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A New Accelerated Third-Order Two-Step Iterative Method for Solving Nonlinear Equations

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Abstract
In the paper a new two-step iterative method has been suggested for estimating nonlinear problems by using quadrature formula. We have also proved that the proposed two-step iterative method has third-order of convergence. For the duration of study, it has been detected that the third-order iterative Method faster than existing cubic Methods such as variant of Newton Raphson Method and Halley Method. Numerous numerical illustrations are specified to demonstrate the efficiency and the performance of the new two-step method. Henceforth, the third-order iterative Method has been considered as an imperative enhancement and refinement to find the root of nonlinear equations.
Keyword: Non-linear equations, cubic methods, order of convergence, error and accuracy.

1. INTRODUCTION

For determining nonlinear problems is an imperious area of research in numerical analysis at it had interested applications in numerous field of pure and applied science have been deliberate in the general framework of the nonlinear problems [1-3], for instance

$$f(x) = 0 \tag{1}$$

To overview these difficulties, countless numerical techniques had planned and examined for estimating nonlinear problems. Such as most useful numerical techniques include Newton Raphson Method [4-5]. Newton Raphson Method are fast converging numerical techniques but are not reliable because keeping a kind of pitfall. However, it is most convenient and useful numerical techniques. In recent years, in literature several modifications had been done by using this technique for finding a single root of a nonlinear equation [6-8]. Furthermore, by combine Newton Raphson Method in literature, which has given a method equation with better accuracy as well as iteration perspective for solving non-linear [9-11]. Correspondingly, Numerous numerical have been illustrated and specified to demonstrate the efficiency and the performance of the new two-step method. These numerical methods have been constructed by using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variation iteration method and decomposition method [12–15]. Respectively, in this paper a Modified cubic iterated method has been suggested. Which is combination of classical Newton Raphson Method and quadrature formula. We have proven that two-step method is third-order convergence and present the comparison of new method with cubic methods such as Variant of Newton Raphson Method [16]

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f`(x_n) + f`(y_n)}$$

and Halley Method [17]

$$x_{n+1} = x_n - \frac{2f(x_n)f`(x_n)}{2f`^2(x_n) + f(x_n)f``(x_n)}$$

From the assessment of existing cubic methods to the new third-order iterated method is that the proposed is fast converging and more efficient to approaching the root.

2. ITERATIVE METHOD

The proposed third-order iterated method is developed by using the quadrature formula and the fundamental theorem of calculus, such as

$$f(x) = f(r) + \frac{h}{3}[f`(r) + 4f`\left(\frac{(x+r)}{2}\right) + f`(x)] \quad (2)$$

Where, the Simson $1/3^{rd}$ rule with n=2, then h become

$$h = \frac{(x-r)}{2}$$

Now, h substitute in (2), then (2) become

$$f(x) = f(r) + \frac{(x-r)}{6}[f`(r) + 4f`\left(\frac{(x+r)}{2}\right) + f`(x)] \quad (3)$$

Here \`r\` is an initial guess sufficiently close to root \`x\` and $f(x) = 0$, then (3) become

$$f(r) + \frac{(x-r)}{6}\left[f`(r) + 4f`\left(\frac{(x+r)}{2}\right) + f`(x)\right] = 0 \quad (4)$$

or

$$x = r - \frac{6f(r)}{f`(r) + 4f`\left(\frac{(x+r)}{2}\right) + f`(x)} \quad (5)$$

For using this relation, we can recommend that the following two-step iterative method as,

$$x_{n+1} = y_n - \frac{6f(y_n)}{f`(y_n) + 4f`\left(\frac{x_n + y_n}{2}\right) + f`(x_n)} \quad (6)$$

$for\ n = 0,1,2 \ldots$

Where $y_n = x_n - \frac{f(x_n)}{f`(x_n)}$

The relationship (6) is a new iterative method

3. CONVERGENCE ANALYSIS

The following section will be shows that the New Developed Method is cubically Convergence.

Proof

Let \`a\` be a simple zero of f. Then, by expanding $f(x_n)$ and $f`(x_n)$ in Taylor's Series about \`a\`, we have

$$f(x_n) = f`(a)(e_n + c_2 e^2{}_n + c_3 e^3{}_n + o(e^4{}_n)) \quad ---(i)$$

$$f`(x_n) = f`(a)(1 + 2c_2 e_n + 3c_3 e^2{}_n + 4c_4 e^3{}_n + o(e^4{}_n)) \quad ---(ii)$$

By using $c_k = \frac{f^k(a)}{k!f^{k-1}(a)}$, k=2,3, 4, … and $e_n = x_n - \mathrm{a}$

From (i) and (ii), we have

$$\frac{f(x_n)}{f`(x_n)} = e_n - c_2 e^2{}_n + 2(c^2{}_2 - c_3)e^3{}_n \quad ---(iii)$$

From (iii), we get

$$y_n = c_2 e^2{}_n - 2(c^2{}_2 - c_3)e^3{}_n \quad ---(iv)$$

expanding $f(y_n)\, f`(y_n)\ and\ f`\left(\frac{x_n+y_n}{2}\right)$ in Taylor's Series about \`a\`, by using (iv), we have

$$f(y_n) = f`(\mathrm{a})[c_2 e^2{}_n + 2(c_3 - c^2{}_2)e^3{}_n]$$

Or

$$f`(y_n) = f`(a)[1 + 2c^2{}_2 e^2{}_n + 4(c_2c_3 - c^3{}_2)e^3{}_n] \qquad --- (v)$$

and

$$f`\left(\frac{x_n + y_n}{2}\right) = f`(a)[1 + c_2e_n + \left(c^2{}_2 + \frac{3}{4}c_3\right)e^2{}_n + \left(\frac{7}{2}c_2c_3 - 2c^3{}_2\right)e^3{}_n] \qquad --- (vi)$$

From(ii), (v), and (vi), we have

$$f`(y_n) + 4f`\left(\frac{x_n + y_n}{2}\right) + f`(x_n)$$

$$= f`(a)\left[6 + 6c_2e_n + (6c^2{}_2 + 6c_3)e^2{}_n + 6\left(3c_2c_3 - \frac{1}{3}c^3{}_2\right)e^3{}_n\right] \qquad --- (vii)$$

From (v) and (vii), we obtain

$$\frac{6f(y_n)}{f`(y_n) + 4f`\left(\frac{x_n + y_n}{2}\right) + f`(x_n)} = \frac{6f`(a)[c_2e^2{}_n + 2(c_3 - c^2{}_2)e^3{}_n]}{6f`(a)\left[1 + c_2e_n + (c^2{}_2 + c_3)e^2{}_n + \left(3c_2c_3 - \frac{1}{3}c^3{}_2\right)e^3{}_n\right]}$$

$$\frac{6f(y_n)}{f`(y_n) + 4f`\left(\frac{x_n + y_n}{2}\right) + f`(x_n)} = c_2e^2{}_n - (3c^2{}_2 - 2c_3)e^3{}_n \qquad --- (viii)$$

From (iv) and $(viii)$ substitute in (6), we get

$$e_{n+1} = c_2e^2{}_n - 2(c^2{}_2 - c_3)e^3{}_n - c_2e^2{}_n + (3c^2{}_2 - 2c_3)e^3{}_n$$

$$e_{n+1} = (3c^2{}_2 - 2c_3 - 2c^2{}_2 + 2c_3)e^3{}_n$$

$$e_{n+1} = c^2{}_2e^3{}_n + o(e^4{}_n) \qquad --- (ix)$$

Error equation (ix) shows that the proposed method (6) is cubic order of convergence.

4. RESULTS AND DISCUSSIONS

In this section, the developed third-order iterative method is pragmatic on few examples of nonlinear equations and developed method compared with the variant of Newton Raphson Method and Halley Method. From the numerical result of table-1, it has been detected that the cubic iterative method is reducing the iterations number which is less than the number of iteration of the existing cubic methods and likewise accuracy side. Mathematical package such as C++ and EXCEL have used to justify the proposed Iterative Method. From the results and comparison of proposed cubic iterative method with the cubic methods that the proposed cubic iterative method is execution healthier than prevailing Techniques.

Table-1

FUNCTION	METHODS	ITERATION	X	A E
Sinx-x+1 **X=2**	Halley Method Variant Newton Method Two-Step Method	2 3 2	1.93456	$4.33922e^{-5}$ $1.19209e^{-7}$ $2.68221e^{-5}$
2x-lnx-7 **X=4**	Halley Method Variant Newton Method Two-Step Method	3 2 2	4.21991	$3.10421e^{-4}$ $1.19209e^{-5}$ $6.19888e^{-6}$
xe^x–2 **X=1**	Halley Method Variant Newton Method Two-Step Method	2 8 2	0.852605	$7.42078e^{-4}$ $2.98023e^{-7}$ $1.52845e^{-2}$
x^3 -9x+1 **X=0**	Halley Method Variant Newton Method Two-Step Method	2 4 2	0.111264	$1.53050e^{-4}$ $1.49012e^{-7}$ $1.56462e^{-7}$
Cosx-x^3 **X=1**	Halley Method Variant Newton Method Two-Step Method	2 6 2	0.865474	$1.28573e^{-3}$ $1.78814e^{-7}$ $1.76102e^{-3}$

Figure-1: Comparative illustration A E % of sinx-x+1

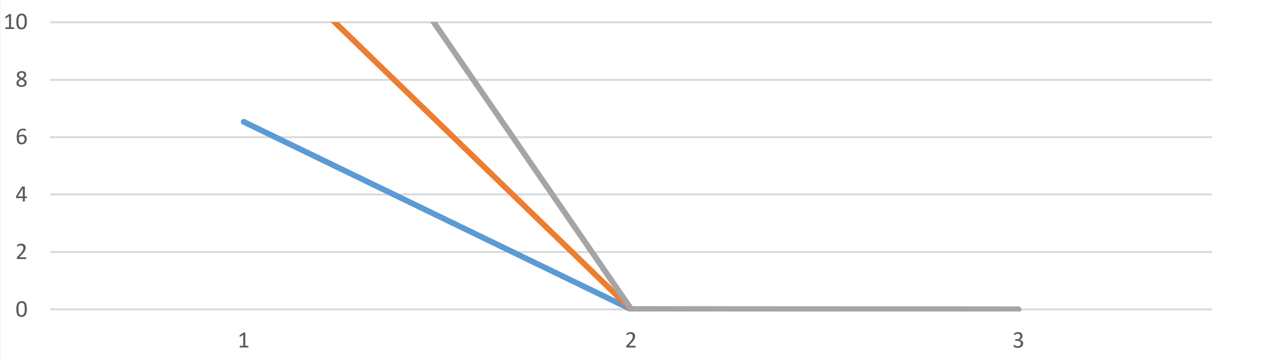
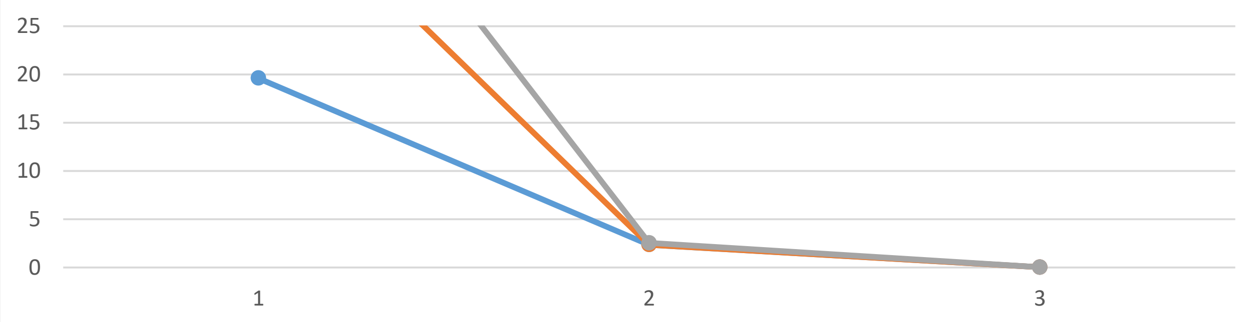


Figure-2: Comparative Illustration A E % of 2x-lnx-7



5. CONCLUSION

In this study, it is obvious that two-step method has been presented. The proposed method has third-order of convergence, which is performing better than the third-order iterated methods such as variant of Newton Raphson Method and Halley Method. Through the Table-1, it is concluding from the fallouts and assessment of suggested third-order method is that the proposed iterated method is loftier than Newton Raphson Method from precision viewpoint as well as iterative interpretation. Henceforth, the new third-order iterated method is a supercilious and a good achievement to find root of nonlinear equations in one variable with the comparison of existing Cubic Methods.

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