ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.8, No.5, 2018

Semicircular Logistic Distribution induced by Simple Projection Method

Sreekanth Y.[1], S.V.S.Girija [2], PhaniY.[3] and A.V. Dattatreya Rao[4]

1. Research Scholar, Dravidian University, Kuppam. Lecturer in Mathematics, Hindu College, Guntur.
2. Associate Professor, Department of Mathematics, Hindu College, Guntur, INDIA.
3. Associate Professor, Department of Basic Science, Shri Vishnu Engineering College for Women, Vishnupur, Bhimavaram, INDIA.
4. Retd. Professor, Department of Statistics, Acharya Nagarjuna University, Guntur, INDIA.

Abstract

For modeling Semicircular data, here we introduce Semicircular Logistic distribution by applying a Simple projection method on Logistic model. We extend it to the l-axial Logistic distribution by a Simple projection for modeling angular data on any arc of arbitrary length. Also derive the characteristic function and the first four trigonometric moments for the evaluation of population characteristics of proposed model. A bivariate version of *l*-axial Logistic distribution is also developed.

Keywords: Circular model, Characteristic function, l-axial data, Projection, Trigonometric moments.

1 Introduction

Phani (2013) proposed new method of constructing circular and semicircular models by applying modified inverse stereographic projection on linear models. By inducing inverse stereographic projection on Weibull and Exponential distributions, Phani et al (2013) constructed Stereographic Semicircular versions of Weibull and Exponential models respectively, whose range spans in $(0,p)$ as the corresponding linear models are in $(0,¥)$ and Girija et al (2014) have constructed Offset Arc Beta model and Offset Arc Exponential type models and this aspect of existence of semicircular/arc models can be viewed as "**natural occurrence**".

The fact that in some practical situations full circular models are not required and it is noted in Guardiola (2004), Jones (1968) and Byoung et al(2008).

ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.8, No.5, 2018

Taking this as cue, on the lines of Byoung et al (2008) the **Semicircular Logistic distribution** (**SCLD**) is constructed by projecting Logistic distribution over a semicircular segment. This paper is devoted in constructing Semicircular Logistic distribution, to derive results on the relationship between modality and parameter. We plot the graphs of the density function and distribution function for various values of parameters. We consider the asymptotic behaviour of the Semicircular Logistic distribution, derive the characteristic function and first four trigonometric moments for proposed model. We extend this model for l-axial also. We concentrate on developing Bivariate Semicircular Logistic model.

2 Semicircular Logistic distribution

A random variable X on the real line is said to have Logistic Distribution with location parameter γ and scale parameter $\lambda > 0$, if the probability density function and cumulative distribution function of X for $x, \gamma \in \mathbb{R}$ and $\lambda > 0$ are given by

$$
\begin{aligned}
f(x) &= \frac{1}{\lambda}\left[1+\exp\left(\frac{-(x-\gamma)}{\lambda}\right)\right]^{-2} \exp\left(\frac{-(x-\gamma)}{\lambda}\right) \\
&= \frac{1}{4\lambda}\sec h^{2}\left(\frac{x-\gamma}{2\lambda}\right) \qquad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= \left[1+\exp\left(\frac{-(x-\gamma)}{\lambda}\right)\right]^{-1} \\
&= \frac{1}{2}\left[1+\tanh\left(\frac{1}{2}\left(\frac{x-\gamma}{\lambda}\right)\right)\right]
\end{aligned}
$$

respectively.

Then by applying Simple projection defined by a one to one mapping

$x = v\tan(\theta)$ or $\theta = \tan^{-1}\left(\frac{x}{v}\right)$ $v > 0 \in \mathbb{R}$, which leads to a Semicircular Logistic distribution.

Definition:

A random variable X_{SC} on the Semicircle is said to have the Semicircular Logistic distribution with location parameter μ scale parameter $\sigma > 0$ denoted by **SCLD**(σ, μ), if the probability density and the cumulative distribution functions are respectively given by

$$g(\theta) = \frac{1}{\sigma}\sec^2(\theta)\left[1+\exp\left(\frac{-(\tan(\theta)-\mu)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan(\theta)-\mu}{\sigma}\right)\right),$$

$$\text{Where } \sigma = \frac{\lambda}{v} > 0, \mu = \frac{\gamma}{v} \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad (2.2)$$

or $$g(\theta) = \frac{1}{4\sigma}\sec^2(\theta)\sec h^2\left[\frac{\tan(\theta)}{2\sigma}\right]$$

$$G(\theta) = \left[1+\exp\left(\frac{-(\tan(\theta)-\mu)}{\sigma}\right)\right]^{-1}, \text{where } \sigma > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Hence the proposed new model **SCLD**(σ, μ) is a Semicircular model.

We consider the asymptotic behavior of the Semicircular Logistic distribution when $\sigma \to 0$.
Suppose θ follows **SCLD** (σ, μ) **.** Let $y = \frac{\theta}{\sigma}$ and then use the change of variable technique.
For sufficiently small σ , we have $\tan(\sigma y) ; \sigma y$ and $\sec(\sigma y) ; 1$ by first order approximation of the Taylor series expansion. Hence, the distribution of θ becomes Logistic distribution (linear). So, for sufficiently small σ , the Semicircular Logistic distribution can be approximated by a linear Logistic distribution.

Theorem 5.2.1: Semicircular logistic distribution is **Symmetric** about $\mu = 0$ and is **unimodal** if $\sigma < 0.5$ and **bimodal** if $\sigma > 0.5$

Proof: The probability density function of Semicircular logistic distribution is

$$g(\theta) = \frac{1}{\sigma}\sec^2(\theta)\left[1+\exp\left(\frac{-(\tan(\theta))}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan(\theta)}{\sigma}\right)\right), \text{where } \sigma > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

equivalently $$g(\theta)=\frac{1}{4\sigma}\sec^2(\theta)\sec h^2\left[\frac{\tan(\theta)}{2\sigma}\right]$$

Differentiating $g(\theta)$ with respect to θ , we get

$$g'(\theta)=\frac{1}{4\sigma}\left[\sec^2(\theta)\tan(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)-\frac{1}{\sigma}\sec^4(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)\tanh\left(\frac{\tan(\theta)}{2\sigma}\right)\right]$$

$$=\frac{1}{4\sigma}\sec^2(\theta)\sec h^2\left(\frac{v\tan(\theta)}{2\sigma}\right)\left[\tan(\theta)-\frac{1}{2\sigma}\sec^2(\theta)\tanh\left(\frac{\tan(\theta)}{2\sigma}\right)\right]$$

For Stationary points $g'(\theta)=0$

$$\Rightarrow \tan(\theta)-\frac{1}{2\sigma}\sec^2(\theta)\tanh\left(\frac{v\tan(\theta)}{2\sigma}\right)=0$$

$$\Rightarrow \tanh\left(\frac{\tan(\theta)}{2\sigma}\right)-\sigma\sin\theta=0$$

$\Rightarrow$ $\theta=0,\ 2\pi,\ 4\pi,...,$ are the stationary points.

$\theta=0$ is the only stationary point which lies in the domain of $g(\theta)$

Differentiating $g'(\theta)$ with respect to θ, we get

$$g''(\theta)=\frac{1}{4\sigma}\left[\begin{array}{l}\sec^2(\theta)\tan^2(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)+\frac{1}{2}\sec^4(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)\\ -\frac{1}{2\sigma}\sec^4(\theta)\tan(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)\tanh\left(\frac{\tan(\theta)}{2\sigma}\right)\end{array}\right]$$

$$-\frac{1}{8\sigma^2}\left[\begin{array}{l}2\sec^2(\theta)\tan(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)\tanh\left(\frac{\tan(\theta)}{2\sigma}\right)\\ -\frac{1}{2\sigma}\sec^6(\theta)\sec h^2\left(\frac{\tan(\theta)}{2\sigma}\right)\tanh^2\left(\frac{\tan(\theta)}{2\sigma}\right)\\ +\frac{1}{4\sigma}\sec^2(\theta)\sec h^4\left(\frac{\tan(\theta)}{2\sigma}\right)\end{array}\right]$$

At $\theta = 0$

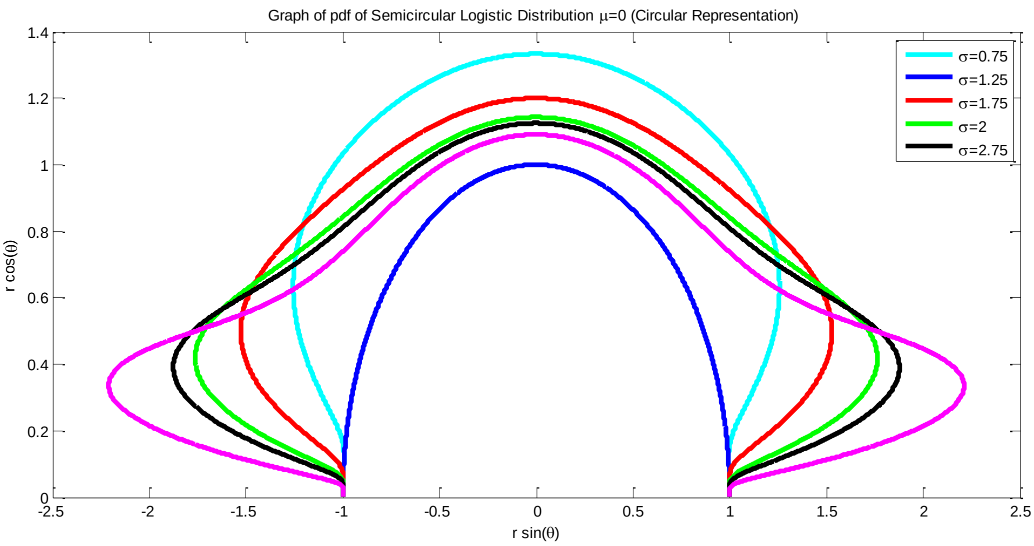
$$g''(0) = \frac{1}{16\sigma} - \frac{1}{32\sigma^3} \qquad = \frac{2\sigma^2 - 1}{32\sigma^3} = \left[\frac{2\sigma^2 - 1}{32\sigma^3}\right]$$

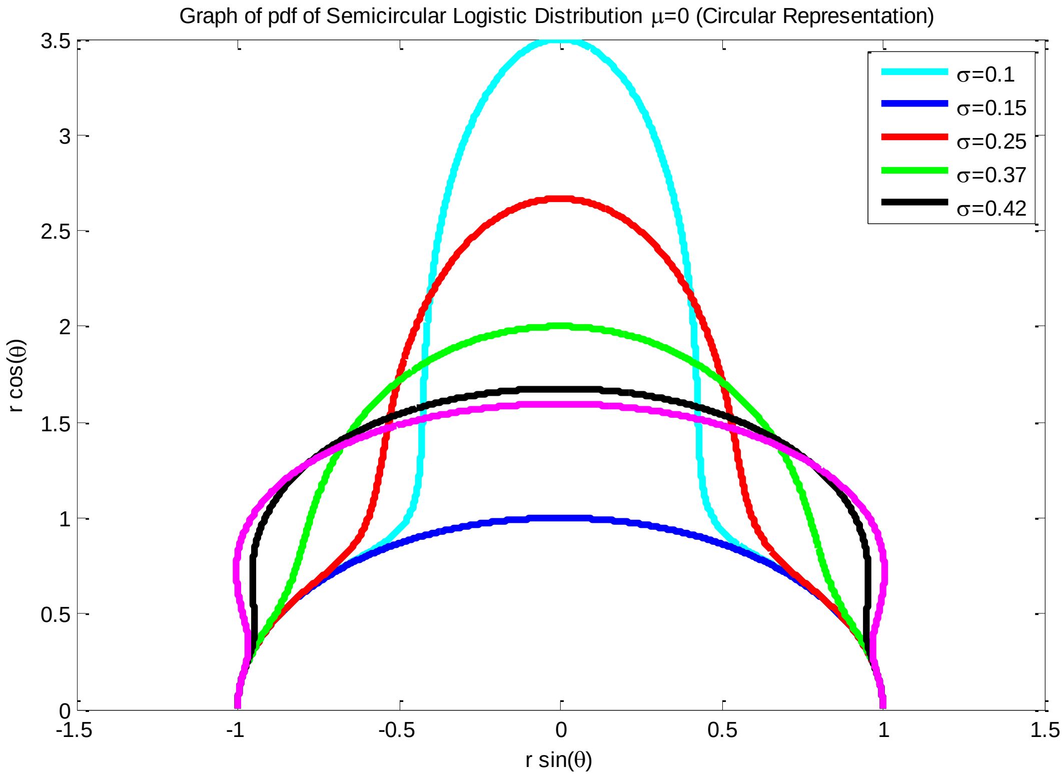
$g(\theta)$ has maximum value at $\theta = 0$ if and only if $g''(0) < 0 \Leftrightarrow \sigma < 0.5$

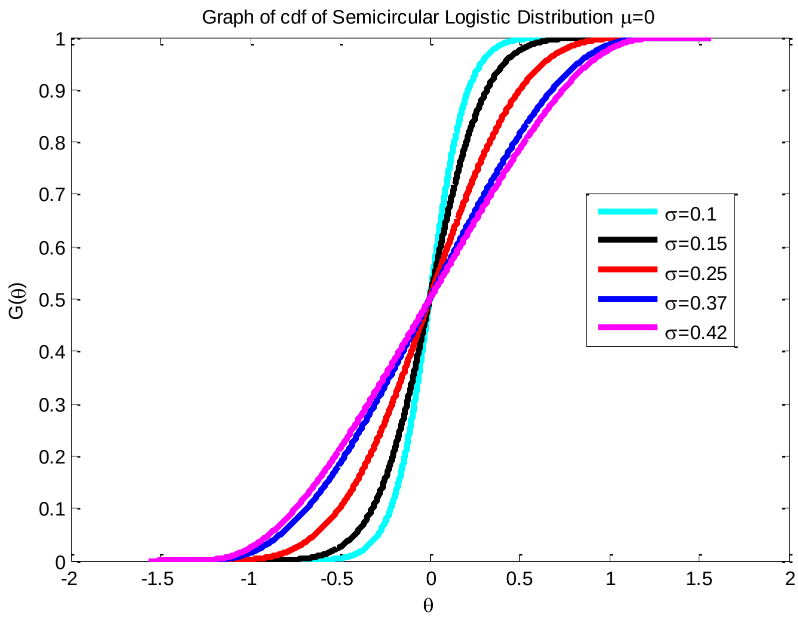
$g(\theta)$ has minimum value at $\theta = 0$ if and only if $g''(0) > 0 \Leftrightarrow \sigma > 0.5$

Hence Stereographic logistic distribution is unimodal if $\sigma < 0.5$ and bimodal if $\sigma > 0.5$

Graphs of the probability density and cumulative distribution function of the Semicircular Logistic distribution for various values of σ and μ




Graph of pdf of Semicircular Logistic Distribution μ=0 (Circular Representation)
σ=0.1
σ=0.15
σ=0.25
σ=0.37
σ=0.42
r cos(θ)
r sin(θ)
3.5
3
2.5
2
1.5
1
0.5
0
-1.5
-1
-0.5
0
0.5
1
1.5


Graph of cdf of Semicircular Logistic Distribution μ=0
σ=0.1
σ=0.15
σ=0.25
σ=0.37
σ=0.42
G(θ)
θ
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

3. The Characteristic function of Semicircular Logistic distribution:

$$\Phi_{X_{SC}}(p)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ip\theta} g(\theta)\,d\theta$$

$$=\frac{1}{\sigma}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ip\theta}\sec^2(\theta)\left[1-\exp\left(-\frac{1}{\sigma}\tan(\theta)\right)\right]^{-2}\exp\left(-\frac{1}{\sigma}\tan(\theta)\right)d\theta$$

$$=\frac{2}{\sigma}\int_{0}^{\frac{\pi}{2}}\cos p\theta\sec^2(\theta)\left[1-\exp\left(-\frac{1}{\sigma}\tan(\theta)\right)\right]^{-2}\exp\left(-\frac{1}{\sigma}\tan(\theta)\right)d\theta,\quad \text{Since}$$

sinpθ is an odd function

$$\Phi_{X_{SC}}(p)=\frac{2}{\sigma}\int_{0}^{\infty}\cos p\left(\tan^{-1}(x)\right)\left[1+\exp\left(-\frac{x}{\sigma}\right)\right]^{-2}\exp\left(-\frac{x}{\sigma}\right)dx,\ \text{putting}\ x=\tan(\theta)$$

As the integral cannot be obtained analytically, MATLAB techniques are applied for the evaluation of the values of the characteristic function.

Trigonometric moments of the Semicircular Logistic distribution

The trigonometric moments of the distribution are given by $\{\varphi_p : p=\pm 1,\pm 2,\pm 3,...\}$, where $\varphi_p=\alpha_p+i\beta_p$, with $\alpha_p=E(\cos p\theta)$ and $\beta_p=E(\sin p\theta)$ being the p^{th} order Cosine and sine moments of the random angle θ, respectively. Because the Semicircular logistic distribution is symmetric about $\mu=0$, it follows that the sine moments are zero. Thus $\varphi_p=\alpha_p$.

Theorem 3.1 Under the pdf of Semicircular logistic distribution is symmetric with $\mu=0$, the first four $\alpha_p=E(\cos p\theta)$, $p=1,2,3,4$ are given as follows

$$\alpha_1=\frac{\pi}{\sigma}\sum_{n=1}^{\infty}(-1)^{n-1}\left[H_0\left(\frac{n}{\sigma}\right)-Y_0\left(\frac{n}{\sigma}\right)\right],$$

$$\alpha_2 = \frac{\sqrt{2\pi}}{\sigma^{\frac{3}{2}}} \sum_{n=1}^{\infty} (-1)^{n-1} \sqrt{n} \left[H_{-1/2}\left(\frac{n}{\sigma}\right) - Y_{-1/2}\left(\frac{n}{\sigma}\right) \right] - 1,$$

$$\alpha_3 = -\frac{\pi}{\sigma^2} \sum_{i=1}^{\infty} (-1)^{n-1} n \left\{ H_{-1}\left(\frac{n}{\sigma}\right) - Y_{-1}\left(\frac{n}{\sigma}\right) \right\} - \frac{6}{\pi\sigma} \sum_{i=1}^{\infty} (-1)^{n-1} G_{13}^{31} \left(\frac{n^2}{4\sigma^2} \middle| \begin{matrix} -\frac{1}{2} \\ 0,\ 0, \frac{1}{2} \end{matrix} \right)$$

$$\alpha_4 = 16 \left[1 - \frac{1}{2\sqrt{\pi}\sigma} \sum_{n=1}^{\infty} (-1)^{n-1} G_{13}^{31} \left(\frac{n^2}{4\sigma^2} \middle| \begin{matrix} -\frac{1}{2} \\ \frac{1}{2},\ 0\ \frac{1}{2} \end{matrix} \right) \right]$$

Where $\int_0^{\infty} \left(u^2 + x^2\right)^{\nu-1} e^{-\mu x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[H_{\nu-\frac{1}{2}}(u\nu) - Y_{\nu-\frac{1}{2}}(u\nu) \right]$

for $|\arg u\pi| < \frac{\pi}{2}, \operatorname{Re}\mu > 0$ and $\operatorname{Re}\nu > 0$ and $G_{13}^{31} \left(\frac{\mu^2 u^2}{4} \middle| \begin{matrix} 1-\nu \\ 1-Q-\nu, 0, \frac{1}{2} \end{matrix} \right)$ is called as Meijer's **G-**function (Gradshteyn and Ryzhik, 2007, formula no. 3.389.2).

4. Extension to l-axial distribution

We extend the proposed model to the l -axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l = 1, 2, \ldots,$ so it is desirable to extend the Semicircular Logistic distribution. To construct the l -axial Logistic distribution, we consider the density function of Semicircular Logistic distribution and use the transformation $\phi = 2\theta/l$, $l = 1, 2, \ldots,$. The probability density function of ϕ is given by

$$g(\phi)=\frac{l}{2\sigma}\sec^2\left(\frac{l\phi}{2}\right)\left[1+e^{-\frac{\tan\left(\frac{l\phi}{2}\right)}{\sigma}}\right]^{-2}e^{-\frac{\tan\left(\frac{l\phi}{2}\right)}{\sigma}}$$

$$\text{where} \quad -\frac{\pi}{l}<\phi<\frac{\pi}{l} \qquad (4.1)$$

Case (1) When $l=2$, the probability density function (5.1) is the same as the probability density

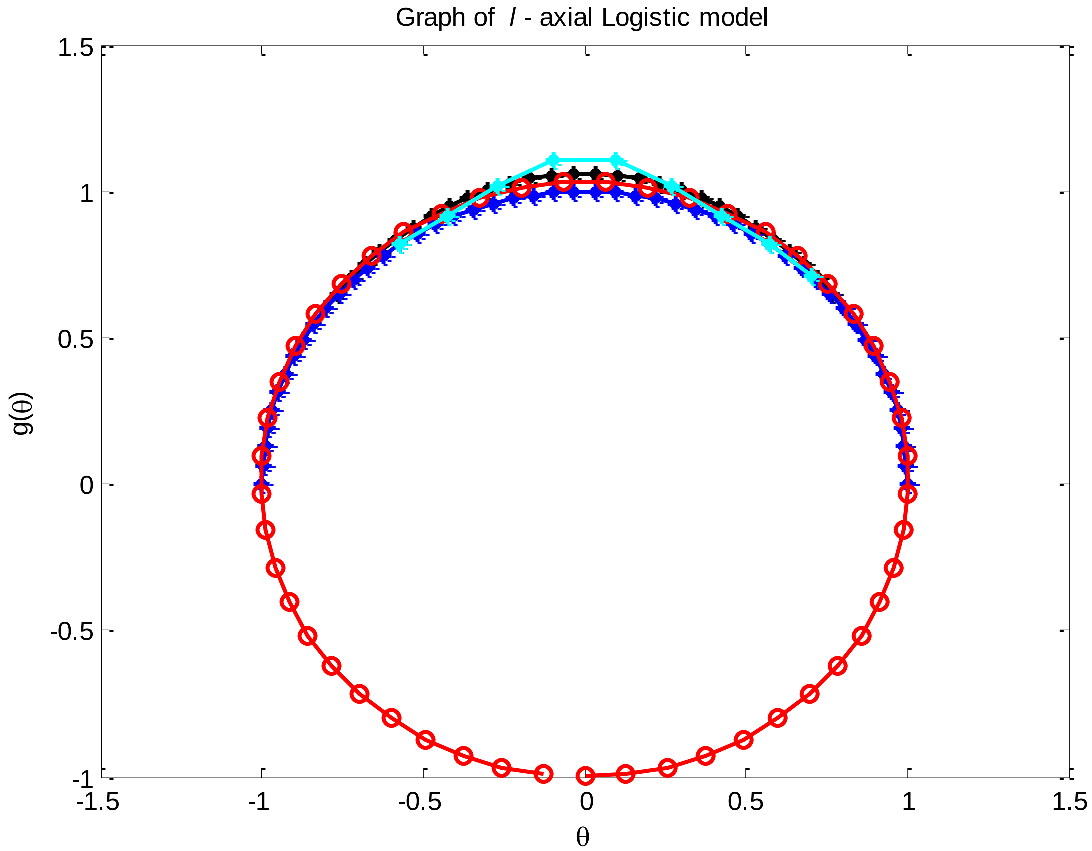
function of **Semicircular Logistic distribution**.

Case (2) When $l=1$, the probability density function (5.1) is the same as that of **Stereographic**

Logistic Distribution[Dattatreyarao et al (2016)] which is circular distribution.

$$g(\phi)=\frac{1}{2\sigma}\sec^2\left(\frac{\phi}{2}\right)\left[1+e^{-\frac{\tan\left(\frac{\phi}{2}\right)}{\sigma}}\right]^{-2}e^{-\frac{\tan\left(\frac{\phi}{2}\right)}{\sigma}}$$

$$\text{where} \quad -\pi<\phi<\pi \qquad (4.2)$$



5. Bivariate Semicircular Logistic Distribution

We can construct a bivariate Semicircular Logistic distribution in a manner similar to the construction of a univariate Semicircular Logistic distribution. We shall use the same semicircular transformation applied in a bivariate case. The probability density function of the bivariate Semicircular Logistic distribution is defined as

$$g(\theta_1,\theta_2)=\frac{2\sec^2(\theta_1)\sec^2(\theta_2)}{\sigma}e^{-\frac{1}{\sigma}(\tan(\theta_1)+\tan(\theta_2))}\left[1+e^{-\frac{1}{\sigma}(\tan(\theta_1))}+e^{-\frac{1}{\sigma}(\tan(\theta_2))}\right]^{-2}$$

$$-\frac{\pi}{2}<\theta_i<\frac{\pi}{2}, i=1,2 \qquad (5.1)$$

To construct this density, we begin with the bivariate Logistic density

$$f(x_1,x_2)=\frac{2}{\lambda}e^{-\frac{1}{\lambda}(x_1+x_2)}\left[1+e^{-\frac{1}{\lambda}x_1}+e^{-\frac{1}{\lambda}x_2}\right]^{-2}, \qquad -\infty<x_i<\infty, \quad i=1,2 \qquad (5.2)$$

Considering the transformation $x_i = v\tan(\theta_i)$, $i = 1, 2$. The Jacobian is $J = v^2 \sec^2(\theta_1)\sec^2(\theta_2)$. Consequently, the probability density function of a bivariate Semicircular distribution is obtained using simple algebra.

Similar to the method of constructing univariate l **- axial Logistic distribution,** it is simple to construct **bivariate** l **- axial Logistic distribution.** Let $\theta_i^* = \frac{2\theta_i}{l}$, $i = 1, 2$ and $l = 1, 2, 3, \ldots$, then the probability function is given by

$$g(\theta_1^*, \theta_2^*) = \frac{2l^2 \sec^2\left(\frac{l\theta_1^*}{2}\right)\sec^2\left(\frac{l\theta_2^*}{2}\right)}{\sigma} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{l\theta_1^*}{2}\right) + \tan\left(\frac{l\theta_2^*}{2}\right)\right)} \left[1 + e^{-\frac{1}{\sigma}\left(\tan\left(\frac{l\theta_1^*}{2}\right)\right)} + e^{-\frac{1}{\sigma}\left(\tan\left(\frac{l\theta_1^*}{2}\right)\right)}\right]^{-2}$$

$$-\frac{\pi}{l} < \theta_i^* < \frac{\pi}{l}, i = 1, 2 \qquad (5.3)$$

Case(1) When $l = 1$ in (5.3) we get

$$g(\theta_1^*, \theta_2^*) = \frac{2\sec^2\left(\frac{\theta_1^*}{2}\right)\sec^2\left(\frac{\theta_2^*}{2}\right)}{\sigma} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta_1^*}{2}\right) + \tan\left(\frac{\theta_2^*}{2}\right)\right)} \left[1 + e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta_1^*}{2}\right)\right)} + e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta_1^*}{2}\right)\right)}\right]^{-2}$$

$$-\pi < \theta_i^* < \pi, i = 1, 2 \qquad (5.4)$$

We call it as bivariate circular Logistic distribution.

Case(2) When $l = 2$ in (5.3) give the **bivariate semicircular Logistic distribution.**

References

1. Byoung, J.A and Hyoung M.K.(2008). A New Family of Semicircular Models: The Semicircular Laplace Distributions,Communications of the Korean Statistical SocietyVol.15, pp. 775-781.
2. Dattatreya RaoA.V., PhaniY., GirijaS.V.S. Stereographic Logistic Model – Application to Noisy Scrub Birds Data, (2016), Chilean Journal of Statistics, , Vol. 7, No. 2, 69-79.
3. Gradshteyn and Ryzhik (2007). Table of Integrals, series and products, 7th edition, Academic Press.

4. Guardiola, J.H. (2004). The Semicircular Normal Distribution, Ph.D Dissertation, Baylor University, Institute of Statistics.
5. Jammalamadaka S. Rao and Sen Gupta, A. (2001). Topics in Circular Statistics, *World Scientific Press, Singapore.*
6. Jones, T.A. (1968). Statistical Analysis of orientation data, Journal of Sedimentary Petrology, 38, 61-67.
7. Girija, S.V.S., Radhika, A.J.V. & Dattatreya Rao, A.V. (2014f). *On Offset l-Arc Models*,Mathematics and Statistics, 2(3): pp. 127-136.
8. Phani, Y. (2013). On Stereographic Circular and Semicircular Models,Unpublished Ph.D. thesis from Acharya Nagarjuna University.
9. Phani, Y., Girija, S.V.S. & Dattatreya Rao, A.V. (2013a). *On Construction of Stereographic Semicircular Models*, Journal of Applied Probability and Statistics, Vol. 8, no. 1, May 2013, pp. 75-90.
10. Toshihiro Abe, Kunio Shimizu & Arthur Pewsey (2010). *Symmetric Unimodal Models for Directional Data Motivated by Inverse Stereographic Projection*, J Japan Statistical Society, 40 (1), 45-61.