GENERALISED OPTIMAL STOPPING STRATEGIES WITH APPLICATIONS TO FINANCE

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Abstract

In this paper we examine the problem of determining the best time to sell an asset, where the stock price is modelled by a hybrid process. In this paper hybrid variable is a mathematical concept that is used to describe a situation in which randomness and fuzziness simultaneously appear in a system or phenomenon. Based on this concept, a hybrid stopping time problem is formulated and investigated. A verification theorem is derived and proved. We illustrate the application of the verification theorem through a practical example in mathematics of finance. A power function with exponent γ, is used as the utility function in the example. This study is extending the model from Oksendal [12] by including the fuzzy component since market value of assets is usually described using vague human language. The theory of hybrid variables provides a more realistic description of the evolution of price processes of financial assets.

Keywords: Randomness, fuzziness, Fuzzy variable, fuzzy process, hybrid variable, hybrid process, stopping time.

1.INRODUCTION

The study of optimal stopping theory has attracted the attention of many mathematicians in recent years. This is due to the necessity of a rigorous approach to optimal stopping problems, see for example [12], [13]. The complexities of the natural and man-made socio-economic systems make the events and processes we encounter indeterminate in various forms. In many cases mathematical models are expressed in terms of stochastic differential equations. For a more elaborate treatment of stochastic differential equations and their applications the reader is referred to [2], [12]. Randomness is a basic type of indeterminacy and probability theory is a branch of mathematics for studying the behaviour of random phenomena. In Harrison [4], it turns out that randomness is appropriate for modelling indeterminacy provided substantial sample data is available. In several cases, we are required to make decisions about a system when sample data about its previous performance are not available. In such cases, it is not appropriate to use probability theory to handle the associated indeterminacy. Fuzziness is a good example of a form of indeterminacy that cannot be examined using probability theory. The notion of fuzziness was introduced by cybernetician Zadeh [15] as another form of indeterminacy that we encounter on a day-to- day basis. According to Purl and Ralescu [14] fuzziness describes events whose measurement is imperfect and vague.

In many cases, fuzziness and randomness simultaneously appear in a system, for example [5]. In order to describe this phenomenon, a fuzzy random variable was introduced by Kwakernaak [6] as a random element taking fuzzy values. More generally, a hybrid variable was introduced by Liu [8] as a tool to describe the quantities with fuzziness and randomness.

In stochastic analysis, the theory of optimal stopping is concerned with the problem of choosing a time to take given action based on sequentially observed random variables in order to maximise an expected payoff or to minimise an expected cost, see [2]. The work in [3], studied problems of this type in the area of statistics, where the action taken may be to test a hypothesis or to estimate a parameter and in the area of operations research, where the action maybe to replace a machine or reorder stock. In finance optimal stopping problems are encountered by investors who need to decide, for example, the best time to sell an asset, see for example [4]. In this paper we examine the optimal stopping problem when the underlying process to be stopped is driven by both fuzziness and randomness and the stopping time is a hybrid variable.

The study of optimal stopping initiated in the 1970s. The reader is referred to [11] for more elaborate discussion of optimal stopping rules. Oksendal [12] studied the optimal stopping of stochastic process with applications in finance. All of the above researches concentrated on stochastic optimal stopping. No research has focused on hybrid optimal stopping. This paper seeks to address the gap.

The following example illustrates the interaction effects of fuzziness and randomness. Consider a trader who wants to export to South Africa, when the value of rand is stronger. Given that South Africa operates within a flexible exchange rate regime, the value of the rand, like any other stock, is determined by the market forces of supply and demand. The demand for a currency relative to the supply will influence its value in relation to another currency. It is common, for example, to hear traders stating that the value of South African rand is weak, strong or very strong compared to other currencies like United States of America dollar (USD). For example, if USD/rand is at R10.05, which means USD1=R10.05. In this case, it is not clear what value of the rand will be in the category of being strong or weak. For instance if the domestic interest rate increases and attract foreign capital and USD/rand increases to R12.02, will be this a strong or very strong value? What about if there is current account deficit and drops to R9.25 or R9.99? The challenge is to determine the demarcation between, for example weak, strong and very strong. For this reason according to [15], the terms weak, strong and very strong are vague and are considered as fuzzy terms. This shows that the market value of rand at time t is a fuzzy variable. For a more extensive study of the notion of fuzziness the reader is referred to [10] and [15]. On the other hand, by consulting financial economics experts, it is possible to estimate the probability distribution of the variable "value of the rand" compared to the USD. Since economists can assign then probability of being weak, strong and very strong to the rand at time t, it follows that the value of rand at any given time t has a random component in addition to fuzzy component. The theory of random variables and their application is relatively well developed and documented, see [1].

From the above discussion, it is abundantly clear that the variable "value of rand" has both attributes of fuzziness and randomness. Such a variable is called a hybrid variable. In [7] the theory of hybrid variables is examined in greater technical details.

In the above example, the trader is faced with the problem of deciding on the best time to export. This is a typical example of an optimal stopping problem. The problem has been presented and solved by several authors where the underlying process is stochastic, see [12]. However, as we have already observed, the value process of a rand normally possess both attributes of fuzziness and randomness.

This paper presents and examines, for the first time in the literature, the notion of hybrid stopping time. One major contribution of this paper is the definition and application of the notion of hybrid filtration. The paper states and proves variational inequalities for stochastic stopping problems. The Dynkin type formula will be used to derive the characteristic operator. The optimal stopping theory is applied to solve the problem of determining the optimal time to sell a stock.

The paper is organised as follows: In section 2, we review some notations and concepts, such as credibility measure, hybrid filtration, hybrid stopping, chance space and hybrid process. In section 3, we introduce the general hybrid stopping problem. A verification theorem for fuzzy stochastic optimal problems is proved in section 4. In section 5, an example is presented to illustrate the application of the main result of the paper.

2. PRELIMINARIES

In this section, we will introduce some essential definitions and properties about random variable, fuzzy variable, hybrid variable, hybrid filtration and hybrid stopping time.

Definition 2.1. ([12]) A probability space is a triple $(\Omega, \mathcal{F}, \mathrm{P})$ where Ω is a sample space, is $\mathcal{F}$ a σ-algebra over Ω and P is probability measure.

Definition 2.2. ([9]) A credibility space is a triple $(\Theta, \mathcal{P}, \mathrm{C})$ where Θ is a non-empty set, $\mathcal{P}$ is the power set of Θ and C is a credibility measure.

Definition 2.3. ([9]) Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space and $(\Theta, \mathcal{P}, \mathrm{C})$ is a credibility space. The product $(\Omega, \mathcal{F}, \mathrm{P}) \times (\Theta, \mathcal{P}, \mathrm{C})$ is called a chance space.

The universal set $\Omega \times \Theta$ is clearly the set of all ordered pairs of the form (ω, θ), where $\omega \in \Omega$ and $\theta \in \Theta$.

Definition 2.4. Let $\{M_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω, that is $M_s \subset M_t$ if $s \leq t$ and suppose $N_t = M_t \times \Theta$ where $M_t \subset \mathcal{F}$. Then we call $\{N_t\}_{t\geq 0}$ a hybrid filtration.

Definition 2.5. A function $\tau : \Omega \times \Theta \to [0;\infty]$ is called a hybrid stopping time with respect to $\{N_t\}_{t\geq 0}$ if $\{(\omega \times \theta) : \tau(\omega \times \theta) \leq t\} \in N_t$ for all t $\geq$ 0.

Theorem 2.1. If τ_1 and τ_2 are hybrid optimal stopping times then $\min\{\tau_1, \tau_2\}$ is hybrid stopping time.

Definition 2.6. ([9]) Let T be an index set and let $(\Theta, \mathcal{P}, \mathrm{C})$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{C})$ to the set of real numbers.

Definition 2.7. ([9]) A hybrid variable is a measurable function from a chance space $(\Omega, \mathcal{F}, \mathrm{P}) \times (\Theta, \mathcal{P}, \mathrm{C})$ to a set of real numbers.

Definition 2.8. Let T be an index set and $(\Omega, \mathcal{F}, \mathrm{P}) \times (\Theta, \mathcal{P}, \mathrm{C})$ a chance space. A hybrid process $\{X_t\}_{t\geq 0}$, is a measurable function from $T \times (\Omega, \mathcal{F}, \mathrm{P}) \times (\Theta, \mathcal{P}, \mathrm{C})$ to the set of real numbers: where $X_t := X(t, \omega, \theta)$ for every

$(t, \omega, \theta) \in [0, \infty)$.

Definition 2.9. ([9]) Let B_t be a Brownian motion, and let C_t be a C process. Then $D_t = (B_t, C_t)$ is called a D process.

Definition 2.10. ([9]) Let $X_t = (Y_t, Z_t)$ where Y_t and Z_t are scalar hybrid processes, and let $D_t = (B_t, C_t)$ be a standard process. For any partition of closed interval [a; v] with $a = t_1 < t_2 < \cdots < t_{k+1} = v$, the mesh is written as

$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|$. Then the hybrid integral of X_t with respect

to D_t is

$$\int_a^b X_t \, dD_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} [Y_{t_i}(B_{t_{i+1}} - B_{t_i}) + Z_{t_i}(C_{t_{i+1}} - C_{t_i})]. \quad (2.1)$$

Remark 2.2. ([9]) The hybrid integral may also be written as follows,

$$\int_a^v X_t \, dD_t = \int_a^v (Y_t dB_t + Z_t dC_t). \quad (2.2)$$

Definition 2.11. ([9]) Suppose $\{X_t\}_{t \ge 0}$ is a hybrid process, B_t is a standard Brownian motion and C_t is a standard C process. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t + h(t, X_t)dC_t, \quad X(0) = X_0 \quad (2.3)$$

is called a hybrid differential equation

$f: [0, T] \times \mathbb{R} \to \mathbb{R}, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $h: [0, T] \times \mathbb{R} \to \mathbb{R}$.

3. BACKGROUND AND PROBLEM FORMULATION

In this section introduce the general hybrid optimal stopping problem. Consider a chance space (Ω, Ƒ, P)× (θ, Ƥ,C). Let X_t be a hybrid process which evolves according to the following differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t + \lambda(t, X_t)dC_t \, ; X_s = B_s + C_s$$

where $B_s + C_s = b + c$, $X_t \in \mathbb{R}$, B_t is a 1-dimensional Brownian motion, C_t is 1-dimensional standard Liu process such that B_s= b and C_s= c. Assume that μ:[s; T]× ℝ → ℝ, σ:[s; T]× ℝ → ℝ, λ:[s; T]× ℝ → ℝ are given functions which satisfy the conditions for the existence and uniqueness of a solution for hybrid differential equations where $0 \le s < T < \infty$. Suppose z is a reward function on ℝ such that $z(\xi) \ge 0$ for all $\xi \in \mathbb{R}$.

Problem 1. The problem is to find the stopping time τ^* and value function Φ(s; b; c) where b and c the values of the stochastic and fuzzy components respectively of the price at time s, such that

Φ(s; b; c) = sup$E^{(s,b,c)}\left[\int_s^{\tau} w(s+t, X_t)dt + z(s+\tau, X_\tau)\right]$

= sup$E^{(s,b,c)}\left[\int_s^{\tau^*} w(s+t, X_t)dt + z(s+\tau^*, X_{\tau^*})\right]$

where w is "profit rate" function.

Definition 3.1. Let $\{X_t\}_{t\geq 0}$ be a hybrid process. The characteristic operator $\mathcal{A}$ $=\mathcal{A}_X$ of $\{X_t\}$ is defined by

$$\mathcal{A}\phi(\mathrm{t,b,c} = \lim_{U\downarrow b+c}\frac{E^{(b,c)}[\phi(\mathrm{X}_\tau)] - \phi(b,c)}{E^{(b,c)}[\tau_U]}$$

where the U's are open sets U_k decreasing to the point b + c which is the price, in the sense that $U_{k+1} \subset U_k$ and $\cap_k U_k = \{b+c\}$, and τ_U=inf$\{t > 0; X_t \notin U\}$ is the first exit time from U for

Lemma 3.1. Let ϕ(t; b; c) $\in C^2([0,\infty) \times \mathbb{R}^2)$. The characteristic operator $\mathcal{A}$ is given by

$$\mathcal{A}\phi(\mathrm{t,b,c}) = \frac{\partial\phi}{\partial t} + \sigma(b+c)\frac{\partial\phi}{\partial b} + \lambda(\mathrm{b+c})\frac{\partial\phi}{\partial \mathrm{c}} + \frac{1}{2}\sigma^2(\mathrm{b+c})^2\frac{\partial^2\phi}{\partial \mathrm{b}^2}.$$

Proof

Since the function ϕ is twice continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t = ϕ(t,$B_{t,}C_t$) has a second order approximation

$$dX_t = \frac{\partial\phi}{\partial t}(\mathrm{t}, B_{t,}C_t)dt + \mu(b+c)\frac{\partial\phi}{\partial b}(\mathrm{t}, B_{t,}C_t)dB_t + \lambda(\mathrm{b+c})\frac{\partial\phi}{\partial \mathrm{c}}(\mathrm{t}, B_{t,}C_t)\mathrm{dC_t} + \frac{\partial^2\phi}{\partial \mathrm{t}^2}(\mathrm{t}, B_{t,}C_t)\mathrm{dt} + \frac{1}{2}\sigma^2(\mathrm{b}+$$

$$\mathrm{c})^2\frac{\partial^2\phi}{\partial \mathrm{b}^2}(\mathrm{t}, B_{t,}C_t)(dB_t)^2 + \frac{1}{2}\lambda^2(\mathrm{b+c})^2\frac{\partial^2\phi}{\partial \mathrm{c}^2}(\mathrm{t}, B_{t,}C_t)(dC_t)^2 + \frac{1}{2}\mu(\mathrm{b+c})\frac{\partial^2\phi}{\partial \mathrm{t}\,\partial \mathrm{b}}(\mathrm{t}, B_{t,}C_t)\mathrm{dt}dB_t + \lambda(\mathrm{b}+$$

$$\mathrm{c})\frac{\partial^2\phi}{\partial \mathrm{t}\,\partial \mathrm{c}}(\mathrm{t}, B_{t,}C_t)\mathrm{dt}dC_t + \lambda(\mathrm{b+c})^2\frac{\partial^2\phi}{\partial \mathrm{b}\,\partial \mathrm{c}}(\mathrm{t}, B_{t,}C_t)dB_t\mathrm{dC_t}.$$

Taking into cognisant that each of the terms $(dt)^2$, $(dC_t)^2$, $\mathrm{dt}dB_t$, $\mathrm{dt}dC_t$ and $dB_t\mathrm{dC_t}$is equal to zero and replacing $(dB_t)^2$ with dt, the following chain rule is obtained

$$\int_0^s \mathrm{d}X_t = \int_0^s \frac{\partial\phi}{\partial t}\,dt + \int_0^s \mu(b+c)\frac{\partial\phi}{\partial b}dB_t + \int_0^s \lambda(\mathrm{b+c})\frac{\partial\phi}{\partial \mathrm{c}}\mathrm{dC_t} + \int_0^s \frac{\sigma^2}{2}(\mathrm{b+c})^2\frac{\partial^2\phi}{\partial \mathrm{b}^2}\mathrm{dt}.$$

This last equation yields

$$X_s = X_0 + \int_0^s \frac{\partial\phi}{\partial t}\,dt + \int_0^s \mu(b+c)\frac{\partial\phi}{\partial b}dB_t + \int_0^s \lambda(\mathrm{b+c})\frac{\partial\phi}{\partial \mathrm{c}}\mathrm{dC_t} + \int_0^s \frac{\sigma^2}{2}(\mathrm{b+c})^2\frac{\partial^2\phi}{\partial \mathrm{b}^2}\mathrm{dt}.$$

By Dynkin's formula

$$E^{(b,c)}[\phi(X_\tau)] = \phi(t,b,c) + E^{(b,c)}\left[\int_0^\tau \frac{\partial\phi}{\partial t}ds + \int_0^\tau \mu(b+c)\frac{\partial\phi}{\partial b}dB_s\right]$$

$$+E^{(b,c)}\left[\int_0^{\tau}\lambda(b+c)\frac{\partial\phi}{\partial c}dC_s+\frac{1}{2}\sigma^2(b+c)^2\int_0^{\tau}\frac{\partial^2\phi}{\partial b^2}ds\right]$$

where $A\phi(t,b,c)$ is the characteristic operator of X_t

$$E^{(b,c)}[\phi(X_\tau)]-\phi(t,b,c)=\int_0^{\tau}E^{(b,c)}\left[\frac{\partial\phi}{\partial t}+A\phi(t,b,c)\right]ds$$

$$E^{(b,c)}[\phi(X_\tau)]-\phi(t,b,c)=E^{(b,c)}\int_0^{\tau}\mathcal{A}\phi ds.$$

Therefore,

$$\mathcal{A}\phi(t,b,c)=\lim_{U\downarrow b+c}\frac{E^{(b,c)}[\phi(X_\tau)]-\phi(t,b,c)}{E^b[\tau_U]}$$

4. MAIN RESULTS

In this section we are going to state and prove the Variational Inequalities.

Theorem 4.1. Variational Inequalities

a. Suppose we can find a function $\phi: S\subset \mathbb{R}\to\mathbb{R}$ such that

   i. $\phi\in C^1(S)\cap C(\bar{S})$.

   ii. $\phi\geq z$ on S where z is the reward function and $\lim_{t\to\tau_{\bar{S}}} z(Y_{\tau_S})\chi_{\{\tau_S<\infty\}}$ where $\tau_S=\inf\{t>0;\ Y_t\notin S\}$ and χ is the characteristic function.

   iii. $\mathcal{A}\phi+w\leq 0$ on S/M where w is profit rate function then $\phi(y)=\Phi(y)$ for all $y\in S$ where $\Phi(y)$ is the value function and $M=\{x\in S;\ \phi(x)>z(x)\}$ is the continuation region such that $M\subset S$.

b. Moreover, assume

   i. $\mathcal{A}\phi+w=0$ on M.

   ii. $\tau_M:=\inf\{t>0, Y(t)\notin M\}<\infty a.s. for\ all\ y$. Then

$$\phi(y)=sup_{\tau\in\mathcal{T}}E^y\left[\int_0^{\tau}w(Y_t)dt+z(Y_t)\right]$$

and

$$\tau^*=\tau_M$$

is the optimal stopping time for this problem.

Proof

Since $\phi\in C^1(S)\cap C(\bar{S})$ and $\phi\in C^2(S/\partial M)$ we can find the sequence of functions

$$\phi_j \in C^2(S) \cap C(\bar{S}); j = 1, 2, 3, \ . \ . \ .$$

such that

1. $\phi_j \to \phi$ uniformly on compact subsets of $\bar{S}$ as $j \to \infty$
2. $\mathcal{A}\phi_j \to \mathcal{A}\phi$ uniformly on compact subsets of $S/\partial M$ as $j \to \infty$
3. $\{\mathcal{A}\phi_j\}_{j=1}^{\infty}$ is locally bounded on S.

Suppose $\{S_R\}_{j=1}^{\infty}$=is a sequence of bounded open sets such that S=$\cup_{R=1}^{\infty} S_R$, $T_R = \min(R, \inf\{t > 0; Y_t \notin S_R\})$ and $\tau \leq \tau_S$ be stopping times. Then by the Dykin`s formula

$$E^y[\phi_j(Y_{\tau \wedge T_R})] = \phi_j(y) + E^y[\mathcal{A}\phi_j(Y_t)dt]$$

where $\tau \wedge T_R = \min\{(\tau, T_R)\}$.

Hence by 1,2,3 and (iii) and the bounded a.e. convergence

$$\phi(y) = \lim_{j\to\infty} E^y\left[\int_0^{\tau\wedge T_R} -\mathcal{A}\phi_j(Y_t)dt + \phi_j(Y_{\tau\wedge T_R})\right].$$

Therefore, by a (ii), a(iii) and b(i)

$$\phi(y) \geq E^y\left[\int_0^{\tau\wedge T_R} w(Y_t)dt + z(Y_{\tau\wedge T_R})\right]. \qquad (4.1)$$

Hence by Fatou lemma, since also $E^y\left[\int_0^{\tau_s} w^-(Y_t)dt\right] < \infty$ and the family $\{g^-(Y_\tau); \tau \leq \tau_s\}$ is uniformly integrable we get

$$\phi(y) \geq E^y\left[\int_0^{\tau} w(Y_t)dt + z(Y_\tau)\right].$$

Since $\tau \leq \tau_S$was arbitrary, conclude that

$$\phi(y) \geq \Phi(y) \in S \qquad (4.2)$$

which proves (a).

We proceed to prove (b): If $y \in M$ then

$$\phi(y) = z(y) \leq \Phi(y) \qquad (4.3)$$

so by (4.2) we have $\phi(y) \geq \Phi(y)$ and

$$\hat{\tau} = \hat{\tau}(y, w) \qquad (4.4)$$

Is optimal for $y \in M$.

Next, suppose $y \in M$. Let $\{M_n\}_{n=1}^{\infty}$ be an increasing sequence of open sets. By Dynkin's formula we have for $y \in M_n$

$$\phi(y) = \lim_{j\to\infty} \phi_j(y) = \lim_{j\to\infty} E^y\left[\int_0^{\tau_n \wedge T_R} -\mathcal{A}\phi_j(Y_t)dt + \phi_j(Y_{\tau_n \wedge T_R})\right]$$

$$= E^y\left[\int_0^{\tau_n \wedge T_R} w(Y_t)dt + \phi(Y_{\tau_n \wedge T_R})\right].$$

So by uniform integrability we get

$$\phi(y) = E^y\left[\int_0^{\tau_M} w(Y_t)dt + z(Y_{\tau_M})\right]$$

$$\phi(y) = J^{\tau_M}(y) \leq \Phi(y) (4.5)$$

where $J^{\tau_M}(y) = \; = E^y\left[\int_0^{\tau_M} w(Y_t)dt + z(Y_{\tau_M})\right]$. Combining (4.2) and (4.5) we get

$$\phi(y) \geq \Phi(y) \geq J^{\tau_M}(y) \coloneqq \tau_M \qquad (4.6)$$

is optimal when $y \in M$. From (4.4) and (4.6) we conclude that $\phi(y) = \Phi(y)$for all $y \in S$. Moreover, the stopping time $\hat{\tau}$ defined by $\hat{\tau}(y, w) = \tau_M$for $y \in M$ is optimal. Therefore, we conclude that τ_M is optimal.

5. APPLICATION

Consider a chance space $(\Omega, \mathcal{F}, P) \times (\Theta, \mathcal{P}, C)$. Suppose the price X_t, at time t, of a person's asset evolves according to a hybrid differential equation of the form

$$dX_t = rX_t dt + \alpha X_t dB_t + \beta X_t dC_t; \;\; X_0 = x = b + c \qquad (5.1)$$

where B_t is a 1-dimensional Brownian motion and C_t is a 1-dimensional standard C process and r, α and β are constants. Suppose that the sale of the asset is associated with a fixed cost

$a > 0$. Then if the owner of the asset decides to sell it at time t the discounted net of the sale is $e^{-\rho t}(B_t + C_t - a)$ where $\rho > 0$ is a given discounting factor.

Problem 2. The problem is to find the optimal stopping time τ^* and the value function $\Phi(s;\ b;\ c)$ such that

$$\Phi(s;\ b;\ c) = \max_{\tau} E^{(s,b,c)}\left[e^{-\rho\tau}(B_\tau + C_\tau - a)\right]$$

$$= E^{(s,b,c)}\left[e^{-\rho\tau^*}(B_{\tau^*} + C_{\tau^*-a}\}\right].$$

Solution. To solve problem 2 we resort to the theory of optimal stopping developed in the previous sections. Specifically, we apply the Variational Inequalities that are stated and proved in section 5.

The characteristic operator $\mathcal{A}$ of the process $Y_t = (s + t,\ B_t,\ C_t)$ is given by

$$\mathcal{A}\varphi(s,b,c) = \frac{\partial\varphi}{\partial s} + r(b+c)\frac{\partial\varphi}{\partial b} + \frac{\alpha^2}{2}(\mathrm{b}+\mathrm{c})^2\frac{\partial^2\varphi}{\partial \mathrm{b}^2} + \beta(\mathrm{b}+\mathrm{c})\frac{\partial\varphi}{\partial \mathrm{c}} \quad (5.2)$$

where $\varphi: [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable function. Now, $(s,b,c) = e^{-\rho s}(b+c-a)$ where a is a constant. Substituting for partial derivatives of φ into (5.2) and simplify we obtain

$$\mathcal{A}\varphi(s,b,c) = e^{-\rho s}[(b+c)(-\rho+\beta+r)+\rho a].$$

If $\mathcal{A}\varphi(s,b,c) > 0$, then

$$e^{-\rho s}[(b+c)(-\rho+\beta+r)+\rho a] > 0.$$

Solving for$b+c$ we get

$$b+c = \frac{\rho a}{\rho-(\beta+r)}.$$

If $r+\beta > \rho$ then the value function is equal to ∞. If $r+\beta = \rho$ then the value function turns out to be $(b+c)e^{-\rho s}$. When $r+\beta \geq \rho$ there is no stopping time, a sequence of better stopping time tends to ∞ instead of converging.

The optimal strategy should be: sell when the underlying reaches the threshold k^*, with k^* depending only on the parameters of the problem, i.e. r, ρ, α, β and a. The selling regions are divided into two: continuation region later region where $b+c < k^*$ and sell now region where $b+c > k^*$. In sell now region clearly $\Phi(s,b,c) = b+c-a$ and in the sell later region it satisfies the P.D.E.

$$\frac{\partial\Phi}{\partial s} + r(b+c)\frac{\partial\Phi}{\partial b} + \frac{1}{2}\alpha^2(\mathrm{b}+\mathrm{c})^2\frac{\partial^2\Phi}{\partial \mathrm{b}^2} + \beta(\mathrm{b}+\mathrm{c})\frac{\partial\Phi}{\partial \mathrm{c}} = 0 \quad (5.3)$$

with the boundary condition $\Phi(s,b,c) = b+c-a$ at $b+c > k^*$. Moreover we have the global inequalities $\Phi(s,b,c) \geq b+c-a$ and

$$\frac{\partial\Phi}{\partial s} + r(b+c)\frac{\partial\Phi}{\partial b} + \frac{\alpha^2}{2}(\mathrm{b}+\mathrm{c})^2\frac{\partial^2\Phi}{\partial \mathrm{b}^2} + \beta(\mathrm{b}+\mathrm{c})\frac{\partial\Phi}{\partial \mathrm{c}} \leq 0. \quad (5.4)$$

To identify the optimal threshold k^* we consider any candidate threshold k. The associated value function Φ_k solves

$$\frac{\partial\Phi_k}{\partial s} + r(b+c)\frac{\partial\Phi_k}{\partial b} + \frac{\alpha^2}{2}(\mathrm{b}+\mathrm{c})^2\frac{\partial^2\Phi_k}{\partial \mathrm{b}^2} + \beta(\mathrm{b}+\mathrm{c})\frac{\partial\Phi_k}{\partial \mathrm{c}} \leq 0. \quad (5.5)$$

for$b+c < k$ with the boundary condition $\Phi_k(s,b,c) = b+c-a$ at $b+c = k$. This can be solved explicitly. If we try the general solution of (5.5) of the form

$$(s,b,c) = e^{-\rho s}\vartheta(b,c). \quad (5.6)$$

Now considering (5.6) we have

$\frac{\partial \Phi_k(s,b,c)}{\partial s} = -\rho e^{-\rho s}\vartheta(b,c)$, $\frac{\partial \Phi_k(s,b,c)}{\partial b} = e^{-\rho s}\vartheta_b(b,c)$, $\frac{\partial \Phi_k(s,b,c)}{\partial b^2} = e^{-\rho s}\vartheta_{bb}(b,c)$, $\frac{\partial \Phi_k(s,b,c)}{\partial c} = e^{-\rho s}\vartheta_c(b,c)$.

Substituting partial derivatives of Φ_k into (5.5) and simplifying we get

$$\rho\vartheta(b,c) + r(b+c)\vartheta_b(b,c) + \frac{1}{2}\alpha^2(b+c)^2\vartheta_{bb}(b,c) + \beta(b+c)\vartheta_c(b,c) = 0 \quad (5,7)$$

Let the solution of (5.7) be of the form

$$(b,c) = A(b+c)^{\gamma_i} \quad (5.8)$$

where A is a constant. So

$\vartheta_b = \gamma_i A(b+c)^{\gamma_i - 1}$, $\vartheta_{bb} = \gamma_i(\gamma_i - 1)A(b+c)^{\gamma_i - 2} and$ $\vartheta_c = \gamma_i A(b+c)^{\gamma_i - 1}$.

Substituting ϑ_b, $\vartheta_{bb}$and ϑ_c into (5,7) and simplifying we get

$$-\rho A(b+c)^{\gamma_i} + r(b+c)\gamma_i A(b+c)^{\gamma_i-1} + \frac{1}{2}\alpha^2(b+c)^2\gamma_i(\gamma_i-1)A(b+c)^{\gamma_i-2} + (b+c)\beta A(b+c)^{\gamma_i-1} = 0.$$

Dividing throughout by $A(b+c)^{\gamma_i - 1}$ we obtain

$$-\rho + r\gamma_i + \frac{1}{2}\alpha^2\gamma_i(\gamma_i - 1) + \beta\gamma_i = 0.$$

This last equation can be written as:

$$-2\rho + (2\beta + 2r - \alpha^2)\gamma_i + \alpha^2\gamma_i^2 = 0.$$

Solving using quadratic formula and simplify we get the solutions

$$\gamma_1 = \frac{-(2r - \alpha^2 + 2\beta) + \sqrt{(2r - \alpha^2 + 2\beta)^2 + 8\rho\alpha^2}}{2\alpha^2}$$

and

$$\gamma_2 = \frac{-(2r - \alpha^2 + 2\beta) - \sqrt{(2r - \alpha^2 + 2\beta)^2 + 8\rho\alpha^2}}{2\alpha^2}.$$

Therefore,

$$\vartheta(b,c) = A_1 b^{\gamma_1} + A_2 b^{\gamma_2}; \quad \gamma_2 < 0 < \gamma_1 \quad (5.9)$$

whereA_1 and A_2 are constants.

To determine Φ_k we must specify $A_1$and A_2. Since u_k should be bounded

as$b \to 0$ and $c \to 0$ we have $A_2 \to 0$ and the boundary condition $u_k = k - a$

atb + c = k gives

$$k - a = A_1(b+c)^{\gamma_1}.$$

Solving this last equation for A_1 we get

$$k^{-\gamma_i}(k - a) = A_1 \quad (5.10)$$

Substituting (5.8) into (5.6) the expected payoff using sales threshold k is

$$\Phi_k(s,b,c) = e^{-\rho s}k^{-\gamma_1}(k-a)(b+c)^{\gamma_1}.$$

Distributing $\left(\frac{b+c}{k}\right)^{\gamma_1}$ and simplify yields

$$\Phi_k(s,b,c) = (k^{1-\gamma_1}(b+c)^{\gamma_1} - a(b+c)^{\gamma_1}k^{-\gamma_1})e^{-\rho s}. \quad (5.11)$$

Differentiating $\Phi(s,b,c)$ with respect to k we get

$$\frac{d\Phi_k}{dk} = e^{-\rho s}(1-\gamma_1)(b+c)^{\gamma_1}k^{-\gamma_1} - e^{-\rho s}a(b+c)^{\gamma_1}(-\gamma_1)k^{-\gamma_1-1}.$$

Substituting k for $b+c$ and simplifying we obtain

$$\frac{d\Phi_k}{dk} = e^{-\rho s}(1-\gamma_1) + e^{-\rho s}\gamma_1 a k^{-1}.$$

When k is optimal $\frac{d\Phi_k}{dk} = 0$. This yields k given by

$$k = \frac{\gamma_1 \alpha}{\gamma_1 - 1}.$$

Therefore,

$$k^* = \frac{\gamma_1 \alpha}{\gamma_1 - 1}. \quad (5.12)$$

One should sell the asset the best time the price reaches the value $(b+c)_{max} = \frac{\gamma_1}{\gamma_1 - 1}$.

The expected discounted profit obtained from this strategy is calculated as follows. We first substitute (5.12)into (5.11) and we obtain

$$\Phi_k = e^{-\rho s}\left(\frac{(b+c)(\gamma_1-1)}{\gamma_1\alpha}\right)^{\gamma_1}\left(\frac{\gamma_1 a - \gamma_1 a + a}{\gamma_1 - 1}\right).$$

Distributing the power γ_1 simplify we get

$$\Phi_k(s,b,c) = e^{-\rho s}\left(\frac{\gamma_1 - 1}{\alpha}\right)^{\gamma_1 - 1}\left(\frac{b+c}{\gamma_1}\right)^{\gamma_1}$$

The conclusion is therefore that one should sell the asset the first time the price reaches the value $(b+c)_{max} = \frac{\gamma_1}{\gamma_1 - 1}$. The expected discounted profit obtained from this strategy is

$$\Phi_{k^*}(s,b,c) = e^{-\rho s}\left(\frac{\gamma_1 - 1}{\alpha}\right)^{\gamma_1 - 1}\left(\frac{b+c}{\gamma_1}\right)^{\gamma_1}.$$

Theorem 5.1. The optimal stopping policy is to sell the asset when the price reaches a certain threshold $k^* = \frac{\gamma_1 \alpha}{\gamma_1 - 1}$ or immediately if the present price is greater than k^* the value achieved by this policy is

$$\Phi_{k^*}(s,b,c) = \begin{cases} e^{-\rho s}\left(\frac{\gamma_1 - 1}{\alpha}\right)^{\gamma_1 - 1}\left(\frac{b+c}{\gamma_1}\right)^{\gamma_1} & ;\ b+c < k^* \\ b+c-a & ;\ b+c > k^* \end{cases}$$

6. CONCLUSION AND RECOMMENDATION

Based on the concept of hybrid process, we studied a hybrid optimal stopping problem: optimising the expected value of an objective function subject to hybrid differential equation. The verification theorem for optimal stopping was then derived. We also formulated and solved optimal stopping problem for hybrid process. As a line for further research the problem of hybrid optimal stopping with jumps can be studied. A hybrid process with jumps is more realistic description of phenomena such as stock price process. The price process of stock may experience jumps due to sudden shift in policy by central bank, war or other natural disasters like floods and drought. For further enquiry problems of optimal stopping rules for hybrid processes can studied in future.

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