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Models for Pricing of Electricity Commodity

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Abstract

In this paper a survey is carry out on models for pricing electricity from market data using the Ornstein – Uhlenbeck process and other time series models like Garch and Arima to calibrate model. We also consider models for the demand and supply of electricity and the associated spike and Equilibrium price for electricity. Simulation experiments are also designed for models studied.
Key words: Models, electricity pricing, simulation and volatility.

1. Introduction

Most models for pricing of electricity make use of models like jump diffusion model and the Box-Cox transformation model. But these models do not capture the spikes in prices of electricity, that is, sudden jumps in prices because of seasonality in the demand and supply of electricity. Non-storable nature of electricity makes it pricing to be characterized by spikes .Electricity prices show strong seasonal fluctuations because of human activity and seasonal climate changes [10].
Electricity prices exhibit mean reversion. This is because of the basic fact that energy prices are driven by supply and demand. Prices of electricity fluctuation about the equilibrium, therefore, electricity prices models will usually have some mean reverting property to capture the mean reverting behavior of electricity prices [8] & [10].

In this work we will consider model for:

1. Pricing electricity from market data using
The Ornstein – Uhlenbeck process and use of other time series models like Garch and Arima to calibrate the model and also consider models for the demand and supply of electricity.
2. Equilibrium price for electricity.
3. Seasonality in the supply of electricity by the use of extended Box-Cox transformation.
4. The utility function for Electricity Company with the possibility of how to optimize it.

1.1 Ornstein-Uhlenbeck process

Traditional financial models start with the Black-Scholes assumption of the Geometric Brownian Motion or log-normal prices. This assumption does not make sense in the context of the electricity prices for many reasons including the non-predictability of the electricity prices. A model which has been used in practice is known as Ornstein-Uhlenbeck process. This is a continuous time model which permits autocorrelation in the series and is written as

$$dX(t) = k[\mu - X(t)]dt + \sigma dw(t) \qquad (1)$$

It is necessary to incorporate mean reversion when modeling electricity prices, because some time we observe that electricity prices jump from 10KWh to 110 KWH due to an unexpected event (e.g. drop in water level at hydropower stations, lack of supply of gasses, transmission constraints, plant brake down, etc).Electricity is non-storable, hence the market is volatile because the variance or volatility will changes with time, that is, the volatility has heteroscedasicity behavour which is a kind of time varying variance([1],[12-13]).

Geometric Brownian motion is a random walk process which is used to model prices based on the assumption that price changes are independent of one another. This means, the historical path of the price follows to achieve its current price is irrelevant for predicting the future price path ([11], [13]). Modification of the random walk is known as the mean reversion, where price changes are not completely independent of one another but rather are related to one another.

In Nigeria there are two groups or contractors as regard the generation, distribution and sale of electricity is concerned. One of the groups is the Power holding of Nigeria (PHCN) which is charged with responsibility of sale of electricity. The second group is the independent power generators (IPG) responsible for the generation and supply of electricity to PHCN.

Electricity forward contracts represent an obligation to buy or sell a fixed amount of electricity at a pre-specified contract price, known as the forward price, at certain time in the future (called maturity or expiration time). Therefore, electricity forward are supply contracts between a buyer (PHCN) and seller Independent Power Generator (IPG), where the PHCN is obligated to take power from IPG and supply it to the public. In finance circle, we say PHCN takes short side of electricity forward where as IPG takes the long position.
Many electricity forward are contracts settled through financial payment based on certain market price index at maturity and others are stated through physical delivery of underlying electricity.

The pay off of a forward contract promising to deliver one unit of electricity at price K at a future time T is $S_T - K$ where S_T the electricity spot price is at time T ([13] & [15]).

1.3 Spark Spread Options

Spark spreads are cross-commodity options which pays out the difference between the price of electricity sold by the generator and the price of the fuels used to generate it.
The payoff at maturity time T is $[S_T - K_H G_T]^+$
Where S_T and G_T are the electricity and fuel prices at time T, respectively.

K_H Fixed heat rate, that is, fuel affects the amount of fuel that a generation assert requires producing one unit of electricity.

1.4 Pricing Power Options

1.4.1 Pricing electricity Derivatives

There are several Research work done on modeling the Pricing of Electricity but the most acceptable one is based technical models directed on the stochastic behaviour of market prices from historical data and statistical analysis.

2. PRELIMINARY NOTES

2.1 Brownian motion ([15])

A scalar standard Brownian motion or standard Wiener process over [0, T] is a random variable W (t) that depends continuously on $t\varepsilon[0,T]$ and satisfies the following conditions:

1. W(0)=0 (with probability 1)
2. For $0 \le s < t \le T$ the random variable given by the increments W (t) – W(s) is normally distributed with mean zero and variance t-s, equivalently, $w(t) - w(s) = \sqrt{t-s}, t, s\varepsilon N(0,1)$ is the normal distribution with zero mean and unit variance.
3. For $0 \le s < t < u < v \le T$ the increments w (t)-w(s) and W (u) – w (v) is independent.

2.2 Stochastic Differential Equations

Most models in Finance, Mathematical Physics (see [4] [9] & [11]) are described by the following Stochastic Differential Equations (SDE).

$$dx(t) = a(x(t),t)dt + b(x(t),t)dw(t), x(a) = x_0$$

Where W is a k-dimensional Brownian motion and $a, b : R^d \times [0,\infty) \to \mathrm{R}^d$
Are maps defined .on $R^d \times [0,\infty)$ taking values in R^d

Definition 1

An R^d – valued process $\{\mathrm{x}(t) : 0 \le t \le T\}$ is said to be an Ito process if it can be represented as

$$x(t) = x(0) + \int_o^t a(u)du + \int_o^t b(u)dw(u), 0 \le t \le T \tag{2}$$

Where $x(0)$ is F - measurable, a is an R^d – valued adapted process satisfying

$$\mathrm{P}\left(\int_o^{\mathrm{T}} |a(t)| dt < \infty\right) = 1, i = 1,2...d \,.$$

In differential form

$$\mathrm{dx(t)} = a(t)dt + b(t)dw(t) \tag{3}$$

Definition 2

A strong solution to the (SDE) on an interval [0,T] is an ito process $\{x(t), 0 \le t \le T\}$ for which $\mathrm{P}(\mathrm{x}(0) = x_o) = 1$ and

$$x(t) = x(0) + \int_o^t a(x(s,s)ds) + \int_o^t b(x(s)s)dw(s), 0 \le t \le T \tag{4}$$

Theorem 1 (Ito's formula) ([11], [13] & [15])

Let x be a $R^d -$ valued Ito's process and let $\mathrm{f} : [0,\mathrm{T}] \times R^d \to R$ be continuously differentiable in its first argument and twice continuous differentiable in the second argument

Let $\Sigma(\mathrm{t}) = b(t)b^T(t)$ then $\mathrm{Y}(t) = f(t, x(t))$ is an ito process with

$$d\mathrm{Y}(t) = \frac{\partial f}{\partial t}(t, x(t))dt + \sum_{i-1}^{d} \frac{\partial f}{\partial x_i}(t, x(t))dx_i(t)$$

$$+ \tfrac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x(t)) \sum_{i,j}(t)dt$$

$$= \left(\frac{\partial f}{\partial t} + \sum_{i-1}^{d} \frac{\partial f}{dx_i} a_i + \tfrac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \sum_{uj} u \right) du$$

$$+ \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} b(t)dw(t) \tag{5}$$

$$\textit{with } b_{\mathrm{i}} \text{ the ith row of } b$$

Integrating the equation (5) we get

$$\mathrm{Y}(t) = f(0, x(0)) + \int_o^t \left(\frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial x_i} a_i + \tfrac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_j} \sum \tfrac{1}{2} \int_0^t \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \sum_{ij}(u) \right) du$$

$$+ \int_o^t \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} b_i(u)dw(u). \tag{6}$$

If x, a and b are scales process, it becomes

$$d\mathrm{Y}(t) = f^{/}(x)a(t) + \tfrac{1}{2} f^{//}(x(t))b^2(t)dt + f^{/}(x(t))dw(t) \tag{7}$$

Theorem 2 (Existence and Uniqueness of Solutions)

Let $\mathrm{E}\left(|x_o|^2\right) < +\infty,$ such that there exists a constant $k>0$ such that

1. $|a(x,t) - a(y,t)| \le k|x-y|$ (Lipchitz constant)
2. $|a(x,(t))| + |b(x,t,)| \le k(1 + |\, x \,|)$ (Linear Growth Property)

For all $t\varepsilon[0,T]$ and all $\mathrm{x}, \mathrm{y}\varepsilon \mathrm{R}^{\mathrm{d}}$.

Then the solution of SDE admits a strong solution $\hat{x}$ and the solution is unique in the sense that if x is a solution

$$p(x(t)=\hat{x},t\varepsilon[0,T])=1.\ \text{For all}\ t\varepsilon[0,T], the\ \text{solution satisfies}\ \ E\left[\|x(t)\|^2\right]<\infty$$

Then

Definition 3([15]):

A. Martingales: An adaptive process $\{x(t),t\geq 0\}$ is a said to be Martingale process if

1. $E[\|x_t\|]<\infty$ for all $t\geq 0$.
2. $E\{x_t \mid F_s\}=X_t$, for all $0\leq s<t<\infty$.

a process is Martingale if it has tendency to rise or fall.

B. Sub-Martingale: If it has no tendency to fall but have tendency to rise, that is, if $E[X_t \mid F_s]\geq X_t$ for *all* $0\leq s\leq t\leq T$

C. Super-Martingale: if it has no tendency to rise but may fall, that is, $E[X_t \mid F_s]\leq X_t$ for *all* $0\leq s\leq t\leq T$

3. METHOD OF ANALYSIS

Approach to characterize market prices include discrete –time series model such as Garch (see [1], [8] & [10]) Market Regime – switching models continuous – time diffusion model such as mean – reversion, jump –diffusion and other diffusion model.

We consider a diffusion process with stochastic volatility governed by a continuous two factors SDE model of the form:

$$\left.\begin{aligned} dS_t &= \mu_1(t_i,s_t)dt+\delta(t,\Sigma)S_t dw_t^1 \\ d\Sigma_t &= \mu_2(t,\Sigma_t)dt+\delta(t,\Sigma_t)dw_t^2 \end{aligned}\right\} \qquad (8)$$

where

$\sum_t = S_t^2, w_l^1, w_t^2$ *are* two Brownian Motions

Correlation coefficients $\mu_1(t,S_t)$ and $\mu_2(t,\Sigma_t)$ may account for some mean reversion either in the spot price or in the spot price volatility, because extreme spikes. Because of extreme power demand, the dynamic of electrical spot prices can be represented by a jump diffusion prices (German 1994) e.g. Merton (1976)(see[15]) model as

$$dS_t = \mu S_T dt + \delta S_t dw_t + \mu S_t dN_t \qquad (9)$$

Where

N_t is a Poisson process whose intensity frequency λ is characterized by jumps, μ is the real random variable from the normal family.

The multi-factor forward curve are defined by the Stochastic Differential Equations (SDEs)

$$\frac{dF(t,T)}{F(t,T)}=\sum_{i=1}^{n}\delta_i(t,T)dz_i(t) \qquad (10)$$

Where $F(t,T)$ the forward price at time t with maturity is date T and $\delta_i(t,T)$ is the volatility function associated with the Brownian motion $z_i^*(t)$.

Assume that

1. $dz_i^{\times}(t)dz_j^{x}(t)=0$ for $i=j$ (independent)
2. Interest rate are deterministic so that future and the forward price are the same

Therefore

$$d(In\,F(t,T))=\sum_{i-1}^{n}\left\{-\tfrac{1}{2}\delta_1^2(u,T)du+\delta i(u,T)dz_i^*(u)\right\} \qquad (11)$$

Integration yields

$$InF(t,T) - In\,\mathrm{F}(0,\mathrm{T}) = \sum_{1}^{n} \left\{ -\tfrac{1}{2}\int_{o}^{t} \delta_i^2(u,T)du + \int_{o}^{T} \delta_i(u,T)dz_i^*(u) \right\} \quad (12)$$

or

$$F(t,T) = F(0,T)\exp\sum_{1}^{n} \left\{ -\tfrac{1}{2}\int_{o}^{t} \delta_1^2(u,T)du + \int_{o}^{T} \delta_o^2(u,T)dz_1^x(u) \right\} dz_i^*(u)$$

The sport rate, $\dfrac{\lim}{T \to t}\mathrm{F}(\mathrm{t},\mathrm{T}) = S(t)$.

Therefore

$$\therefore$$

$$\ln \mathrm{S}(\mathrm{t}) = \ln F(0,t) - \tfrac{1}{2}\sum_{1=1}^{n} \int_{o}^{t} \delta_1^2(u,t)dt + \sum_{i=1}^{n} \int_{o}^{t} \delta_i(u,t)dz_i^*(u)$$

Assume that

1. $E^*\left[\left(\int_{o}^{t} \delta_i(u,T)dz_i^*(u)\right)^2\right] < \infty, \text{for all t.}$
2. *Volatility* is square integrable martingale process and expections are zero. such that

$$E^*(\ln \mathrm{S}(\mathrm{t})) = \ln \mathrm{F}(0,\mathrm{t}) - \tfrac{1}{2}\sum_{i=1}^{n}\int_{o}^{t} \delta_1^2(u,t)du$$

And variance

$$Var(InS(t)) = E^*\left(In\,\mathrm{S}(\mathrm{t})^2\right) - E[Ins(t)]^2$$

$$Var(S(t)) = \sum_{1}^{n} \int_{o}^{t} \delta_1^2(u,t)du$$

The Dynamic of sport price is finally given by

$$\frac{dS(t)}{dt} = \left[\frac{\partial F(0,t)}{\partial t} - \sum_{i=1}^{n}\int_{o}^{t} \delta_i(u,t)\frac{\partial \delta_i(u,t)}{\partial t} + \sum_{i=1}^{n}\int_{o}^{t} \frac{\partial \delta_i(u,t)}{\partial t} dz_i^*(u.)\right]dt$$

$$+\left[\sum_{i-1}^{n} \delta_i(t,t)dz_i^*(t)\right] \quad (13)$$

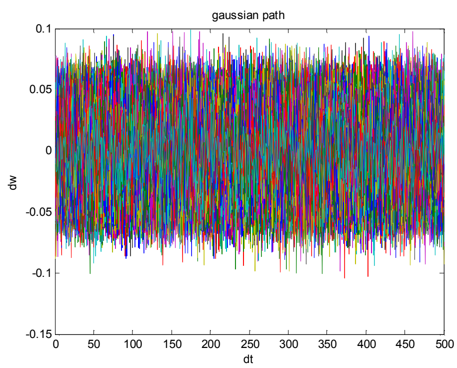

Figure 1: Simulation of Gaussian process

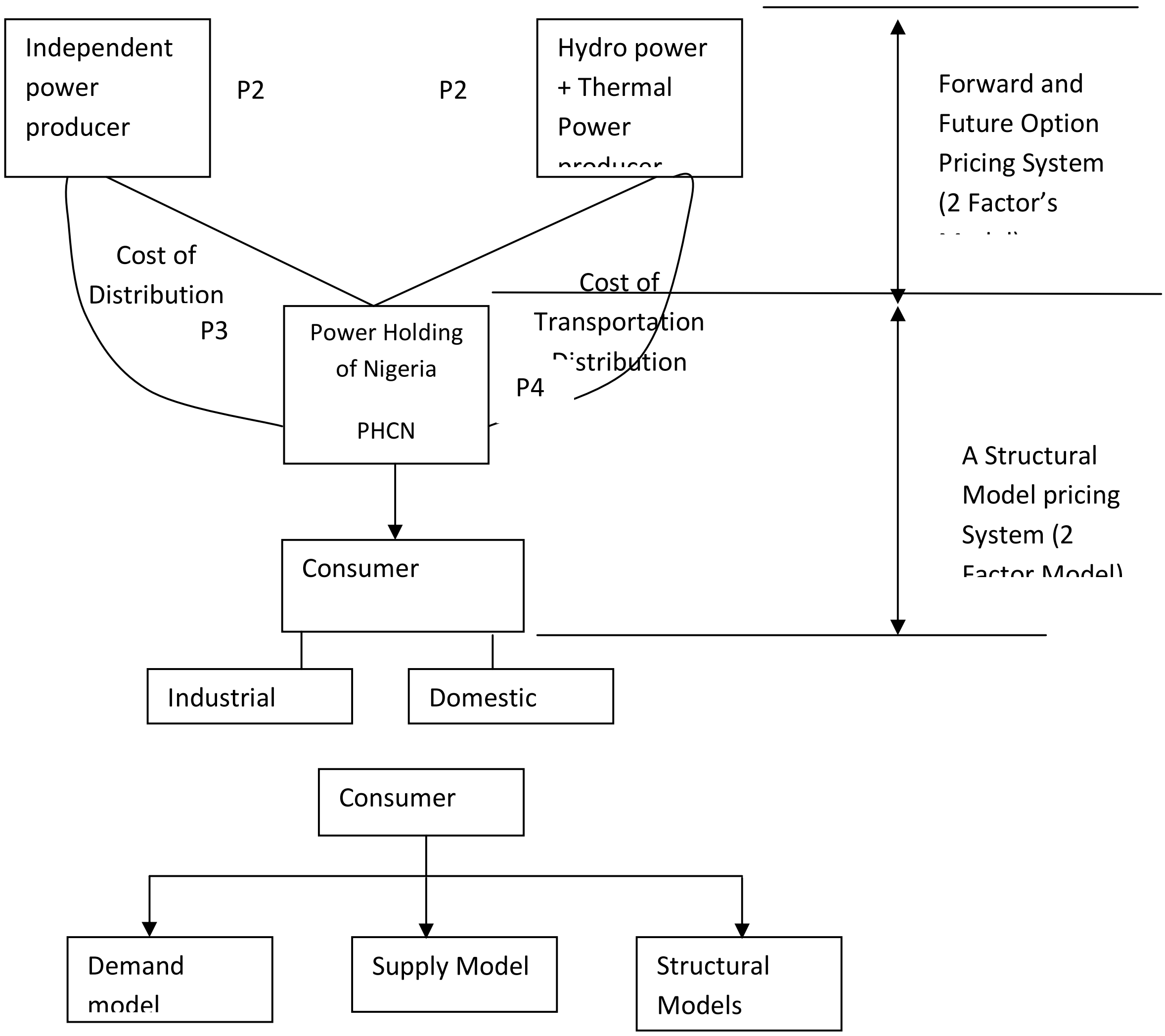

Figure 2: ARCHITECTURAL DESIGN FOR PRICING OF ELECTRICITY

4 Numericalization of stochastic differential equations

4.1 Stochastic Integrals

Given a suitable function in the integral $\int_o^T h(s)ds$ can be approximated using remaining sum (see [4], [6] & [7]) as

$$\sum_{K=0}^{N-} h(t_k)(t_{k+1} - t_k) \tag{14}$$

Where the discrete points $t_k = k\delta t$, thus

$$\int_o^T h(s)ds = \lim_{\delta t \to 0} \sum_{K=0}^{N-1} h(t_k)(t_{k+1} - t_k) \tag{15}$$

(In Riemann sense).For stochastic Integral $\int_{to}^t h(s)dw(s)$ can be approximated by Ito sum as

$$\int_{to}^t h(s)dw(s) \approx \lim_{\delta t \to 0} \sum_{k=0}^{N-1} h(t_k)(w(t_{k+1})) - w(t_k) \tag{16}$$

The Ito version is the limiting case of

$$\sum_{j-0}^{n-1} w(t_j)(w)(t_{j+1}) - w(t_j)$$

$$= \tfrac{1}{2} \sum_{j-0}^{N-1} \left(w(t_{j+1})^2 - w(t_j)^2 - \left(w(t_{j+1}) - w(t_j)\right)^2 \right] \tag{17}$$

$$= \tfrac{1}{2} \left(w(T)^2 - w(0)^2 - \sum_{j-0}^{N-1} w(t_{j+1}) - w(t_j)^2 \right)$$

$$\mathrm{E}\left(\sum_{j-0}^{N-1} \left(w(t_{j-1}) - w(t_j)\right)^2 \right) = T$$

$$Var\left(\sum_{j-o}^{N-1} \left(w(t_{j-1}) - w(t_j)\right)^2 \right) = O(\delta t)$$

We can show that

$$\int_o^T w(s)ds(s) = \tfrac{1}{2} w(T)^2 - \tfrac{1}{2} T$$

For the ito integral. For Stratonvich integral

$$\int_0^T h(s)dw(s) = \lim_{\delta t \to 0} \sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2} \right) \tag{19}$$

And the Stratonvich $\left(w(t_{j+1})w(t_{j-1})\right)$ version is the limiting case of

$$\sum_{j-0}^{N-1} \left(\frac{w(t_j) + w(t_{j+1})}{2} + \Delta Z \right) \left(w(t_{j+1}) - w(t_j)\right)$$

Where each ΔZ_j is independent $N(0, \Delta t / 4)$

The sum becomes

$$\tfrac{1}{2}\left(w(T)^2 - w(0)^2\right) + \sum_{j-0}^{N-1} \Delta Z_j \left(w(t_{j+1}) - w(t_j)\right)$$

Seen that (see [12] & [13])

$$\mathrm{E}\left(\sum \Delta Z_j w(t_{j+1}) - w(t_j)\right) = 0$$
$$Var\left(\sum DZ_j \left(w(t_{j+1} w(t_j))\right)\right) = O(\delta t)$$

Therefore

$$\int_0^T w(t)dw(s) = \tfrac{1}{2} w(T)^2$$

George Box and Gwilyn Jenkins introduced Autoregressive Integrated Moving Average (ARIMA) time series models in 1970. These models are mathematical models and used for short term forecast of 'well behaved' data and find the best fit of time series in order to get a forecast. Generalized Auto Regressive Conditional Heteroscedastic (GARCH) model, Bollerseve ([1]) generalized the conditional δ (voluntary) to Garch (p.q) model as

$$\delta_t^2 = w + \sum_{l-1}^{q} \alpha_i \varepsilon_{i-1} + \sum_{j=1}^{p} \beta_j \delta_{j-1}^2 \qquad (20)$$

With the initial conditions: $w > 0, \alpha_i \geq 0$
$(for\ \mathrm{i} = 1,...,\mathrm{q}), \beta_j \geq 0, (for\ \mathrm{j} = 1,...\mathrm{p})$ and $\beta_j + \alpha_i < 1$,
To ensure that the condition variance is nonnegative and stationary for all t.
Loosely speaking, we can think of heteroscedasicity as time-varying variance (i.e., volatility). Conditional implies a dependence on the observations of the immediate past, and autoregressive describes a feedback mechanism that incorporates past observations into the present. GARCH then is a mechanism that includes past variances in the explanation of future variances. If the process has the mean, variance and autocorrelation structure constant over time then process is known as stationary process.

More specifically, GARCH is a time-series technique that allows users to model the serial dependence of volatility. The series is heteroscedastic, i.e., its variances vary with time. If its variances remain constant with time, the series is homoscedastic.

$$GARCH(p,q) = w + \delta(L)\varepsilon_t^2 + \beta(L)\delta_t^2$$
$$\delta(L) = \sum_{L=1}^{q} \alpha, L^i, \beta(L) = \sum_{L-1}^{p} \beta i L^i$$
$$\alpha(0) = \beta(0) = 0$$

Condition for stationary. The process is stationery if $[1 = \beta(L)] = 0$ must be all the roots i.e. outside the of the polynomials unit circle.
Hence

$$\delta_t^2 = w(1-\beta(L))^{-1} + \alpha\left(L[1-\beta(L)]\varepsilon_t^{-1}\right) \qquad (21)$$

The GARCH Model can be expressed without lost of generality as an ARMA process.
Suppose Garch Model can be expressed without loss of generality as an ARMA process.
Suppose $n_1 = \varepsilon_t^2 - \delta_t^2$ is an martingale difference sequence, then

$$\varepsilon_t^2 = w + (\alpha_t + \beta_t)\varepsilon_{t-1}^2 + 2_t - \beta_t R_{t-1} \qquad (22)$$

This is an ARMA process.

Estimation of Model Parameters
Estimation parameters of models is a very crucial aspect of modeling, there are many algorithms for estimation in the literature, many of which do not yield unbiased, efficient and consistent estimates.The estimation of ARCH models are usually done using maximum likelihood (ML) method, using the standard prediction error decomposition type, the log-likelihood function for the Garth Model is

$$L\left(w,\alpha_t,\beta_t,\wp\middle|r_1,r_2,\ldots,r_t\right)$$
$$=\frac{1}{\sqrt{2\pi\delta_t^2}}\exp\left(\frac{-(r_1-\gamma)^2}{2\delta_t^2}\right)\frac{1}{\sqrt{2T\delta_{t-1}^2}}\exp-\left(\frac{(r_t-\gamma)^2}{2\delta_{t-1}^2}\right)$$
$$\therefore\frac{1}{\sqrt{2\pi\delta_t^2}}\exp-\sum_{i=1}^{t}\left(\frac{r_i-\gamma}{2\delta_t^e}\right)^2 \tag{23}$$

This function is monotonically increasing without lost of generality, if we truncate the likelihood function for the GARCH Model assuming a normal error distribution therefore

$$\log w(I)=-\tfrac{I}{2}\log(2\pi)-\tfrac{1}{2}\sum_{i=}^{I}\log\delta+\left(\frac{\varepsilon_t^2}{\delta_4^2}\right)$$

Where ($\varepsilon_t=r_t-m_t$) and the parameter vector

$$\phi\left(w,\delta_1,\sigma_2,\ldots,\sigma_{r,}B_1,\ldots B_q\right)^T$$

The first derivatives with respect to their various parameters are:

$$\frac{\partial Ii}{\partial\sigma}=-\tfrac{1}{2}\left[\frac{1}{\sigma_t^2}-\frac{\varepsilon_t^2}{(\sigma_t^2)^2}\right]\frac{\partial\sigma_t^2}{\partial\theta}$$
$$=\frac{1}{2}\left(\frac{1}{\sigma_t^2}\right)\frac{\partial\sigma_t^2}{\partial\sigma}\left(\frac{\varepsilon_t^2}{\sigma_t^2}-1\right) \tag{24}$$

3. The Euler –Murayama method([4]&[13])

Consider the stochastic differential equation

$$dX(t)=f(X(t))dt+g(X(t))dw(t),X(0)=x_0,0\le t\le T$$

To apply numerical methods (SDEs), we first discretize the interval let $\Delta t=\dfrac{T}{L}$ for some positive integer $L,t_k=k\delta t$,let the numerical approximation to $X(t_k)$ be X_k ,the Euler- Maruyama (EM) method takes the form

$$X_k=X_{k-1}+f(X_{k-1})\delta t+g(X_{k-1})(w(t_k)-w(t_{k-1}))$$

Remark 1

For deterministic case, g=*0;* the EM becomes the classical Euler's method

Example 1

Applying EM method to the following

$$dX(t)=\lambda X(t)dt+\mu X(t)dw(t),X(0)=x_0$$

From above $g(X(t))=\mu X(t)$ and $f(X(t))=\lambda X(t)$ .The analytic solution to the problem is $X(t)=X(0)\exp\left[(\lambda-\dfrac{1}{2}\mu^2)t+\mu w(t)\right]$.Truncating Ito-Taylor expansion at an appropriate point such that

$$X_k=X_{k-1}+\Delta tf(X_{k-1})+g(X_{k-1})(w(t_k)-w(t_{k-1}))$$
$$\frac{1}{2}g(X_{k-1})g'(X_{k-1})(w(t_k)-w(t_{k-1}))^2 \tag{26}$$

In the logistic equation we can apply Milstein's method to the SDE

$$dX(t)=rX(t)(K-X(t))dt+\beta X_k(w(t_k)-w(t_{k-1}))+\frac{1}{2}\beta^2X_{k-1} \tag{27}$$

Applying Milstein's algorithm, we get

$$X_k = X_{k-1} + r\delta X_k(K - X_k) + \beta X_k(w(t_k) - w(t_{k-1})) + \frac{1}{2}\beta^2 X_{k-1} \quad (28)$$

4. Utility, Demand and Supply Functions

The general utility function electricity is of the form

$$g(\pi_1, q_1, k_1, k_2) + h(\pi_2, q_2, k_2) + p_2 q_2 = x - z,$$

where

$$g(\pi_1, q_1, k_1, k_2) = \begin{cases} 0 \text{ if } q_1 \le k_1 \\ \pi_1(q_1 - k_1) \quad \text{if } k_1 < q_1 \le k_2 \\ \pi(k_2 - k_1) \text{ if } q_1 > k_1 \end{cases}$$

$$h(\pi_2, q_2, k_2) = \begin{cases} 0 \text{ if } q_1 \le k_2 \\ \pi(q_2 - k_1) \text{ if } q_1 > k_2 \end{cases}$$

Denote the goods by q_1 and q_2 ,and assume that q_1 can be purchases in unlimited quantities at price p_2 ,but that electricity q_1 is purchased according g to a two-part tariff with decreasing block rate as follows: $\pi_2 < \pi_1$, assume that the consumer possesses a utility function $\phi(q_1, q_2)$ that is maximized subject to his level of income x.

In short run

$$q = u(x, \pi, z)s$$

Where $u(.)$ is the utilization rate of s .and assume to depend upon the level of income (x) ,the price of electricity (π) and other factors such as (economic, social, or demographic) that might be relevant.

For short run demand the utility function is

$$u = a_0 + a_1 x + a_2 \pi + a_3 z$$

or

$$u = a_0 + a_1 \ln x + a_2 \ln \pi + a_3 \ln z$$

The short run demand function for electricity becomes

$$q = (a_0 + a_1 x + a_2 \pi + a_3 z)s$$

or

$$q = (a_0 + a_1 \ln x + a_2 \ln \pi + a_3 \ln z)s$$

Long run

Stock of electricity consuming capital goods () is given by

$$\hat{S} = b_0 + b_1 x + b_2 \pi + b_3(r + \delta) + b_4 z$$

Where r and δ denotes the market rate of interest and rate depreciation of the capital stock respectively p denotes the price per kilowatts of addition to the capital stock.

5. Model for Supply and Demand of electricity

The model for supply ($S_t(q)$) and Demand ($D_t(q)$) of electricity at time t as modeled by Manuda et.al (see[10]) in their study on dynamic supply-demand model for electricity prices was given as

$$S_t(q) = a_{0,t} \exp\left(a_{1,t} q + a_{2,t}(q - QH_t) I_{q > QH_t} GP_t\right) + a_{3,t} \sum_{t-30(60,90) \le \tau < t} GQ_\tau$$

And

$$D_t(q) = b_{0,t} + b_{1,t}(T_t - 65)^2 + b_{2,q} + b_{3,t}SE_t + R_t$$

The equilibrium model $E_t(q) = S_t(q) - D_t(q)$ and the equilibrium price q* such that $E_t(q^*) = 0.$
The sport price process at time t is defined by the equilibrium of supply and demand at time t so we have $u_t(p_t) = D_t$ demand follows

Ornstein-Uhlenbeck process, Balow (2002) (see [10]) suggests cappy p_t at some maximum price whenever demand exceeds the maximum supply.
Suppose supply is non random and independent of t defined

$$p_t = \begin{cases} \left(\dfrac{a_0 - D_t}{b_0}\right)^{\frac{1}{\alpha}} & , D_t < a_0 - \varepsilon_0 b_0 \\ \varepsilon_0^{\frac{1}{2}}, D_t \geq a_0 - \varepsilon_0 b_0 & \end{cases}$$

Let the inverse of Box-Cox transformation be

$$f_\alpha(x) = \begin{cases} (1+\alpha x)^{1/\alpha} \\ e^\alpha, \alpha = 0 \end{cases}, \alpha \neq 0$$

Design of Simulation Experiments ([4], [6] & [7])

The price process p_t is a function of normalized demand x_t which follows an Ornstein-Uhlenbeck process (OUP).

The exact solution to OUP x_t for any $0 < t < T, X_T \sim N(\mu, \sigma^2)$

The mean $E(X_t) = \mu = e^{-a(T-t)}X_t + b(1 - e^{-a(T-t)})$

The variance= $\mathrm{Var}(X_t) = \dfrac{\sigma^2}{2a}(1 - e^{-2a(T-t)})$

For time-stepping equation Simulate X_t at time $0 < t_0 < t_1 < ... < t_n$.

$$X_{t_i+1} = e^{-a(t_{i+1}-t_i)}X_{t_i} + b(1 - e^{-a(t_{i+1}-t_i)}) + \sigma\sqrt{\frac{1}{2a}(1 - e^{-a(t_{i+1}-t_i)})}z_{i+1}$$

$z_i \sim N(0,1)$

Monte Carlo Simulation ([2], [5], [6] & [14]) for OUP model

Algorithm 1
Input: a, b, T, σ
For $i = 1,2,...,n$
Generate X_{t_i}

Set $X_{t_i+1} = e^{-a(t_{i+1}-t_i)}X_{t_i} + b(1 - e^{-a(t_{i+1}-t_i)}) + \sigma\sqrt{\dfrac{1}{2a}(1 - e^{-a(t_{i+1}-t_i)})}z_{i+1}$

Set $\hat{P}_n = \dfrac{(X_1 + X_2 + ... + X_n)}{n} = E(X_n)$

Then $\hat{P}_n \to P$ with probability 1 as $n \to \infty$.Let the sample deviation of $\hat{P}_1, \hat{P}_2, ...\hat{P}_n$ and let z_δ denotes the

$1-\delta$ quartile of the standard normal distribution (i.e. $\phi(z_0)=1-\delta$ then $\hat{P}_n \pm z_{\delta/2}\frac{s_{\hat{P}_n}}{\sqrt{n}}$ is an asymptotical (as $n\to\infty$) valid for $s_c=\sqrt{\frac{\sigma^2(1-e^{-2a(T-t)})}{2a}}$ ([2], [9] & [15])

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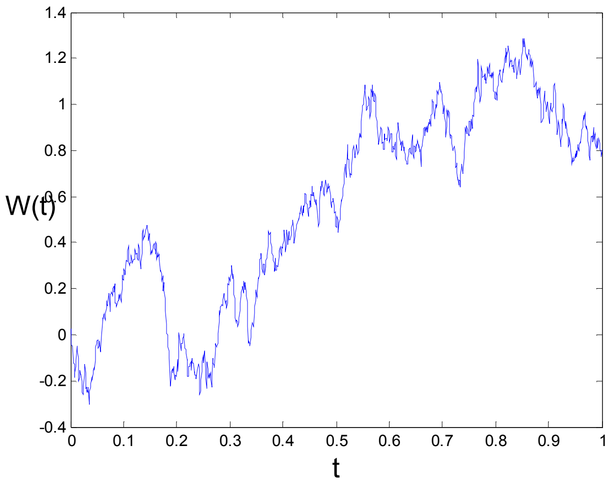
Appendix

Experiment 1

Simulation of a non-linear Ornstein-Uhlenbeck process

```
%Barlow's model
%dX=a*(b-X)*dt+sigma*dW
%P= (1+alpha*X) ˆ (1/alpha) if 1+alpha*X>epsilon
%P=epsilonˆ (1/alpha) if 1+alpha*X<=epsilon
a=1.0055; b=1.020; sigma=0.042; alpha=1.234; epsilon=0.008; % calibration of model
N=1000; %number of time steps
T=10; %time interval
dt=T/N;
tvec= [1: N];
X (1) =1; %X_0
%time stepping algorithm
for i=2: N
X(i)=exp(-a*dt)*X(i-1)+b*(1-exp(-a*dt))+sigma*...
...sqrt ((1-exp (-2*a*dt))/ (2*a))*randn (1);
```

```
end
for i=1: N
if 1+alpha*X (i)>epsilon
P (i) = (1+alpha*X (i)) ˆ (1/alpha);
else
P (i) =epsilonˆ (1/alpha);
end
end
%generate the plot of the simulated NLOU
plot (tvec, P) xlabel (’Time index’) ylabel (’Price P_t’)
title (’Simulation of the NLOU process P_t’)
```



Experiment 2

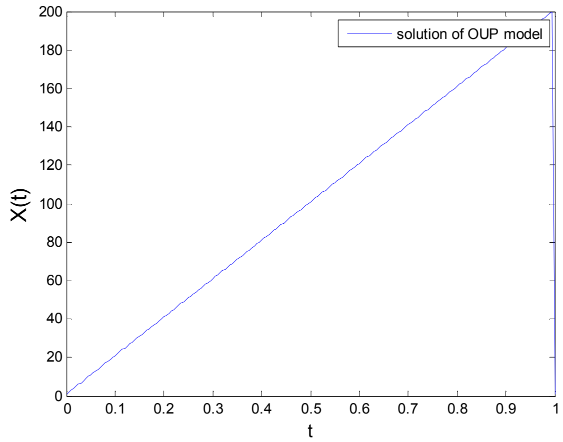
```
randn ('state', 100)
>> T=1; N=1000; dt=T/N;
randn ('state', 100) % activate random generator
>> T=1; N=1000; dt=T/N; % activate the counter
>> dw=sqrt (dt)*randn (1, N);
>> W=cumsum (dw);
>> plot (0: dt: T, [0, W],'b-') % plot W (t) using marker
>> xlabel (‘W (t)','FontSize', 17) % labeling of x-axis with the given fontsize
>> ylabel (‘W (t)', FontSize’, 16,'Rotation', 0) % labeling of x-axis with the given fontsize
```

Experiment 3

```
% solve dX (t) =k [mu-X (t)] dt+sigmadw (t)
>> randn ('state', 100)
>> k=2.023; mu=0.54; sigma=0.044;
>> Xzero=1; T=1; N=200; dt=T/N;
>> Dt=dt;
>> Xem=zeros (1, N);
>> Xtemp1=Xzero;
>> j=1: N
winc=sigma*randn;
>> g=k*(mu-Xtemp1);
>> Xtemp1=Xtemp1+Dt*g+winc;
>> Xtemp (j) =Xtemp1;
>> end
plot ([0: Dt: T], [j, Xtemp1],'b-')
```


>> legend ('solution of OUP model')
>> xlabel ('t','FontSize, 12)

ylabel (‘X (t)','FontSize', 14)



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