

Finite Time Stability Criteria for Nonlinear Fractional Order Dynamical System

Prebo Clifford Jackreece
Department of Mathematics and Statistics, University of Port Harcourt
prebo.jackreece@uniport.edu.ng

Abstract

In this paper finite time stability criteria for a class of nonlinear fractional order delayed system is addressed. By using the generalized and classical Bellman-Gronwall's approach sufficient conditions that guarantees system trajectories to stay within the a priori given set is obtained.

Keywords: Nonlinear system, multiple time delays, Finite time stability, fractional order system

1. Introduction

The question of stability of linear and nonlinear dynamical system is an important and active area of research and of main interest in control theory. In recent years there has been a substantial increase of activities on time delay systems. Time delays are very often encountered in different technical systems such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, control systems etc. (Williams and Otto 1960; Hong et al. 2002; Hong 2002). These delays are due to transportation of materials, energy or information, (Zavarei and Jamshidi, 1987; Dugard and Verriest, 1997). The existence of time delays regardless of its presence in the control or/and state, may cause undesirable system transient response and frequently the source of instability. Consequently, the question of stability of these class of systems is of theoretical and practical importance, see (Boukas and Al-Muthairi 2006; Liu 2003; Zhou and Lam 2003).

Numerous remarkable results relating to the stability of time delay systems have been published with particular emphasis on the application of Frequency domain techniques (Olgac and Sipahi 2002; Jia et al. 2007), Lyapunov methods, or idea of matrix measure (Lee and Diant, 1981; Mori 1985), small-gain-base methods (Gu et al., 2003).

When dealing with stability of systems, a distinction is made between classical Lyapunov stability and finite time stability. However, from the practice point of view, finite time stability has advantage over the classical Lyapunov stability. The concept of Lyapunov stability is largely known to the control community, conversely a system is said to be finite time stable if, given a bound on the initial condition, its state does not exceed a certain threshold within a specified time interval. To verify the finite time stability of systems, several authors have developed different techniques to investigate stability criteria, see (Amato et al. 2003; Debeljkovi et al., 2014; Debeljkovic et al., 2013; Shen et al. 2007; Moulay et al. 2008; Liu 2003a; Liu 2003b; Kablar and Debeljkovic, 1998; Jackreece 2016, 2017) and references therein.

Fractional calculus was first introduced 300 years ago and dates back to the works of Leibnitz, Liouville, Riemann, Grunwald, and Letnikov (Oldham and Spanier 1974; Samko et al., 1993; Kilbas et al., 2006). Fractional calculus is branch of Mathematics that focus on investigating the properties of derivatives and integrals of non-integer order and it is a generalization of the traditional (integer order) calculus in which the order of derivatives and integrals can be any real or complex number. In recent years, the study of fractional order derivative has attracted increasing interest due to an important role it plays in mathematics, physics, engineering, control systems, dynamical systems and in particular in the modeling of many natural phenomena, (Kulish and Lage, 2002; Soczkiewicz, 2002; Sabatier et al., 2007; Kilbas et al., 2006; Yu, 2011). There are two aspects that essentially differentiates fractional order models and integer order, which makes it more realistic to characterize real world physical systems by fractional order state equation. First for the integer order derivative indicates a variation of a certain attributes at a particular time for a physical or mechanical process, while fractional order is concerned with the whole-time domain. Second, integer order derivatives describe the local properties of a certain position for physical process, while fractional order derivative is related to the whole space, (Caputo, 1967; Hilfer, 2000).

However, in many real world physical systems, fractional calculus is more feasible than integer calculus to model the behaviors of such system. For example, fractional electrical networks (Mohd et al., 2016; Zahra et al. 2017), fractional order Schrodinger equation (Naber, 2004), fractional oscillator equation (Ryabov and Puzenko, 2002), fractional Lotka-Volterra equation (Hilfer, 2000), robotics ((Ferreira et al,2008). In particular stability analysis is one of the most fundamental and important issues for systems.

Different approaches have been used to investigate the stability of various fractional dynamical system, such as analytical approach (Chen and Moore, 2002; Kaslik and Sivasundaram, 2012), fixed point theorem ((Deng & Li, 2012; Deng, 2010), the Lyapunov method (Li et al., 2010; Momani and Hadid 20004; Liu and Jiang 2014), linear matrix inequality (Sabatier et al., 2010), Bellman-Gronwall inequality (Denghao and Wei, 2014; Lazarevic, 2006; Jackreece and Aniaku 2018).

Recently there have been advances in control theory of fractional order dynamical systems for different kinds of stability. (Matignon, 1996) considered the structural stability result of fractional differential equations with applications to control processing from both algebraic and analytical point of views. In their paper (Deng et al. 2007) analyzed the stability of linear fractional differential system with multiple time delays. Based on the characteristic equation defined by using Laplace transform, several stability criterions are derived. Jackreece and Aniaku 2018 developed sufficient condition for the finite time stability of a class of linear fractional order systems with time varying delays using generalized Bellman-Gronwall inequality.

In Lazarevic and Debeljkovic 2005, they extend some basic results of finite and practical stability of linear, continuous fractional order invariant time delay systems with delay in the state by proposing a stability test procedure using Bellman-Gronwall theorem. Denghao and Wei 2014, obtained sufficient condition for the finite time stability of a class of fractional singular time delay systems by giving the Mittag-Leffler estimate of the solution for an equivalent system. Also, finite time stability analysis of fractional order time delay systems is studied in (Lazarevic, 2006; Lazarevic et al., 2010). Our aim in this paper is to develop sufficient condition for the finite time stability of a class of nonlinear fractional order systems with multiple time varying delays in the state.

2. Preliminaries

Consider the following control system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x\left(t-\tau_i(t)\right) + Bu(t) + f(t,x) \tag{2.1a}$$

$$x(t) = \psi(t) \qquad t \in [-\tau_M, 0] \tag{2.1b}$$

Where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input control vector, $A_0, A_i \in R^{n\times n}$, $B \in R^m$ are constant system matrices, $\tau_i(t), i = 1,2,...,k$ are time varying delays which satisfies the condition

$$0 < \tau_i(t) \le \tau_M \quad \forall i \in \{1,2,...,k\}, \forall t \in T .$$

$\psi \in C\left([-\tau_M, 0], R^n\right)$ is an admissible initial state and $C\left([-\tau_M, 0], R^n\right)$ is a Banach space of continuous function mapping the interval $[-\tau_M, 0]$ into R^n which converges uniformly and the norm defined by

$$\|\psi\| = \sup_{-\tau_M \le \theta \le 0} |\psi(\theta)|$$

The system behavior is defined over the time interval $I=[0,T]$, where T is a positive number. $f(t,x)_{n\times 1}:R\times R^n \to R^n$ satisfies Lipschitz condition w.r.t. x, that is

$$\|f(t,x)-f(t,y)\|\le L\|x-y\|$$
$$\Rightarrow \|f(t,x)\|\le L\|x\|+m$$

Where

$$m=\|f(t,\theta)\|,\theta \text{ is a null vector}$$

For the time invariant sets $S_{(\ )}$, used as bounds of the system trajectories are assumed to be bounded, open and connected. Let S_β be a given set of all allowable states of the system for $\forall t\in I$. Let S_α be the set of all initial states of the system such that $S_\alpha \subseteq S_\beta$ and S_γ denote the set of all allowable control actions. The sets S_α and S_β are connected and a priori known.

Let $\sigma_{\max}(.)$ be the largest singular value of matrix A and

$$\sigma_1=\max_{1\le i\le k}\{\sigma_{\max}(A_i)\},\sigma=\max\{\sigma_{\max}(A_0),\sigma_1\}\Rightarrow\|A_i\|<\sigma\ \forall i=1,2,...,k \tag{2.2}$$

$$b=\sigma_{\max}(B) \tag{2.3}$$

Before proceeding further, we will introduce the following definitions and theorems which will be used in the next section.

Matrix measures have been extensively studied in (Desoer, and Vidyasagar, 1975;Hu and Liu, 2004) and it is used to estimate upper bounds of matrix exponential. The following theorem relates an upper bound of a matrix exponential to its matrix measures.

Theorem 2.1: (Desoer, and Vidyasagar, 1975; Hu and Mitsui, 2012) For any matrix $A\in R^{n\times n}$ the estimate

$$\|\exp(A(t))\|\le\exp(\mu(A)(t))$$

holds.

Theorem 2.2: The matrix norm or Lozinskii logarithm norm of a $n\times n$ matrix A is

$$\mu(A)=\lim_{h\to 0}\frac{|I+hA|-I}{h}$$

Where $\|(.)\|$ is any matrix norm compatible with some vector norm $|x|_{(.)}$. The matrix measure defined in theorem 2.2 has three variants depending on the norm utilized in the definition.

It is assumed that the usual smoothness condition is satisfied by system (2.1) so that there will be no difficulty with the question of existence, uniqueness and continuity of solutions with respect to initial data.

Before stating our results, we introduce the concept of finite-time stability for time-delay system (2.1). This concept can be formalized through the following definition.

Definition 2.1: Time delayed control system is finite time stable with respect to $\{S_\alpha, S_\beta, T, \|(.)\|, \mu(A_0) \neq 0\}$, $\alpha < \beta$ if and only if :

$$\psi(t) \in S_\alpha,\ \forall t \in [-\tau, 0] \text{and } u(t) \in S_\gamma, \forall t \in T$$

implies

$$x(t : t_0, x_0) \in S_\beta, \forall t \in [0, T]$$

See (Debeljkovi, et al., 2001; Lazarevic, and Debeljkovi, 2005)

Definition: 2.2: (Hu et al. 2004); For any real matrix $A = (a_{ij})_{n\times n}$, its matrix measure is defined as

$$\mu_\rho(A) = \lim_{\varepsilon \to 0^+} \frac{\|I - \varepsilon A\| - 1}{\varepsilon} \tag{2.5}$$

Where $\|.\|_\rho$ denotes the matrix norm in $R^{n\times n}$, I is the identity matrix and $\rho = [1, 2, \infty]$ norm. The matrix norms are defined as follows

$$\|A\|_1 = \max_j \left\{ \sum_{i=1}^{n} |a_{ij}| \right\},\ \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} \text{ and } \|A\|_\infty = \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$$

Lemma 2.1: For the definition of matrix measure, for any $A, B \in R^{n\times n}, \rho = 1, 2, \infty$, we have

1. $-\|A\|_\rho \leq \mu_\rho(A) \leq \|A\|_\rho$
2. $\mu_\rho(\alpha A) = \alpha \mu_\rho(A), \quad \forall \alpha > 0$
3. $\mu_\rho(A + B) \leq \mu_\rho(A) + \mu_\rho(B)$

Definition 2.3: System given by eq. (2.1) satisfying the initial condition $x(t) = \psi_x(t),\ -\tau_M \leq t \leq 0$ is finite time stable w.r.t. $\{\delta, \varepsilon, \gamma, t_0, T\}$, $\delta < \varepsilon$ if and only if $\|\psi_x\|_C < \delta$ and $\|u(t)\| < q_u,\ \forall t \in T$ imply $\|x(t)\| < \varepsilon,\ \forall t \in T.$

3. Fundamentals of Fractional Calculus

At first, the differential and integral operators can be generalized into one fundamental D_t^α operator t which is known as fractional calculus, Oldham and Spanier 1974; Podlubny 1999; Kilbas et al. 2006; Das 1999)

$$_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0 \\ 1 & \Re(\alpha) = 0 \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0 \end{cases}$$

There are many ways to define fractional derivatives and integrals. The definition generally used in recent studies are, Grunwald-Letinkov, Riemann-Liouville and Caputo definitions.

Definition 3.1

The Grunwald-Letnikov (GL) fractional derivative of order $\alpha,\ \alpha > 0$ and fractional integral of order $\alpha,\ \alpha > 0$ of a continuous function $f(t)$ defined on the interval $[a,b]$ are defined by

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{[(t-a)/h]}(-1)^k\binom{\alpha}{k}f(t-kh),\ \binom{\alpha}{k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \tag{3.1}$$

and the fractional integral defined as,

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \sum_{k=0}^{n-1}\frac{f^{(k)}(0)t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds \tag{3.2}$$

Where a and t are limits of the operator, $[(.)]$ denotes the integer part of $(.)$ and $\Gamma(.)$ is the Euler's gamma function that generalizes factorial for non-integer arguments:

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt,\quad \Gamma(z+1) = \Gamma(z),\ z = x+iy$$

One basic property of the gamma function is that it satisfies the functional equation

$$\Gamma(z) = z\Gamma(z), \Rightarrow \Gamma(n+1) = n(n+1)! = n!$$

Definition 3.2

Let $[a,b]$ be a finite interval, $-\infty < a < b < \infty,\ [a,b] \subset R$ and $f(t)$ be a continuous function defined on $[a,b]$, the Riemann-Liouville fractional derivative of order α is given by

$$ {}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds \tag{3.3}$$

For $(n-1 < \alpha < n)$ and $\Gamma(.)$ is Euler's gamma function.

Closely related to Riemann-Liouville Fractional order derivative is the fractional integral defined by

$$ {}_{a}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,\qquad \alpha < 0 \tag{3.4}$$

Caputo 1967, has proposed that the integer order (classical) derivative of function x, as are commonly used in initial value problem with integer order equations be incorporated.

Definition 3.3 (Das 1999)

The Caputo fractional derivative of order $\alpha < 0$ of a function $f : (0,\infty) \to R$ can be written as

$$ {}_{0}^{C}D_t^{\alpha}[f(t)] = \frac{d^{\alpha}f}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{n-\alpha-1}}ds \tag{3.5}$$

$$n-1 < \alpha < n,\qquad f^{(n)}(s) = \frac{d^n f}{ds^n}$$

Some properties of Riemann-Liouville and Caputo derivatives are recalled here, (Zhang 2008; Momani and Hadid 2004).

Property 3.1

When $0 < \alpha < 1$, we have

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}D_t^{\alpha}x(t) - \frac{x(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}$$

In particular, if ${}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}D_t^{\alpha}x(t)$

Property 3.2

For $v > 1,$ we have

$$ {}_{t_0}D_t^{\alpha}(t-t_0)^{v} = \frac{\Gamma(1+v)}{(1+v-\alpha)}(t-t_0)^{v-\alpha}$$

In particular, if $0 < \alpha < 1$ and $x(t) = (t - t_0)^v$ then from property 3.1, we have

$$ {}_{t_0}^{C}D_t^{\alpha}(t - t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-\alpha)}(t - t_0)^{v-\alpha} $$

Property 3.3
From the definition of Caputo derivation eq. (3.5) when $0 < \alpha \le 1$ we have

$$ I_{t_0}^{\alpha} \, {}_{t_0}^{C}D_t^{\alpha} x(t) = x(t) - x(t_0) $$

where

$$ \left(I_{t_0}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > t_0 $$

Property 3.4
Fractional order differentiation is a linear operator:

$$ D^{\alpha}\left(\lambda f(t) + \mu g(t)\right) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t) $$

Also, the chain rule has the form

$$ \frac{d^{\beta} f(g(t))}{dt^{\beta}} = \sum_{k=0}^{\infty} \binom{\beta}{k}_{\Gamma} \left(\frac{d^{\beta-k}}{dt^{\beta-k}} I \right) \frac{d^k}{dt^k} f(g(t)) $$

Where $k \in N$ and $\binom{\beta}{k}_{\Gamma}$ are the coefficients of the generalized binomial

$$ \binom{\beta}{k}_{\Gamma} = \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} $$

There are also two functions that play an important role in the study of stability of FDE's

Definition 3.4
The Mittag-Leffler function is defined by

$$ E_{\alpha}(z) = \sum_{k-0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} $$

where $\mathrm{Re}(z) > 0,\ z \in C.$ the two parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} $$

where $\mathrm{Re}(z) > 0,$ and $\beta \in C, z \in C$
Definition 3.5

The $\alpha -$ exponential function is defined by

$$ e_{\alpha}^{\lambda z} = z^{\alpha-1} E_{\alpha,\alpha}\left(\lambda z^{\alpha}\right) $$

where $z \in C \setminus 0,\ \mathrm{Re}(\alpha) > 0$ and $\lambda \in C.\ E_{\alpha,\alpha}(.)$ is the two parameter Mittag-Leffler function. Mittage-Leffler function is frequently used in the solution of fractional order system and is a generalization of the exponential function.

4. Main result
Here, a class of linear dynamical system with time varying delays in the state of the form

$$D_t^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}} = A_0 x(t) + \sum_{i=1}^{k} A_i x\left(t - \tau_i(t)\right) + Bu(t) + f(t, x) \tag{4.1}$$

with initial condition $x(t) = \psi_x(t)$, where the time varying delays satisfy eq. (2.2) and D_t^{α} denotes Caputo fractional derivative of order α, $0 < \alpha < 1$ is considered.

Lemma 4.1 (Bellman-Gronwall inequality (Hale 1971; Ye et at. 2007)
Suppose $x(t)$ and $a(t)$ are nonnegative and locally integrable on $0 \le t < t$, $T \le \infty$, and $g(t)$ is nonnegative continuous function defined on $0 \le t < t$, $g(t) \le M$, M a constant, $\alpha > 0$ with

$$x(t) = a(t) + g(t)\int_0^t (t-s)^{\alpha-1} x(s) ds$$

On this interval, then:

$$x(t) \le a(t) + \int_0^t \left(\frac{(g(t)\Gamma(\alpha))}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right) ds, \quad 0 \le t < T$$

Theorem 4.1 The dynamical system eq. (4.1) satisfying the initial condition $x(t) = \psi_x(t)$, $-\tau_M \le t \le 0$ is finite time stable w.r.t. $\{\delta, \varepsilon, \alpha_u, J\}$, $\delta < \varepsilon$ if the following condition is satisfied

$$\left(1 + \frac{L + \sigma(k+1)(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right) e^{\frac{\sigma(k+1)(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}} + \frac{\gamma^*}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \le \varepsilon / \delta$$

where $\gamma^{\bullet} = bq_u + m/\delta$, $\Gamma(.)$ Euler's gamma function.

Proof
In accordance with the property of fractional order $0 < \alpha < 1$, the solution of eq. () can be obtained in the form of an equivalent Volterra integral equation

$$x(t) = x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left(A_0 x(s) + \sum_{i=1}^{k} A_i x\left(s - \tau_i(s)\right) + Bu(s) + f(s, x) \right) ds \tag{4.2}$$

To obtain an estimate of the solution we apply the norm $\|(.)\|$ to eq. (4.2) and using appropriate property of the norm, the following applies:

$$\|x(t)\| \le \|x(t_0)\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left\| A_0 x(s) + \sum_{i=1}^{k} A_i x\left(s - \tau_i(s)\right) + Bu(s) + f(s, x) \right\| ds \tag{4.3}$$

Also, applying the norm $\|(.)\|$ to eq. (4.1), it holds:

$$\left\| \frac{d^{\alpha} x(t)}{dt^{\alpha}} \right\| \le \|A_0\| \|x(t)\| + \sum_{i=1}^{k} \|A_i\| \left\| x\left(t - \tau_i(t)\right) \right\| + \|B\| \|u(t)\| + \|f(t, x)\|$$

$$\le \sigma \|x(t)\| + \sum_{i=1}^{k} \sigma \left\| x\left(t - \tau_i(t)\right) \right\| + \|B\| \|u(t)\| + \|f(t, x)\| \tag{4.4}$$

where $\|A\|$ denotes the induced norm of matrix A, considering

$$\left\| x\left(t - \tau_i(t)\right) \right\| \le \sup_{t^* \in [t - \tau_M, t]} \left\{ \left\| x\left(t^*\right) \right\|, \forall i \in \{1, 2, .., k\} \right\} \tag{4.5}$$

Applying the inequality eq. (4.5), eq. (4.4) can be written as

$$\left\| \frac{d^{\alpha} x(t)}{dt^{\alpha}} \right\| \le \sigma(k+1) \left(\sup_{t^* \in [t - \tau_M, t]} \left\| x\left(t^*\right) \right\| + \|\psi_x\|_C \right) + \|b\| \|u(t)\| + m + L \|x(t)\|$$

$$\le \sigma(k+1)\left(\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|+\left\|\psi_x\right\|_C\right)+bq_u+m+L\left\|x(t)\right\| \tag{4.6}$$

where $\left\|u(t)\right\|\le q_u$

So,

$$\left\|\frac{d^\alpha x(t)}{dt^\alpha}\right\|\le \sigma(k+1)\left(\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|+\left\|\psi_x\right\|_C\right)+bq_u+m+L\left\|x(t)\right\| \tag{4.7}$$

Combining eq. (4.7) with eq. (4.3) yields

$$\begin{aligned}
&\left\|x(t)\right\| \\
&\le \left\|x(t_0)\right\|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)\right|^{\alpha-1}\left\{\sigma(k+1)\left(\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|+\left\|\psi_x\right\|_C\right)+bq_u+m+L\left\|x(t)\right\|\right\}ds \\
&\le \left\|\psi_x\right\|_C+\frac{L+\sigma(k+1)}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|ds \\
&\qquad +\frac{1}{\Gamma(\alpha)}\left\{\sigma(k+1)\left\|\psi_x\right\|_C+bq_u+m\right\}\times\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|ds
\end{aligned} \tag{4.8}$$

On expanding eq. (4.8) and integrating yields

$$\begin{aligned}
\left\|x(t)\right\|\le \left\|\psi_x\right\|_C\left(1+\frac{L+\sigma(k+1)(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right)+\frac{bq_u+m}{\Gamma(\alpha+1)}(t-t_0)^\alpha \\
+\frac{\sigma(k+1)}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|ds
\end{aligned} \tag{4.9}$$

Let

$$a(t)=\left\|\psi_x\right\|_C\left(1+\frac{L+\sigma(k+1)(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right)+\frac{bq_u+m}{\Gamma(\alpha+1)}(t-t_0)^\alpha,\quad g(t)=\frac{\sigma(k+1)}{\Gamma(\alpha)} \tag{4.10}$$

By eq. (4.9), we have

$$\left\|x(t)\right\|\le a(t)+g(t)\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|ds \tag{4.11}$$

Obviously, the right-hand side of eq. (4.11) is a nondecreasing continuous function defined on [0, T], hence we have

$$\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|\le a(t)+g(t)\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|ds \tag{4.12}$$

Applying the generalized Bellman-Gronwall inequality, lemma 4.1, leads to

$$\left\|x(t)\right\|\le\sup_{t^*\in[t-\tau_M,t]}\left\|x(t^*)\right\|\le a(t)e^{g(t)\int_{t_0}^{t}\left|(t-t_0)^{\alpha-1}\right|ds}=a(t)e^{\frac{\sigma(k+1)(t-t_0)^\alpha}{\Gamma(\alpha+1)}}$$

And the relation

$$\left\|x(t)\right\|\le\delta\left(1+\frac{L+\sigma(k+1)(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right)e^{\frac{\sigma(k+1)(t-t_0)^\alpha}{\Gamma(\alpha+1)}}+\frac{bq_u+m}{\Gamma(\alpha+1)}(t-t_0)^\alpha$$

Hence by the basic conditions of theorem (4.1), eq. (4.1) yields

$$\left\|x(t)\right\|<\varepsilon\quad\forall t\in J$$

This completes the proof.

Reference

Amato, F., Ariola, M., Cosentino, C., Abdallah, C. T., & Dorato, P. (2003). Necessary and sufficient conditions for finite-time stability of linear systems. *Proc. American Control Conference, Denver, Colorado, USA,* 4452–4456.

Boukas, E. K. and Al-Muthairi, N. F. (2006). Delay-dependent Stabilization of Singular Linear Systems with Delays. *International Journal of Innovative Computing, Information and Control*, *2*(2), 283–291.

Caputo, M., (1967), Linear models of dissipation whose Q is almost frequency independent, II. *Geophys. J. R. Astron. Soc. 13, 529-539.*

Chen, Y. Q. and Moore, K. L., (2002), Analytical Stability bound for a class of delayed fractional order dynamic system, *Nonlin. Dyn. Vol. 29, 191-200*

Das, S., (1999), *Functional fractional calculus,* academic press, San Diego.

Debeljkovi, D. Lj, Lazarevic, M. P., Koruga, Dj., Milinkovic, S. A., Jovanovic, M. B. and Jacic, L. A. (2001). Further Results on Non-Lyapunov Stability of the Linear Nonautonomous Systems with Delayed State. *Facta Universitatis, Mechanics, Automatic Control and Robotic Series*, *3*(11), 231–241.

Debeljkovi, D Lj, Stojannovic, S. B., Simeunovic, G. V. and Dimitrijevic, N. J. (2014). Further results on stability of singular time delay systems in the sense of non-Lyapunov: A new delay dependent conditions. *Automatic Control and Information Sciences*, *2*(1), 13–19.

Debeljkovic Dr., D. L., Stojanovic, S. B., & Jovanovic, A. M. (2013). Further results on finite timeand practical stability of linear continuous time delay systems. *FME Transactions*, *41*(3), 241–249.

Deng, W., (2010), Smoothness and stability of the solution for nonlinear fractional differential equation, *Nonlinear Anal. 72(3-4), 1768-1777.*

Deng, W., & Li, C. (2012). Numerical schemes for fractional ordinary differential equations. *Numerical Modelling*. Retrieved from http://cdn.intechopen.com/pdfs-wm/33091.pdf

Deng, W., Li, C., & Lü, J. (2007). Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dynamics*, *48*(4), 409–416. http://doi.org/10.1007/s11071-006-9094-0

Denghao, P. and Wei, J., (2014), Finite time stability analysis of fractional singular time delay systems, *Advances in difference equation,2014/1/259.*

Desoer, C. A., and Vidyasagar, M. (1975). *Feedback Systems: Input-Output properties*. Academic Press, New York.

Dugard, L. and Verriest, E. I. (1997), *Stability and control of time delay systems.* Springer-Verlag.

Ferreira, N. M. F., Duarte, F. B., Miguel, F. M., Marcos, M. G., & Machado, J. A. T. (2008). Application of fractional calculus in the Dynamical analysis and control of mechanical manupulators. *Fractional calculus and applied analysis,11(1),91-113.*

Hale, J. K., (1971), *Functional differential equations,* Springer, New York

Gu, K., Kharitonov, V. L. and Chen, J., (2003), S*tability of time delay systems*, Boston, Birkhauser.

Hilfer, R., (2000), *Application of fractional calculus in Physics*, World Scientific, Singapore.

Hong, Y., Xu, Y. and Huang, J. (2002). Finite time control for robot manipulation. *Systems and Control Letters*, *46*, 243–253.

Hong, Y. (2002). Finite time Stabilization and Stabilizability of a class of controllable systems. *Systems and Control Letters*, *46*, 231–236.

Hu, G. Da and Liu, M. Z. (2004). The weighted logarithm matrix norm and bounds of the matrix exponential. *Linear Algebra Appl.*, *390*, 145–154.

Hu, G. Da, & Mitsui, T. (2012). Bounds of the matrix eigenvalues and its exponential by lyapunov equation. *Kybernetika*, *48*(5), 865–878.

Jackreece P. C., (2016). Finite time stability of linear control system with multiple delays, *Control Theory and Informatics, Vol. 6(3), 69-72.*

Jackreece, P. C., (2017). Finite time stability of linear control system with constant delay in the state, *International Journal of Mathematics and Statistics Invention(IJMI), Vol.5(2), 49-52*.

Jackreece P. C. and Aniaku, S. *E.*, Finite time stability of linear fractional order dynamical system with varying delays, *international Journal of Applied Mathematical Research, 7(2), 2018,27-32.*

Jia, H., Cao, X., Yu, X. and Zhang, P., (2007), A simple approach to determine power system delay margin, in Proceedings of the IEEE PES general meeting, Montreal, Quebec, 1-7

Kablar, N. A. and Debeljkovic, D. L. (1998). Non-Lyapunov stability of linear singular systems: Matrix measure approach. *Preprints 5th IFAC Symposium on Low Cost Automation, Shenyang, China, September 8-10, TS13,* 16–20.

Kaslik, E. and Sivasundaram, S., (20120. An analytical and numerical methods for the stability analysis of linear fractional delaydifferential equations, Journal of computational and Applied Mathematics, 236(16), 4027-4041.

Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., (2006), Theory and applications of fractional differential equations, Elsevier, New York.

Kulish, V. V. and Lage, J. L., (2002), Application of fractional calculus to Mechanics, *Journal of Fluids Engineering, Vol. 124, No.3, 803-806.*

Lazarevic, M. P., (2006), Finite time stability analysis of PD^a fractional control of robotic time delay systems, *Mech. Res. Commun.33(2), 269-279.*

Lazarevic, M. P., and Debeljkovi, D. L. (2005). Finite Time Stability Analysis of Linear Autonomous Fractional Order Systems With Delayed State. *Asian Journal of Control*, *7*(4), 440–447.

Lazarevic, M. P., Obradovic, A. and Vasic, V., (2010). Robust finite time stability analysis of fractional order time delay systems: New results, WSEAS2010 Control10, May Tunisia, 101-106.

Lee, T. N., and Diant, S., (1981), Stability of time delay system, *IEEE trans. Automat. Control AC 31(3), 951-953.*

Li, C., Chen, K., Lu, J. and Tang, R., (2017). Stability and Stabilization analysis of fractional order linear systems subject to actuator saturation and distribution, IFAC paper online 50-1, 9718-9723.

Li, Y., Chen, Y., and Podlubny, I., (2010), stability of fractional order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl. 59(5), 1810-1821.*

Liu, P. L. (2003). Exponential Stability for Linear Time Delay Systems with Delay Dependence. *Journal of the Franklin Institute*, *340*, 481–488.

Liu, K. and Jiang, W., (2014). Stability of fractional neutral system, Adv. Differ. Equ. 2014, 78(1014)

Matignon, D. (1996). Stability results for fractional differential equations with applications to control processing. *Computational Engineering in Systems Applications*, 963–968.

Mohd, F. A., Manoj, S. and Renu, J., (2016). An application of fractional calculus in electrical engineering, *Advanced Engineering Technology and Application, 5(2), 41-45.*

Momani, S. and Hadid, S. (2004), Lyapunov stability solution of fractional integrodifferential equations, *Int. J. math.Math. Sc. 47, 2503-2507.*

Mori, T., (1985), Criteria for Asymptotic stability of linear time delaysystems, *IEEE trans. Automat. Control, AC 30, 158-161.*

Moulay, E., Dambrine, M., Yeganefar, N. and Perruquetti, W. (2008). Finite Time Stability and Stabilization of time delay systems. *Systems and Control Letters*, *57*, 561–566.

Naber, M., (2004), Time fractional Schrodinger equation, *J. Maths. Phys. 45, 3339-3352.*

Olgac , N. and Sipahi, R. An exact method for the stability analysis of time delayed LTI systems, IEEE trans. Autom control, 47(5), 793-797. 2002.

Oldham, K. B., and Spanier, J., (1974), *The fractional calculus*, Academic Press, New York.

Podlubny, I., (1999), *Fractional differential equation,* Academic Press, San Diego.

Ryabov, Y. and Puzenko, A., (2002), Damped oscillation in view of the fractional oscillator equation, *Phys. Rev., 66, 184-201.*

Sabatier, J., Moze, M., and Farges, C., (2010), Stability conditions for fractional order systems, *Comput. Math. Appl.59(5), 1594-1609.*

Samko, S. G., Kilbas, A. A. and Marichev O. I., (1993), *Fractional integrals and derivatives theory and applications*, Gordon and Breach, New York.

Shen, Y., Zhu, L. and Guo, Q. (2007). Finite Time boundedness analysis of uncertain neural networks with time delay: An LMI approach. *Proc. 4th Int Symp. Neural Networks, Nanjing, China.*, 904–909.

Soczkiewicz, E. (2002), Application of fractional calculus in the theory of viscoelasticity, *Molecular and Quantum Acoustics, Vol. 23, 397-404.*

Williams, J.and Otto, E., A generalized chemical processing model for the investigation of computer model AIEE trans (Commun. Electron) Vol. 79, 1960, 458-473.

Ye, H., Gao, J. and Ding, Y., (2007), A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl., 328(2007), 1075-1081.*

Yu, F., (2011), Integrable coupling system of fractional differential equations, *Eur. Phys. J. Spec. Top., 193, 27-47*.

Zavarei, M. and Jamshidi, M., (1987), *Time Delay systems: Analysis, Optimization and Applications*, North-Holland, Amsterdam.

Zahra W. K., Hikal, M. M., Taher, A. Bahnasy, (2017). Solution for fractional order electrical circuits in Laplace transform and on standard finite difference method, *Journal of the Egyptian Mathematical Society, 1-10*

Zhang, X., (2008), Some results of linear fractional order time delay system, *Appl. Math. Comp., 197, 407-411*.

Zhou, S. and Lam, J. (2003). Robust Stabilization of Delay Singular Systems with Linear Fractional Parametric Uncertainties. *Circuits Systems Signal Processing*, *22*(6), 578–588.