_n + \sum_{j=1}^{R} b_{ij} H_j\right) \quad (6)$$

with

$$g(x_n, z_n) = -Z_n^2 f(x_n, y_n),\ Z_n = 1/y_n \quad (7)$$

$$di = \sum_{j=1}^{R} b_{ij}, i = 1(1)R. \quad (8)$$

is called the R-stage Inverse Runge –Kutta method.

In this paper, we consider the scheme for the case R = 3 for numerical solution of initial value problems in ODEs.

To facilitate this, the parameters d_i, b_{ij} and V_i are estimated from the local truncation error equation

$$T_{n+1} = y_{n+1}\left[1 + y_n \sum_{i=1}^{4} V_i H_i\right] - y_n \quad (9)$$

The numerical values of these coefficients can be obtained from the set of non-linear equation generated by adopting the following steps.

Step 1: Take the Taylor series expansion of y_{n+1} and H_i, i = 1(1)R about point (x_n, y_n).

Step 2: Inserts the results of the expansion in step 1 into (9)

Step 3: Rearrange the final expansion about x_n in the power series of h so that T_{n+1} can be expressed in the form

$$T_{n+1} = C_o y_n + C_1 h y'_n + C_2 h^2 y''_n + ... C_p h^p y_n^{(p)} + C_{p+1} h^{p+1} y_n^{(p+1)} + 0h^{p+2} \quad (10)$$

and the order of accuracy of the scheme can be identified.

The scheme (5) is said to be of order P. If

$C_p = 0$, p = 0, 1, 2….P and $C_{p+1} \neq 0$, P = 0,1, 2, 3 …. P (11)

Thus to obtain the values of the above coefficients, we solve the set of non-linear system of equation (11) for P = 0, 1, 2……..P with $C_{p+1} \neq 0$ so as to ensure that the value of the parameters yield computational method that have

(a) Minimum bound of local truncation error (Ralston (1962).
(b) Maximum attainable order of accuracy (King 1966)
(c) Minimum computer storage space.
(d) Large interval of absolute stability.

The derivation of the scheme is taken up in next section.

2. Derivation of the scheme

Now setting R = 3 in equation (5), then Implicit three stage Inverse R -K scheme is of the form

$$y_{n+1} = \frac{y_n}{1 + y_n (V_1 H_1 + V_2 H_2 + V_3 H_3)} \quad (12)$$

Where

$$H_1 = hg(x_n + d_1 h,\ Z_n + b_{11} H_1 + b_{12} H_2 + b_{13} H_3)$$

$$H_2 = hg(x_n + d_2 h,\ Z_n + b_{21} H_1 + b_{22} H_2 + b_{23} H_3) \quad (13)$$

$$H_3 = hg(x_n + d_3 h,\ Z_n + b_{31}H_1 + b_{32}H_2 + b_{33}H_3)$$

Subject to the constraints $d_i = \sum_{j=1}^{3} b_{ij},\ i = 1,2,3$ (14)

From equation (9), its local truncation error is

$$T_{n+1} = y_{n+1}[1 + y_n(V_1H_1 + V_2H_2 + V_3H_3)] - y_n \quad (15)$$

Expanding y_{n+1} in Taylor series about x_n, we have

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(iv)} + \frac{h^5}{5!}y_n^{(v)} + 0h^6 \quad (16)$$

where,

$$y'_n = f_n, \quad y''_n = f_x + f_n f_y = Df_n$$
$$y'''_n = f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy} + f_y(f_x + f_n f_y) = D^2 f_n + f_y Df_n$$
$$y_n^{(iv)} = D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n \quad (17)$$
$$y_n^{(v)} = D^4 f_n + f_y D^3 f_n + Df_y D^2 f_n + Df_y D^2 f_y + 3Df_n D^2 f_y$$

The Taylor series of H_i, i = 1, 2, 3, is

$$H_i = hN_i + h^2M_i + h^3R_i + h^4D_i + h^5P_i + 0h^6 \quad (18)$$

where, i = 1, 2, 3.

$$N_i = \frac{f_n}{y_n^2}, \quad M_i = -\frac{d_i}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right)$$

$$R_i = \frac{1}{y_n^2}(b_{i1}d_1 + b_{i2}d_2 + b_{i3}d_3)\left(Df_n - \frac{2f_n^2}{y_n}\right) - \frac{1}{2}\frac{d_i^2}{y_n^2}\left(D^2 f_n + \frac{2f_n}{y_n}\left(\frac{f_x}{y_n} - f_n^2\right)\right)$$

$$D_i = \frac{1}{y_n^2}(b_{i1}R_1 + b_{i2}R_2 + b_{i3}R_3 + b_{i4}R_4)\left(-\frac{2f_n}{y_n} + f_y\right) + d_i(b_{i1}M_1 + b_{i2}M_2 + b_{i3}M_3)\left(\frac{-2f_x}{y_n} + f_{xy}\right)$$
$$+\frac{1}{6}\frac{d_{ii}^3}{y_n^2}\left[D^3 f_n + \frac{6f_n f_{xx}}{y_n} - \frac{6f_n^2 f_x}{y_n} - \frac{4f_n^3 f_x}{y_n}\right] + f_n^3 f_{yy}$$

$$P_i = (b_{i1}D_1 + b_{i2}D_2 + b_{i3}D_3)\left(\frac{-2f_n}{y_n^2} + f_y\right) + d_i(b_{i1}R_1 + b_{i2}R_2 + b_{i3}R_3)$$
$$+\frac{1}{24}\frac{d_1^4}{y_n^4}\left[D^4 f_n + f_n f_{xxx} - 12f_n^2 f_{xx} + 24f_n^2 f_{xx} + 24f_n^3 f_{xyy} - f_n^4 f_{yyy}\right] \quad (19)$$

Using equations (16) and (18), T_{n+1} becomes

$$T_{n+1} = C_o + C_1 hy'_n + C_2 h^2 y''_n + C_3 h^3 y'''_n + C_4 h^4 y_n^{(iv)} + C_5 h^5 y_n^{(v)} + 0h^6 \quad (20)$$

where $C_o = 0$, $C_1 = y'_n + y_n^2(V_1N_1 + V_2N_2 + V_3N_3)$

$$C_2 = \frac{Df_n}{2} + y_n^2(V_1M_1 + V_2M_2 + V_3M_3) + y_n y'_n(V_1N_1 + V_2N_2 + V_3N_3)$$

$$C_3 = y_n''' + y_n^2(V_1R_1 + V_2R_2 + V_3R_3) + y_ny_n'(V_1M_1 + V_2M_2 + V_3M_3)$$
$$+ y_ny_n''(V_1N_1 + V_2N_2 + V_3N_3)$$
$$C_4 = y_n^{(iv)} + y_n^2(V_1D_1 + V_2D_2 + V_3D_3) + y_ny_n'(V_1R_1 + V_2R_2 + V_3M_3) +$$
$$y_ny''(V_1M_1 + V_2M_2 + V_3M_3)$$
$$C_5 = y_n^{(v)} + y_n^2(V_1P_1 + V_2P_2 + V_3P_3) + y_ny_n'(V_1D_1 + V_2D_2 + V_3D_3)$$
$$+ y_ny_n''(V_1R_1 + V_2R_2 + V_3R_3) \quad (21)$$

where, N_i, M_i, R_i, D_i, and P_i are defined in equation (19).
Imposing accuracy of order five the system of non-linear equation below are obtained as:

$$V_1 + V_2 + V_3 = 1$$
$$V_1d_1 + V_2d_2 + V_3d_3 = \tfrac{1}{2}$$
$$V_1d_1^2 + V_2d_2^2 + V_3d_3^2 = \tfrac{1}{3}$$
$$(V_1d_1^3 + V_2d_2^3 + V_3d_3^3 = \tfrac{1}{4}$$
$$V_1(b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + V_2(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) +$$
$$V_3(b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{6}$$
$$V_1(b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + V_2d_2(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) = \tfrac{1}{2}$$
$$(V_1b_{11} + V_2b_{21} + V_3b_{31})(b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + (V_1b_{12} + V_2b_{22} + b_3b_{32})$$
$$(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) + (V_1b_{12} + V_2b_{23} + V_3b_{33})\ (b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{24}$$

Subject to the constraints

$$b_{11} + b_{12} + b_{13} = d_1 \quad (22)$$
$$b_{21} + b_{22} + b_{23} = d_2$$
$$b_{31} + b_{32} + b_{33} = d_3 \quad (23)$$

solving these equations for the parameters V_1, V_2, V_3
b_{11}, b_{12}, b_{13}, b_{21}, b_{23}, b_3, b_{32}, b_{33}, d_1, d_2, d_3, we get:

$$V_1 = \tfrac{1}{9},\ V_2 = \frac{16+\sqrt{6}}{36},\ V_3 = \frac{16-\sqrt{6}}{36},\ d_1 = 0,\ d_2 = \frac{6-\sqrt{6}}{10}$$

$$d_3 = \frac{6+\sqrt{6}}{10},\quad b1_1 = b_{12} = b_{13} = 0 \quad b_{21} = \frac{9+\sqrt{6}}{75},\ b_{22} = \frac{24+\sqrt{6}}{120},$$

$$b_{23} = \frac{168-73\sqrt{6}}{600},\quad b_{31} = \frac{9+\sqrt{6}}{75},\quad b_{32} = 168 + 73\sqrt{6},$$

$$b_{33} = 24 - \sqrt{6/120}$$

Yielding a 3rd stage method of order five of the form

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{36}\left(4H_1 + \left(16+\sqrt{6}\right)\right)H_2 + 16 - \sqrt{6}\big)H_3} \quad (24)$$

where,

$$H_1 = hg(x_n,\ z_n)$$

$$H_2 = hg\left(x_n + \left(\frac{6-\sqrt{6}}{10}\right)h,\quad Z_n + \left(\frac{9+\sqrt{6}}{75}\right)H_1\left(\frac{24-\sqrt[7]{6}}{120}\right)H_2 + \left(\frac{168-73\sqrt{6}}{600}H_3\right)\right.$$

$$H_3 = hg\left(x_n + \left(\frac{6+\sqrt{6}}{10}\right)h, Z_n + \left(\frac{9-\sqrt{6}}{75}\right)H_1 + \left(\frac{168+73\sqrt{6}}{600}\right)H_2 + \frac{(24-\sqrt{6})}{120}H_3 \right) \quad (25)$$

For the case

$$V_1 = \frac{16-\sqrt{6}}{36}, \quad V_2 = \frac{16+\sqrt{6}}{36}, \quad V_3 = \frac{1}{9}, \ d_1 = \frac{4-\sqrt{6}}{10}, d_2 = \frac{4+\sqrt{6}}{10},$$

$$d_3 = 1, \ b_{11} = \frac{24-\sqrt{6}}{120}, \ b_{12} = \frac{24-\sqrt{6}}{120}, \ b_{13} = b_{23} = b_{33} = 0$$

$$b_{21} = \frac{24+\sqrt{6}}{120}, \ b_{22} = \frac{24+\sqrt{6}}{120}, \ b_{31} = \frac{6-\sqrt{6}}{12}, b_{32} = \frac{6+\sqrt{6}}{12}$$

We get 3-stage method of order five as

$$y_{n+1} = \frac{y_n}{1 + {}^{y_n}\!/_{36}\left[(16-\sqrt{3})H_1 + (16+\sqrt{6})H_2 + 4H_3\right]} \quad (26)$$

where,

$$H_1 = hg\left(x_n + \left(\frac{4-\sqrt{6}}{10}\right)h, Z_n + \left(\frac{24-\sqrt{6}}{120}\right)H_1 + \frac{(24-11\sqrt{6})}{12}H_2 \right)$$

$$H_2 = hg\left(x_n + \left(\frac{4-\sqrt{6}}{10}\right)h, Z_n + \left(\frac{24-11\sqrt{6}}{120}\right)H_1 + \frac{(24-11\sqrt{6})}{120}H_2 \right)$$

$$H_3 = hg\left(x_n + h, Z_n + \left(\frac{6-\sqrt{6}}{12}\right)H_1 + \left(\frac{6+\sqrt{6}}{12}\right)H_2 \right) \quad (27)$$

$$V_1 + V_2 + V_3 = 1$$

$$V_1 d_1 + V_2 d_2 + V_3 d_3 = \tfrac{1}{2}$$

$$V_1 d_1^2 + V_2 d_2^2 + V_3 d_3^2 = \tfrac{1}{3}$$

$$(V_1 d_1^3 + V_2 d_2^3 + V_3 d_3^3 = \tfrac{1}{4}$$

$$V_1\left(b_{11}d^2{}_1 + b_{12}d_2^2 + b_{13}d_3^2\right) + V_2(b_{21}d_1^2 + b_{22}d_2^2 + b_{23}d_3^2) + V_3\left(b_{31}d_1^2 + b_{32}d_2^2 + b_{32}d_2^2 + b_{33}d_3^2\right) = \tfrac{1}{12}$$

$$V_1(b_{11} + V_2 b_{21} + V_3 b_{31}) + (b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + (V_1 b_{12} + V_2 b_{22} + V_3 b_{32})$$

$$(b_{21}d_1 + b_{22}d_2 + b_{32}d_3) + (V_1 b_{13} + V_2 b_{23} + V_3 b_{33})(b_{31}d_1 + b_{32}d_2 + b_{33}d_3) = \tfrac{1}{24}$$

$$V_1 d_1(b_{11}d_1 + b_{12}d_2 + b_{13}d_3) + V_2 d_2(b_{21}d_1 + b_{22}d_2 + b_{23}d_3) + V_3 d_3\left(b_{31}d_1 + b_{32}d_2 + b_{33}d_3\right) = \tfrac{1}{8}$$

$$V_3(b_{11}d_2^2 + b_{12}d_2^2 + b_{13}d_3^2) + V_2(b_{21}d_1^2 + b_{22}d_2^2 + b_{23}d_3^2) + V_3\left(b_{31}d_1^2 + b_{32}d_2^2 + b_{33}d_3^2\right) = \tfrac{1}{48} \quad (28)$$

With the constraints in equation (23). Solving these equations that is (28) and (23), we obtained 3 stage Implicit Inverse R-K schemes of order six with cases

$$V_1 = \tfrac{5}{18}, \ V_2 = \tfrac{8}{18}, \ V_3 = \tfrac{5}{18}$$

$$d_1 = \frac{5-\sqrt{15}}{10}, \ d_2 = \tfrac{1}{2},$$

$$d_3 = \frac{5-\sqrt{15}}{10}, \ b_{11} = \frac{5}{36}, \ b_{12} = \frac{10-3\sqrt{15}}{45}, \ b_{13} = \frac{25-6\sqrt{15}}{180}$$

$$b_{21} = \frac{10+3\sqrt{15}}{72},\quad b_{22} = {}^{2}\!/_{9},\ b_{23} = \frac{10-3\sqrt{15}}{72},\quad b_{31} = \frac{25+6\sqrt{15}}{180}$$

$$b_{32} = \frac{10-3\sqrt{15}}{45},\ b_{33} = {}^{5}\!/_{36}$$

as

$$y_{n+1} = \frac{y_n}{1 + {}^{y_n}\!/_{18}\left(5H_1 + 8H_2 + 5H_3\right)} \tag{29}$$

Where,

$$H_1 = hg\left(x_n + \left(\frac{5-\sqrt{15}}{10}\right)h,\quad Z_n + \frac{5}{36}H_1 + \left(\frac{10-3\sqrt{15}}{45}\right)H_2 + \left(\frac{25-6\sqrt{15}}{180}\right)H_3\right)$$

$$H_2 = hg\left(x_n + {}^{1}\!/_{2}h,\ Z_n + \left(\frac{10-3\sqrt{15}}{72}\right)H_1 + 2/9\,H_2 + \left(\frac{10-3\sqrt{15}}{72}\right)H_3\right)$$

$$H_3 = hg\left((x_n + \left(\frac{15-\sqrt{15}}{10}\right)h, Z_n + \left(\frac{25-6\sqrt{15}}{180}\right)H_1 + \left(\frac{10-3\sqrt{15}}{45}\right)H_2 + {}^{5}\!/_{18}H_3\right) \tag{30}$$

3. Analysis of the Basic Properties

The basic properties required of a good computational method include accuracy, consistency, convergence and stability.

Therefore, in this section, we shall consider the analysis of the error, consistency , convergence and stability properties of the proposed schemes.

3.1 Consistency

A scheme is said to be consistent if the difference equation of the computation formulae exactly approximate the differential equation it intends to solve (Ademiluyi & Babatola (2001)).

To prove that the equation (12) is consistent.

Recall that

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{3} V_i H_i} \tag{31}$$

Subtracting y_n on both sides of equation (31), we get

$$y_{n+1} - y_n = \frac{y_n}{1 + y_n \sum_{i=1}^{3} V_i H_i} - y_n \tag{32}$$

Simplifying:

$$y_{n+1} - y_n = \frac{-y_n{}^2 \sum_{i=1}^{3} V_i H_i}{1 + y_n \sum_{i=1}^{3} V_i H_i} \tag{33}$$

But

$$H_i = hg\left(x_n + d_i h,\ Z_n + \sum_{j=1}^{3} b_{ij} H_j \right) \tag{34}$$

Hence,

$$y_{n+1} - y_n = -\frac{-y_n^2 h \sum_{i=1}^{3} V_i g\left(x_n + d_i h, Z_n + \sum_{j=1}^{3} b_{ij} H_j \right)}{1 + y_n \sum_{i=1}^{3} V_i hg\left(x_n + d_i h, Z_n + \sum_{j=1}^{3} b_{ij} H_j \right)} \tag{35}$$

Dividing through by h and taking limit as h tends to zero

$$\lim_{h \to o} \frac{y_{n+1} - y_n}{h} = y_n^2 g(x_n, Z_n) \tag{36}$$

$$\text{but } g(x_n, z_n) = -\frac{1}{y_n^2} f(x_n, y_n) \tag{37}$$

Hence,

$$\lim_{h \to o} \frac{y_{n+1} - y_n}{h} = -y_n^2 \left(\frac{-1}{y_n^2} f(x_n, y_n)\right) \tag{38}$$

$$y'(x_n) = f(x_n, y_n) \tag{39}$$

Hence the method is consistent.

3.2 Convergence

A numerical scheme (12) is said to be convergent, if when applied to initial value problem (1) generate a corresponding approximation y_n which tends to the exact solution $y(x_n)$ as n approaches infinity.

To show the convergence of the three stage implicit inverse R-K scheme

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{r} V_i H_i(y_n)} \tag{40}$$

while that of the exact equation is of the form

$$y(x_{n+1}) = \frac{y(x_n)}{1 + y(x_n) \sum_{i=1}^{3} V_i H_i(y(x_n)} + T_{n+1} \tag{41}$$

Subtracting equation (40) from (41), we have

$$y(x_{n+1}) - y_{n+1} = \frac{y(x_n)}{1 + y(x_n) \sum_{i=1}^{3} V_i H_i(y(x_n)} - \frac{y_n}{1 + y_n \sum_{i=1}^{3} V_i H_i(y_n)} + T_{n+1} \tag{42}$$

$$e_{n+1} = e_n \frac{\left[1 - y(x_n)y_n \sum_{i=1}^{3} V_i \frac{\partial H_i}{\partial y}\right]}{\left(1 + y(x_n)\sum_{i=1}^{3} V_i H_i(y(x_n))\right)\left(1 + y_n \sum_{i=1}^{3} V_i H_i(y_n)\right)} + T_{n+1} \quad (43)$$

Setting

$$P = 1 + y(x_n)\sum_{i=1}^{3} V_i H_i(y(x_n))$$

$$Q = 1 + y_n \sum_{i=1}^{3} V_i H_i(y_n)$$

$$R = 1 + y_n y(x_n)\sum_{i=1}^{3} V_i \frac{\partial H_i}{\partial y} \quad (44)$$

Then equation (43) becomes

$$e_{n+1} = \frac{R}{PQ} e_n + T_{n+1} \quad (45)$$

Set $\frac{R}{PQ} = K$ and, $T = \max_{0 \le n \le \infty} T_{n+1}$ (46) $E_n = \max_{0 \le n \le \infty} e_n$

(47)

Then

$E_{n+1} = KE_n + T$

$E_1 \le KE_o + T$ (48)

$E_2 \le KE_1 + T$

$E_2 \le K\,[KE_o + T] + T$

$K^2E_o + KT + T$

$E_3 \le KE_2 + T = K[K^2E_o + KT + T]$ K^3E_o + $K^2T + KT$

Therefore

$$E_{n+1} \le K^{n+1}E_o + \sum_{i=1}^{3} K^{1+1}T + T \quad (49)$$

since

$$\frac{R}{PQ} = K < 1$$

It is easy to see that as $n \to \infty$, then $E \to 0$. This shows that the schemes converges.

3.3 Stability

Since a consistent and convergent one-step scheme is stable, then the scheme is stable. However, to show that the method is A-stable and P-stable, it is adopted for solution of the A-stability test model equation.

$$y' = \lambda y, \; y(x_o) = y_o \quad (50)$$

Applying the scheme (5) for the numerical solution of equation (50), we obtain a recurrent equation

$y_{n+1} = P(z)y_n$ (51)

Where,

$$\frac{1}{1 - V^T(I + Bz)_e^{-1} e} \quad (52)$$

Where

$V^T = (V_1, V_2, V_3)$

$B = \{b_{ij}\}$, $i, j = 1(1)R$

$e\,(1, 1, 1, 1, 1)^T$, $z = \lambda h$

With the scheme (29) and (30), the recurrent equation is

$$P(z) = \frac{1 + 0.62z + 14z^2 + 8.6z^3}{z + 0.43z^2 - 0.079z^3} \qquad (53)$$

The difference equation (51) yields A-stable and P-stable solution,

iff $|P(z)| <|$ (54)

That is $-1 < |P(z)| < 1$ (55)

Simplifying, it is found that the interval of Absolute stability is [-∞, 0] and
[-∞, ∞], which implies that the scheme is A-stable and P-stable.

3.4 Numerical Experiments

In order to confirm the applicability and suitability of the scheme for solution of ODEs, the scheme was computerized.

Its performance was checked by comparing its accuracy with three stage Rational R – K method.

Example 1: Consider initial value problem (IVP)

$$y' = 2x + y, \quad y(o) = 1 \qquad (56)$$

Whose theoretical solution is

$$y(x) = -2\,(x+1) + 3e^x \qquad (57)$$

This problem is solved using equation (30) compare the result with the results of three stage Rational R-K.
The results are shown in Table 1.

Problem 2:

The second example consider non-linear initial value problem

$$y' = 10(y-1)^2, \quad y(o) = -1 \qquad (58)$$

With exact solution $$y(x) = 1 + \frac{1}{(1+10x)} \qquad (59)$$

This problem is solved using scheme (29) compare the result with results of three-stage Rational R-K scheme. The results are shown in Table 2.

Table 1: Accuracy of Comparism of inverse Runge-Kutta Method and Rational Runge-Kutta Scheme on Problem (54)

H	**YEXACT** $y(x_n)$	**IMPLICIT INVERSE ERROR** e_I	**RATIONAL R-K ERROR** e_R
0.10000000D-01	0.10101510D+01	0.11920930D-06	0.11920930D-05
0.50000000D -02	0.10050380D+01	0.45670012D-06	0.45700230D-05
0.25000000D-02	0.10025090D+01	0.25670013D-06	0.67800012D-05
0.12500000D-02	0.10012520D+01	0.11920930D-06	0.11920930D-06
0.62500000D-03	0.1006260D+01	0.23841860D-06	0.23841860D-06
0.31250000D-03	0.10003130D+01	0.11920930D-06	0.11920930D-06
0.15625000D-03	0.10001560D+01	0.11920930D-06	0.11920930D-06
0.78125000D-04	0.10000780D+01	0.11920930D-06	0.11920930D-06
0.39062500D-04	0.10000390D+01	0.16780045D-06	0.67800123D-07
0.19531250D-04	0.10000200D+01	0.14570009D-07	0.45670012D-07
0.97656250D-05	0.10000200D+01	0.112500000D-08	0.34560123D-07
0.48828120D-05	0.10000100D+01	0.131920930D-08	0.11920930D-08
0.24414060D-05	0.10000050D+01	0.12350000D-09	0.12567800D-09
0.12207030D-05	0.10000010D+01	0.14570009D-10	0.45680123D-10

Table 2: Numerical Solution Of Inverse Runge-Kutta Scheme and Rational Runge Kutta Scheme On Problem (56)

X_n	YEXACT $yY(x_n)$	Y IMPLICIT e_I	RATIONAL R-K ERROR e_R
0.50000000D-01	0.16005570D+01	0.16087730D+01	0.16085630D+01
0.10000000D+00	0.26887940D+01	0.26890410D+01	0.27186930D+01
0.15000000D+00	0.42650510D+01	0.42644700D+01	0.42838940D+01
0.20000000D+00	0.71333530D+01	0.71357520D+01	0.72948030D+01
0.25000000D+00	0.11710000D+02	0.11795410D+02	0.12093870D+02
0.25000000D+00	0.19787060D+02	0.19787160D+02	0.20004610D+02
0.30000000D+00	0.73072390D+02	0.73047950D+02	0.73043760D+02
0.35000000D+00	0.82315650D+02	0.82342970D+02	0.82435650D+02
0.40000000D+00	0.87418700D+02	0.87473200D+02	0.87461140D+03
0.45000000D+00	0.14473800D+03	0.14477470D+03	0.14745480D+03
0.50000010D+00	0.23754000D+03	0.24474440D+03	0.24871160D+03
0.55000010D+00	0.39157880D+03	0.40344160D+03	0.41338760D+03
0.60000010D+00	0.64512860D+03	0.66505600D+03	0.66496690D+03
0.65000010D+00	0.10691200D+04	0.10963400D+04	0.10961940D+04
0.70000010D+00	0.17542830D+04	0.18073470D+04	0.18071050D+04
0.80000010D+00	0.28933570D+04	0.29795150D+04	0.29791150D+04
0.85000010D+00	0.47632880D+04	0.49119720D+04	0.49113140D+04
0.90000020D+00	0.78512380D+05	0.80978800D+04	0.80967950D+04
0.95000020D+00	0.12945030D+05	0.13345520D+05	0.13348510D+05
0.10000000D+01	0.21300460D+05	0.21993930D+05	0.21998850D+05

Discussion

A cursory observation of results of the approximate solution y_n in Table 1 shows that as the step size decreases as the error of the results tends to zero. The error of the new scheme decreases at faster rate than that of Rational R-K schemes. This shows that the new scheme converges faster than Rational R-K (RR-K) method.

In Table 2, it can be seen that the result of new scheme is more accurate than that RR-K.

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