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On Some Analytic Functions With Negative Coefficients

Ezekiel Oyekan* Adetunji Olayiwola Adedolapo Lawal
Department of Mathematics and Statistics, Bowen University, PMB 284 Iwo, Osun State, Nigeria
* E-mail of the corresponding author: shalomfa@yahoo.com

Abstract

In this note we employ the Salagean differential operator to the familiar Hadamard product (or convolution) in order to introduce and investigate two new subclasses of analytic functions. The results presented include coefficient estimates and extreme properties for functions belonging to these subclasses.

Keywords: Salagean operator, Hadamard product (or convolution), coefficient estimates, extreme points

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of *analytic* and *univalent functions* $f(z)$ in the open unit disk $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$, the class of starlike functions of order α and the class of convex functions of order α, respectively.

Let D^n be the Salagean differential operator (see [1]) $D^n: A \longrightarrow A, n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D\big(D^{n-1} f(z)\big)$$

We note that;

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

Definition 1 (Hadamard Product or Convolution)

Given two functions $f, g \in \mathcal{A}$ where $f(z)$ is given by (0.0.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (2)$$

The Hadamard product (or convolution) $f * g$ is defined by (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \qquad z \in \mathbb{U}$$

Definition 2 (Subordination Principle)

Let $f(z)$ and $g(z)$ be analytic in the unit disk $\mathbb{U}$ then $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U})$$

if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}),$$

In particular, it the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

See also Duren [2].

Let $\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$

analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $\lambda_k \geq 0, \mu_k \geq 0$ and $\lambda_k \geq \mu_k$.
We note that

$$(f * \Phi)(z) = z + \sum_{K=2}^{\infty} k^m \lambda_k z^k \text{ and } (f * \Psi)(z) = z + \sum_{K=2}^{\infty} k^n \mu_k z^k \tag{3}$$

By using the Binomial expansion on (3) we have the following:

$$\gamma(z) = (f * \Phi)^\beta(z) = z + \sum_{k=2}^{\infty} a_k(\beta)\lambda_k(\beta) z^{k+\beta-1}$$

and (4)

$$\eta(z) = (f * \Psi)^\beta(z) = z + \sum_{k=2}^{\infty} a_k(\beta)\mu_k(\beta) z^{k+\beta-1}$$

Where $\lambda_k(\beta), \mu_k(\beta)$ *and* $a_k(\beta)$ are coefficients λ_k, μ_k *and* a_k respectively depending on β (for all $k \in \mathbb{N}$ *and* $\beta \in \mathbb{N}$). We give the following definitions:

Definition 3. A function $f(z) \in \mathcal{A}$ is said to be in the class $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ if and only if

$$\frac{D^m \gamma(z)}{D^n \eta(z)} \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha \tag{5}$$

where $\prec$ denotes subordination, $(f * \Psi)^\beta(z) \neq 0$, A and B are arbitrary fixed numbers $-1 \leq B < A \leq 1, -1 \leq B < 0, \alpha(0 \leq \alpha < 1), \beta, m \in \mathbb{N}, n \in \mathbb{N}_0$ $(m > n)$, $\gamma(z)$ and $\eta(z)$ are as defined in (4). It is to see that when $\beta = 1$ we obtain the class $E_{m,n}(\Phi, \Psi; A, B, \alpha)$ introduced by Eker and Seker [3].

In other words, $f(z) \in E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ if and only if there exists an analytic function $\omega(z)$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$ such that

$$\frac{D^m \gamma(z)}{D^n \eta(z)} = (1-\alpha)\frac{1+A\omega(z)}{1+B\omega(z)} + \alpha \tag{6}$$

The condition (6) is equivalent to

$$\left| \frac{\frac{D^m \gamma(z)}{D^n \eta(z)} - 1}{(A-B)(1-\alpha) - B\left(\frac{D^m \gamma(z)}{D^n \eta(z)} - 1\right)} \right| < 1, \qquad z \in \mathbb{U}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ whose elements can be expressed in the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0.$$

We shall denote by $\tilde{E}_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$, the subclass of functions in $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ that have their non-zero coefficients from second onwards, all negative. Thus, we can write

$$\tilde{E}_{m,n}(\gamma, \eta; A, B, \alpha, \beta) = E_{m,n}(\gamma, \eta; A, B, \alpha, \beta) \cap \mathcal{T}$$

It is easy to check that various subclasses of $\mathcal{T}$ referred to above can be represented as $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha)$ for suitable choices of Φ, Ψ. For example,

1. $\tilde{E}_{0,0}\left(\frac{1}{(1-z)^2}, \frac{z}{1-z}; 1, -1, \alpha, 1\right) = \mathcal{S}^*(\alpha)$ and $\tilde{E}_{0,0}\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1, -1, \alpha, 1\right) = \mathcal{K}(\alpha)$ which were studied by Silverman ([4]),
2. $\tilde{E}_{0,0}\left(\frac{z}{(1-z)^2}, z; 1, -1, \alpha, 1\right) = \mathcal{P}^*(\alpha)$ which was studied by Bhoosnurmath et al. and Gupta et al. ([5,6]),
3. $\tilde{E}_{0,0}\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; 1, -1, \alpha, 1\right) = \mathcal{R}(\alpha)$ which was studied by Silverman and Silvia ([7])

2. Coefficient Inequalities

Theorem 1:
If $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{k=2}^{\infty}\Big[(1-B)\left(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\Big]|a_k(\beta)| \le (A-B)(1-\alpha)\beta^n \tag{9}$$

for some $\lambda_k(\beta)\ge 0, \mu_k(\beta)\ge 0, \lambda_k(\beta)\ge\mu_k(\beta), \alpha(0\le\alpha<1),\ \beta, m\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$ *,then* $f(z)\in E_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$.

Proof. Let condition (9) hold. Then we have;

$$|D^m\gamma(z)-D^n\eta(z)|-|(A-B)(1-\alpha)D^n\eta(z)-B(D^m\gamma(z)-D^n\eta(z))|$$

$$= \left|\beta^m z^\beta+\sum_{k=2}^{\infty}(k+\beta-1)^m a_k(\beta)\lambda_k(\beta)z^{k+\beta-1}-\left(\beta^n z^\beta+\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k+\beta-1}\right)\right|$$
$$-\Bigg|(A-B)(1-\alpha)\left(\beta^n z^\beta+\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k+\beta-1}\right)$$
$$-B\Bigg(\beta^m z^\beta+\sum_{k=2}^{\infty}(k+\beta-1)^m a_k(\beta)\lambda_k(\beta)z^{k+\beta-1}-\left(\beta^n z^\beta+\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k+\beta-1}\right)\Bigg)\Bigg|$$

$$\le \left|z^\beta(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k+\beta-1}\right|$$
$$-\Bigg|(A-B)(1-\alpha)\beta^n z^\beta+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k+\beta-1}$$
$$-B\left(z^\beta(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k+\beta-1}\right)\Bigg|$$

$$\le |z|^\beta(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]|a_k(\beta)||z|^{k+\beta-1}-(A-B)(1-\alpha)\beta^n|z|^\beta$$
$$+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+\beta-1)^n\mu_k(\beta)|a_k(\beta)||z|^{k+\beta-1}-|B|\Bigg(|z|^\beta(\beta^m-\beta^n)$$
$$+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]|a_k(\beta)||z|^{k+\beta-1}\Bigg)$$

$$\le \Bigg[(1-B)\left((\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]\right)$$
$$+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+\beta-1)^n\mu_k(\beta)\Bigg]|a_k(\beta)|-(A-B)(1-\alpha)\beta^n$$

$$\le \sum_{k=2}^{\infty}\big[(1-B)\big(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\big) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\big]|a_k(\beta)| - (A-B)(1-\alpha)\beta^n \le 0$$

This completes the proof of Theorem 1.

In the following theorem it is shown that the condition (9) is also necessary for functions $f(z)$ of the form (8).

Theorem 2: *Let* $f(z)\in\mathcal{T}$ *(as defined above).* $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty}\Big[(1-B)\Big(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\Big) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\Big]\,a_k(\beta) \le (A-B)(1-\alpha)\beta^n \tag{10}$$

for some $\lambda_k(\beta)\ge 0, \mu_k(\beta)\ge 0, \lambda_k(\beta)\ge\mu_k(\beta), \alpha(0\le\alpha<1),\ \beta, m\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$

Proof. Since $\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)\subset E_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$, we only need to prove the "only if" part of theorem. For function $f(z)\in\mathcal{T}$, we can write:

$$\left|\frac{\frac{D^m\gamma(z)}{D^n\eta(z)}-1}{(A-B)(1-\alpha)-B\left(\frac{D^m\gamma(z)}{D^n\eta(z)}-1\right)}\right| =$$

$$= \left|\frac{D^m\gamma(z)-D^n\eta(z)}{(A-B)(1-\alpha)D^n\eta(z)-B(D^m\gamma(z)-D^n\eta(z))}\right|$$

$$= \left|\frac{(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k-1}}{(A-B)(1-\alpha)\beta^n-(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k-1}+B[(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k-1}]}\right| < 1$$

Since $Re(z)\le|z|$ for all $z\in\mathbb{U}$,

$$Re\left\{\frac{(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k-1}}{(A-B)(1-\alpha)\beta^n-(A-B)(1-\alpha)\sum_{k=2}^{\infty}(k+\beta-1)^n a_k(\beta)\mu_k(\beta)z^{k-1}+B[(\beta^m-\beta^n)+\sum_{k=2}^{\infty}[(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]a_k(\beta)z^{k-1}]}\right\} < 1$$

If we choose z real and letting $z\to 1^-$, we have;

$$\sum_{k=2}^{\infty}\Big[(1-B)\Big(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\Big) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\Big]\,a_k(\beta) \le (A-B)(1-\alpha)\beta^n$$

This completes the proof of theorem 2.
To prove our next result, we shall need the following theorem.

Corollary A: *Let* $f(z)\in\mathcal{T}$ *(as defined above) and* $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ *then,*

$$a_k(\beta)\le\frac{(A-B)(1-\alpha)\beta^n}{(1-B)\Big(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\Big)+(A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)} \tag{11}$$

for some $\lambda_k(\beta)\ge 0, \mu_k(\beta)\ge 0, \lambda_k(\beta)\ge\mu_k(\beta), \alpha(0\le\alpha<1),\ \beta, m\in\mathbb{N}, n\in\mathbb{N}_0$.

Proof. Since $f(z)\in\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$, then condition (3.2.3) gives

$$|a_k(\beta)|\le\frac{(A-B)(1-\alpha)\beta^n}{(1-B)\Big(\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)\Big)+(A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)}$$

This completes the proof.

3 Extreme Points of the Class $\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$

Theorem 3: *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)\beta^n}{(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)} z^{k+\beta-1}$$

for some $\lambda_k(\beta) \geq 0, \mu_k(\beta) \geq 0, \lambda_k(\beta) \geq \mu_k(\beta), \alpha(0 \leq \alpha < 1), m \in \mathbb{N}, n \in \mathbb{N}_0$ *and* $\beta \in \mathbb{N}, k = 2,3,\ldots$

Then $f(z) \in \tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z)$$

where $\delta_k \geq 0$ *and* $\sum_{k=1}^{\infty} \delta_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z)$$

$$= z - \sum_{k=2}^{\infty} \frac{\delta_k (A-B)(1-\alpha)\beta^n}{(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)} z^{k+\beta-1}$$

Then from Theorem 2, we have

$$\sum_{k=2}^{\infty}\left[(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\right]$$
$$\cdot \frac{\delta_k (A-B)(1-\alpha)\beta^n}{(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)}$$

$$= (A-B)(1-\alpha)\beta^n \sum_{k=2}^{\infty} \delta_k$$

$$= (A-B)(1-\alpha)\beta^n(1-\delta_1) \leq (A-B)(1-\alpha)\beta^n$$

Then, $f(z) \in \tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ by Theorem 2.

Conversely, suppose that $f(z) \in \tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$.

Since

$$a_k(\beta) \leq \frac{(A-B)(1-\alpha)\beta^n}{(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)}, (\beta \in \mathbb{N}, k = 2,3,\ldots)$$

We may set

$$\delta_k = \frac{\left[(1-B)\left(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\right) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\right]|a_k(\beta)|}{(A-B)(1-\alpha)\beta^n}, (\beta \in \mathbb{N}, k = 2,3,\ldots)$$

and

$$\delta_1 = 1 - \sum_{k=2}^{\infty} \delta_k$$

Then,

$$f(z) = z - \sum_{k=2}^{\infty} \delta_k f_k(z)$$

This completes the proof of the theorem.

Corollary B: *The extreme points of* $\tilde{E}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ *are the functions* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)\beta^n}{(1-B)\big(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\big) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)} z^{k+\beta-1}$$

for some $\lambda_k(\beta) \geq 0, \mu_k(\beta) \geq 0, \lambda_k(\beta) \geq \mu_k(\beta), \alpha(0 \leq \alpha < 1),\ \beta, m \in \mathbb{N}, n \in \mathbb{N}_0, k = 2,3,\ldots$

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