Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.8, No.5, 2018

On the Efficient Estimator of Exponential Parameter

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Abstract

Exponential distribution plays an important role in modeling real-life data relating to the continuous waiting time. In this article, a new estimator of the exponential parameter has been proposed. Some statistical properties of the proposed estimator have been studied. The performance of the new estimator has been compared theoretically, and empirically with the maximum likelihood estimator. Simulation and examples to real-life data reveal that the new estimator has higher relative efficiency as compared to the maximum likelihood estimator. The R program utilized in this study has also been provided.

Keywords: Moment generating function; method of moments; exponential parameter; relative efficiency; simulation.

1. Introduction

Exponential distribution has widely been used in modeling distributions in areas ranging from studies on the lifetimes of manufactured items [Davis, (1952), Epstein and Sobel (1953), Epstein (1958)] to research involving survival or remission times in chronic diseases [Feigl and Zelen (1965), Shanker, Fesshaye and Selvaraj (2015)]. The wide applicability of exponential distribution in lifetime modeling is due to the availability of simple statistical methods for it [Epstein and Sobel (1953)] and it represents the lifetimes of many things such as various types of manufactured items [Davis, (1952)]. Exponential distribution and of its parameter estimation appear in any standard book of statistics, for example, see*:* Hogg, McKean, and Craig (2013), Casella and Berger (2002), and Rohatgi (1984). We say that a continuous random variable X follows an exponential distribution with parameter θ (mean) if the probability density function is given by

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}; x > 0;\ \theta > 0$$

In general, θ is unknown and estimated using sample data. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n. Then, the maximum likelihood function of $f(x)$ is given by

$$L(\theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \tag{1}$$

Taking logarithm on both sides of the likelihood function in (1), we get

$$l(\theta) = lnL(\theta) = -nln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

Taking derivative of l with respect to θ, and setting equal to zero, a maximum likelihood estimator (MLE) of θ, $\hat{\theta}$ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \tag{2}$$

It is easy to see that $\hat{\theta}$ is an unbiased estimate of θ, i.e.,

$$E(\hat{\theta}) = \theta$$

with variance of $\hat{\theta}$ given by

$$V(\hat{\theta}) = \frac{\theta^2}{n}$$

In the next section, we propose a new estimator of the exponential parameter θ and study some statistical properties of the proposed estimator.

2. Proposed estimator

The proposed estimator is derived using the moment generating function. It is easy to show that the moment generating function of $X \sim Exp(\theta)$ is

$$M_X(t) = E(e^{Xt}) = (1 - \theta t)^{-1}$$

Given a random sample $X_1, X_2, \cdots, X_n$ of size n, it appears that the moment generating function of $\sum_{i=1}^{n} X_i$ is given by

$$M_{\sum_{i=1}^{n} X_i}(t) = E\left(e^{\sum_{i=1}^{n} X_i t}\right) = (1 - \theta t)^{-n}$$

Then, by the method of moments, a new estimator of θ, $\tilde{\theta}$ follows from solving the equation

$$e^{\sum_{i=1}^{n} X_i t} = (1 - \tilde{\theta} t)^{-n}$$

After an algebraic manipulation, the new estimator $\tilde{\theta}$ of θ takes the following form

$$\tilde{\theta} = \frac{1 - e^{-t\bar{x}}}{t}; t \neq 0 \qquad (3)$$

3. Statistical properties of the new estimator

We study some statistical properties of the proposed estimator $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$, which we state in terms of the theorems 3.1-3.5. The proofs of the theorems are given in the Appendix.

THEOREM 3.1 The expected value of $\tilde{\theta}$ is $E(\tilde{\theta}) = \frac{1}{t}\left[1 - \left(1 + \frac{t\theta}{n}\right)^{-n}\right]$ and if $t \to 0$, then $\tilde{\theta}$ is an unbiased estimate of θ.

THEOREM 3.2 The bias of $\tilde{\theta}$ is $B(\tilde{\theta}) = \frac{1}{t}\left[1 - \left(1 + \frac{t\theta}{n}\right)^{-n}\right] - \theta$ and if $t \to 0$, then bias of $\tilde{\theta}$ is 0.

THEOREM 3.3 The variance of $\tilde{\theta}$ is $V(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n}\right]$ and if $t \to 0$, then variance of $\tilde{\theta}$ is the same as the variance of $\hat{\theta}$.

THEOREM 3.4 The mean square error (MSE) of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$MSE(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2$$

and if $t \to 0$, then MSE of $\tilde{\theta}$ is the same as the variance of $\hat{\theta}$.

THEOREM 3.5 The relative efficiency (RE) of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with respect to $\hat{\theta}$ is

$$RE = \frac{\theta^2/n}{\frac{1}{t^2}\left[\left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2} \times 100\%$$

It is easy to see that as $t \to 0$, $\tilde{\theta}$ and $\hat{\theta}$ are the same. If $t \neq 0$, then there may exist a non-zero t such that

$$MSE(\tilde{\theta}) < V(\hat{\theta})$$

or,

$$\frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2 < \frac{\theta^2}{n} \tag{4}$$

which leads to the efficient estimator of θ.

We can easily search for values of t, for selected values of θ and n, satisfying the relation (4) and estimate the percent relative efficiency of the proposed estimator $\tilde{\theta}$ with respect to $\hat{\theta}$. We have utilized R code in examples and simulation to search for such t and evaluated the estimated relative efficiency and distributional fit.

4. Fitting of an exponential distribution

In this section, we provide algorithms that have been used to compare the goodness of fit (GOF) of the exponential distributions with estimators $\tilde{\theta}$ and $\hat{\theta}$.

4.1 Chi-squared GOF

For chi-squared goodness of fit [Rohatgi (1984), D'Agostino (1986)], we compare observed and expected frequency under the fitted exponential models given by the two estimators $\hat{\theta}$ and $\tilde{\theta}$ using a chi-squared test. The algorithm for the goodness of fit is as follows:

Given an observed sample, we divide the range of the observed values into k equal intervals and evaluate the observed and expected frequency in each of the k intervals. Let the k intervals be designated by $[u_0, u_1]$, $(u_1, u_2]$, …, $(u_{i-1}, u_i]$, …, $(u_{k-1}, u_k]$, where u_i is the upper end-point of ith interval, $i = 1, 2, \dots, k$. An observation from the given sample can be observed in any of the intervals with probability $p = 1/k$. Then, the cumulative distribution function (cdf) of observed sample is

$$F(u_i) = ip \tag{5}$$

Also, by the property of the exponential distribution,

$$F(u_i) = \int_0^{u_i} \frac{1}{\theta} e^{-\frac{t}{\theta}} dt = 1 - e^{-\frac{u_i}{\theta}} \tag{6}$$

By solving (5) and (6), it follows that

$$u_i = \theta \ln\left(\frac{1}{1-ip}\right)$$

The values of u_i for two underlying estimators $\hat{\theta}$ and $\tilde{\theta}$ are then given by

$$\hat{u}_i = \hat{\theta} \ln\left(\frac{1}{1-ip}\right) \tag{7}$$

$$\tilde{u}_i = \tilde{\theta} \ln\left(\frac{1}{1-ip}\right) \tag{8}$$

The observed frequency for the ith interval for two estimators are given by

$$\hat{o}_i = \#(\hat{u}_{i-1} < x \le \hat{u}_i) \tag{9}$$

$$\tilde{o}_i = \#(\tilde{u}_{i-1} < x \le \tilde{u}_i) \tag{10}$$

The expected frequency for the ith interval is

$$e_i = n \times p_i;\ p_i = \frac{1}{k} = 0.125, if\ k = 8$$

The goodness of fit is evaluated by implementing the test statistic $\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$ for two estimators $\hat{\theta}$ and $\tilde{\theta}$, which follow a chi-squared distribution with $(k-1-\nu)$ degrees of freedom (DF), where ν is the number of parameters estimated. Because we estimate one parameter θ by $\hat{\theta}$ or $\tilde{\theta}$, therefore $\nu = 1$.

4.2 Using AIC and BIC criteria

In order to apply AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) [Schwarz (1978), Wit, Heuvel, and Romeijn (2012), Burnham and Anderson (2004)], we find the estimated likelihood functions given by two estimators $\hat{\theta}$ and $\tilde{\theta}$ of θ. The likelihood function of exponential distribution is given by

$$L = \prod_{j=1}^{n} \frac{1}{\theta} e^{-\frac{x_j}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\theta}}$$

Then, using the likelihood estimator $\hat{\theta}$,

$$\hat{L} = \left(\frac{1}{\hat{\theta}}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\hat{\theta}}} = \left(\frac{1}{\bar{x}}\right)^n e^{-n} \tag{11}$$

Using the new estimator $\tilde{\theta}$,

$$\tilde{L} = \left(\frac{1}{\tilde{\theta}}\right)^n e^{-\frac{\sum_{j=1}^{n} x_j}{\tilde{\theta}}} = \left(\frac{1}{\frac{1-e^{-t\bar{x}}}{t}}\right)^n exp\left(-\frac{n\bar{x}}{\frac{1-e^{-t\bar{x}}}{t}}\right) \tag{12}$$

Then, AIC and BIC due to the estimator $\hat{\theta}$ are

$$\widehat{AIC} = (-2)\big(ln\hat{L}\big) + 2\nu \tag{13}$$

$$\widehat{BIC} = (-2)\big(ln\hat{L}\big) + \nu\big(ln(n)\big) \tag{14}$$

Similarly, AIC and BIC due to the estimator $\tilde{\theta}$ are

$$\widetilde{AIC} = (-2)\big(ln\tilde{L}\big) + 2\nu \tag{15}$$

$$\widetilde{BIC} = (-2)\big(ln\tilde{L}\big) + \nu\big(ln(n)\big) \tag{16}$$

The method with the lower of values of AIC and BIC provides the better fit.

5. Examples and applications

We consider two real-life examples (Examples 1 and 2) towards assessing and comparing the goodness of fit using the two estimators $\hat{\theta}$ and $\tilde{\theta}$ of θ. In both examples, we consider 8 intervals (i.e., $k = 8$) designated by $[u_0, u_1]$, $(u_1, u_2]$, …, $(u_6, u_7]$, …, $(u_7, u_8]$, where u_i is the upper end-point of ith interval, $i = 1, 2, \ldots, 8$. Therefore, an observation from a given sample of size n can be observed in any of the intervals with probability $p = \frac{1}{8} = 0.125$, and the expected frequency of each interval is $n \times 0.125$.

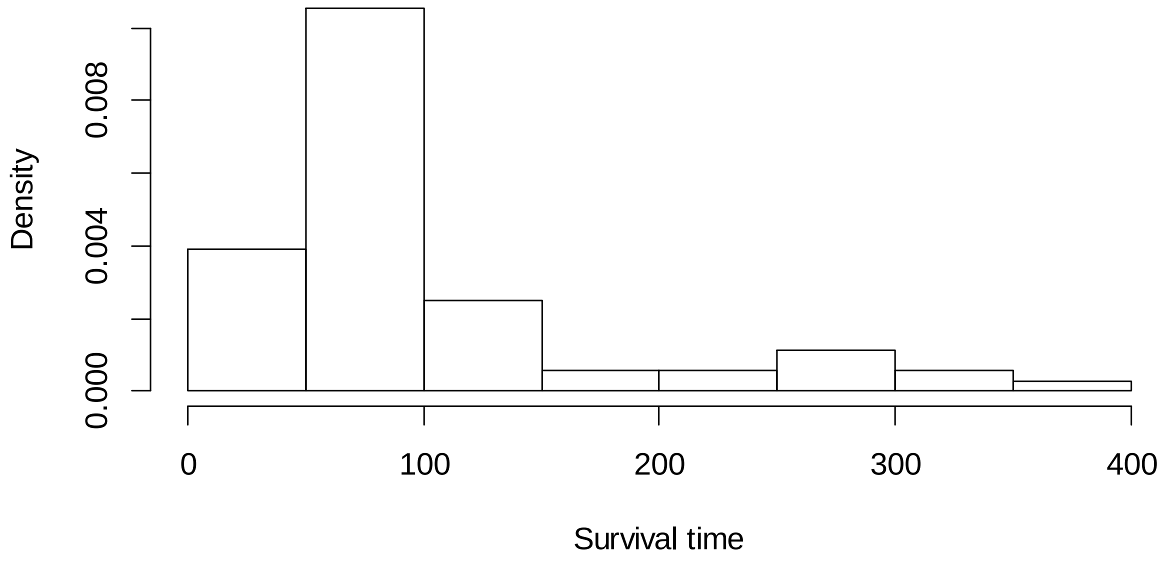
Example 1

Let us consider the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960), and Shanker, Fesshaye and Selvaraj (2015).

12	15	22	24	24	32	32	33	34	38	38	43
44	48	52	53	54	54	55	56	57	58	58	59
60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91
95	96	98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341	341	376

The density histogram in Figure 1 suggests that the shape of the survival time distribution is positive skewed.

Figure 1: Density of survival time of guinea pigs for data in Example 1



It appears that the mean survival time is 99.82 days. We test the null hypothesis that the data come from an exponential distribution with mean 100. Table 1 below provides 8 intervals and corresponding observed frequencies evaluated using equations (7)-(10).

Table 1. Intervals, observed and expected frequencies of survival times of guinea pigs evaluated using equations (7)-(10) using estimators $\hat{\theta}$ and $\tilde{\theta}$ with $t = 0.00051$

Intervals for $\hat{\theta}$	$\hat{o}_i$	Intervals for $\tilde{\theta}$	$\tilde{o}_i$	e_i
[0,13.329)	1	[0, 12.995)	1	9
[13.329, 28.716)	4	[12.995, 27.998)	4	9
[28.716, 46.915)	8	[27.998, 45.741)	8	9
[46.915, 69.189)	22	[45.741, 67.458)	21	9
[69.189, 97.905)	15	[67.458, 95.455)	15	9
[97.905, 138.379)	8	[95.455, 134.916)	9	9
[138.379, 207.568)	5	[134.916, 202.373)	5	9
[207.568, ∞)	9	[202.373, ∞)	9	9
Total	72	Total	72	72

The observed value of the Chi-square test statistic under the MLE estimate $\hat{\theta} = \bar{x}$ is $\chi^2_{\hat{\theta}} = \sum_{i=1}^{8} \frac{(\hat{o}_i - e_i)^2}{e_i} = 34.667$ and the p-value is 0.000005 with $DF = (8-1-1) = 6$. It also follows that $AIC = 808.8836$ and $BIC = 811.1603$.

For the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with $t = 0.00051$, the observed value of the Chi-square test statistic is

$\chi^2_{\tilde{\theta}} = \sum_{i=1}^{8} \frac{(\tilde{o}_i - e_i)^2}{e_i} = 31.778$ and the p-value is 0.00002 with $DF = (8-1-1) = 6$. It also follows that $AIC = 808.9303$ and $BIC = 811.2069$.

Given above analyses, at 5% level of significance we reject the null hypothesis that the data come from an exponential distribution with mean 100 using the both estimates $\hat{\theta}$ and $\tilde{\theta}$. However, it appears that the relative efficiency of the proposed estimate $\tilde{\theta}$ compared to $\hat{\theta}$ is 105.49%.

It is to be noted that we can search for values of t for selected values of θ and n, satisfying the relation in equation (4) in order to compute the percent relative efficiency of the proposed estimator $\tilde{\theta}$ with respect to $\hat{\theta}$, using R program. The R code that was used to search for values of t and computing relative efficiency of $\tilde{\theta}$ with respect to $\hat{\theta}$, along with other computations aspects for data in Example 1 has been reported in Appendix.

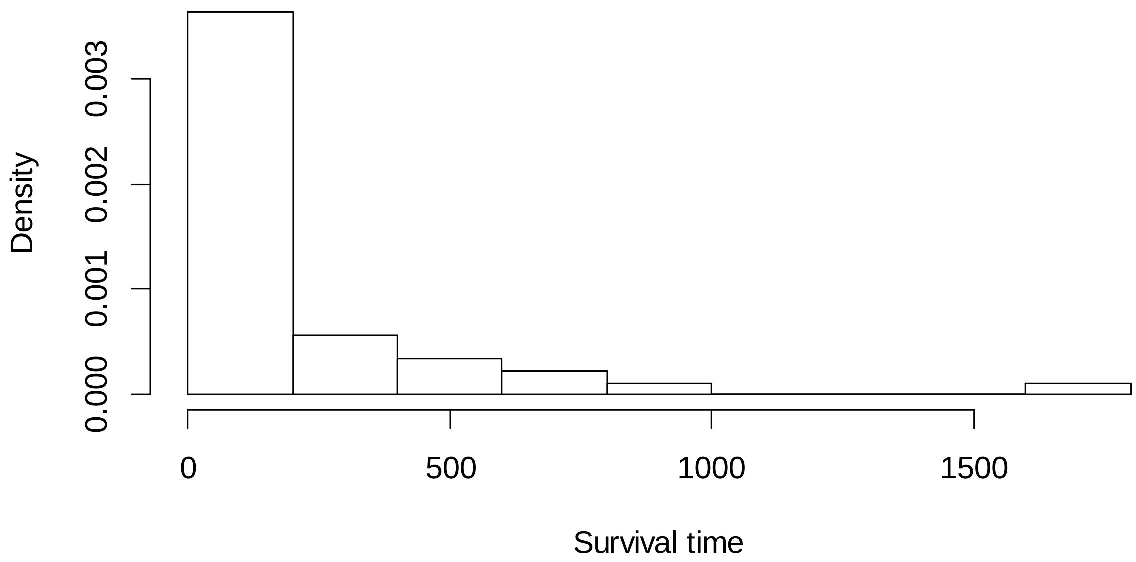
Example 2

The data set reported by Efron (1988) and Shanker, Fesshaye and Selvaraj (2015) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.2	23.6	23.7	25.9	32	37	41.35	47.38	55.46	58.4	63.5
68.46	78.3	74.5	81.4	84	92	94	110	112	119	127
130	133	140	146	155	159	173	179	194	195	209
249	281	319	339	432	469	519	633	725	817	1776

The density histogram in Figure 2 demonstrates that the shape of the distribution of the data is positive skewed.

Figure 2: Density of survival time of patients with head and neck cancer for data in Example 2



The mean survival time is 223.477. We test the null hypothesis that the data come from an exponential distribution with mean 223. Table 2 below provides 8 intervals and corresponding observed frequencies evaluated using equations (7)-(10).

The observed value of the Chi-square test statistic under the MLE estimate $\hat{\theta} = \bar{x}$ is $\chi^2_{\hat{\theta}} = \sum_{i=1}^{8} \frac{(\hat{o}_i - e_i)^2}{e_i} = 6.182$ and the p-value is 0. 40311 with $DF = (8 - 1 - 1) = 6$. It also follows that $AIC = 566.0191$ and $BIC = 567.8033$.

Table 2. Intervals, observed and expected frequencies of survival times of treated Head and Neck cancer patients evaluated using equations (7)-(10) for estimators $\hat{\theta}$ and $\tilde{\theta}$ with $t = 0.00041$

Intervals for $\hat{\theta}$	$\hat{o}_i$	Intervals for $\tilde{\theta}$	$\tilde{o}_i$	e_i
[0, 29.841)	4	[0, 28.515)	4	5.5
[29.841, 64.290)	7	[28.515, 61.433)	6	5.5
[64.290, 105.035)	7	[61.433, 100.367)	8	5.5
[105.035, 154.902)	8	[100.367, 148.018)	8	5.5
[154.902, 219.193)	7	[148.018, 209.451)	7	5.5
[219.193, 309.805)	2	[209.451, 296.036)	2	5.5
[309.805, 464.707)	3	[296.036, 444.054)	3	5.5
[464.707, ∞)	6	[444.054, ∞)	6	5.5
Total	44	Total	44	44

For the new estimate $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ with $t = 0.00041$, the observed value of the Chi-square test statistic is

$$\chi^2_{\tilde{\theta}} = \sum_{i=1}^{8} \frac{(\tilde{o}_i - e_i)^2}{e_i} = 6.545$$

and the p-value is 0.36498 with $DF = (8 - 1 - 1) = 6$.

It also follows that $AIC = 566.1115$ and $BIC = 567.8957$.

Given above analyses, at 5% level of significance there is a strong evidence that the data come from an exponential distribution with mean 223 using the both estimates $\hat{\theta}$ and $\tilde{\theta}$. However, the relative efficiency of the proposed estimate $\tilde{\theta}$ compared to $\hat{\theta}$ is 108.7%.

6. Relative efficiency of the new estimator

In this section, we investigate relative efficiency of the proposed estimator $\tilde{\theta}$ compared to $\hat{\theta}$ for given values of t, θ and n, using R code.

We consider various values of the parameter θ fixed at 0.5, 2.5, 5, 10, 15, 20, 25, 30, 50, 100, arbitrarily and sample size at 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and 100. For each combination of θ and n, we consider values of t between a and b with an increment of 0.0001, notationally expressed as $t \in [a, b, @\, 0.0001]$, where $a = 0.0001$ and values of b are evaluated using the search so as to satisfy (4) and are reported along with the relative efficiency for a given combination of θ and n in the Table 3.

Table 3. Relative efficiency of proposed estimate compared to the maximum likelihood estimate for varying sample size n and t

θ	n	$t \in [0.0001, b, @\, 0.0001]$	Range of RE
0.50	5	$0.0001 \le t \le 2.1600$	$100.01 \le RE \le 164.85$
	10	$0.0001 \le t \le 1.2320$	$100.00 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.8789$	$100.01 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.6862$	$100.01 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.5637$	$100.00 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.4786$	$100.00 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.4159$	$100.01 \le RE \le 111.02$

	40	$0.0001 \le t \le 0.3679$	$100.00 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.3298$	$100.00 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.2989$	$100.00 \le RE \le 107.80$
	100	$0.0002 \le t \le 0.1544$	$100.00 \le RE \le 103.95$
2.5	5	$0.0001 \le t \le 0.4330$	$100.06 \le RE \le 164.85$
	10	$0.0001 \le t \le 0.2464$	$100.00 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.1757$	$100.04 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.1372$	$100.02 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.1127$	$100.02 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.0957$	$100.01 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.0831$	$100.04 \le RE \le 111.02$
	40	$0.0001 \le t \le 0.0735$	$100.04 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.0659$	$100.03 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.0597$	$100.04 \le RE \le 107.80$
	100	$0.0001 \le t \le 0.0308$	$100.04 \le RE \le 103.95$
5	5	$0.0001 \le t \le 0.2166$	$100.01 \le RE \le 164.85$
	10	$0.0001 \le t \le 0.1232$	$100.00 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.0877$	$100.11 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.0686$	$100.02 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.0563$	$100.06 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.0478$	$100.05 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.0415$	$100.09 \le RE \le 111.02$
	40	$0.0001 \le t \le 0.0367$	$100.08 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.0329$	$100.08 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.0298$	$100.09 \le RE \le 107.80$
	100	$0.0001 \le t \le 0.0154$	$100.04 \le RE \le 103.95$
10	5	$0.0001 \le t \le 0.1083$	$100.01 \le RE \le 164.85$
	10	$0.0001 \le t \le 0.0616$	$100.00 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.0439$	$100.07 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.0343$	$100.02 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.0281$	$100.14 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.0239$	$100.05 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.0207$	$100.18 \le RE \le 111.02$
	40	$0.0001 \le t \le 0.0183$	$100.17 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.0164$	$100.17 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.0149$	$100.09 \le RE \le 107.80$
	100	$0.0001 \le t \le 0.0077$	$100.04 \le RE \le 103.95$
15	5	$0.0001 \le t \le 0.0722$	$100.01 \le RE \le 164.85$
	10	$0.0001 \le t \le 0.0410$	$100.13 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.0292$	$100.22 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.0228$	$100.18 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.0187$	$100.23 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.0159$	$100.14 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.0138$	$100.18 \le RE \le 111.02$
	40	$0.0001 \le t \le 0.0122$	$100.17 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.0109$	$100.26 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.0099$	$100.18 \le RE \le 107.79$
	100	$0.0001 \le t \le 0.0051$	$100.14 \le RE \le 103.95$
20	5	$0.0001 \le t \le 0.0541$	$100.10 \le RE \le 164.85$
	10	$0.0001 \le t \le 0.0308$	$100.00 \le RE \le 135.59$
	15	$0.0001 \le t \le 0.0219$	$100.22 \le RE \le 124.59$
	20	$0.0001 \le t \le 0.0171$	$100.18 \le RE \le 118.79$
	25	$0.0001 \le t \le 0.0140$	$100.31 \le RE \le 115.21$
	30	$0.0001 \le t \le 0.0119$	$100.23 \le RE \le 112.78$
	35	$0.0001 \le t \le 0.0103$	$100.35 \le RE \le 111.02$
	40	$0.0001 \le t \le 0.0091$	$100.35 \le RE \le 109.68$
	45	$0.0001 \le t \le 0.0082$	$100.17 \le RE \le 108.64$
	50	$0.0001 \le t \le 0.0074$	$100.27 \le RE \le 107.79$

	100	$0.0001 \leq t \leq 0.0038$	$100.23 \leq RE \leq 103.95$
25	5	$0.0001 \leq t \leq 0.0433$	$100.06 \leq RE \leq 164.85$
	10	$0.0001 \leq t \leq 0.0246$	$100.13 \leq RE \leq 135.59$
	15	$0.0001 \leq t \leq 0.0175$	$100.30 \leq RE \leq 124.59$
	20	$0.0001 \leq t \leq 0.0137$	$100.10 \leq RE \leq 118.79$
	25	$0.0001 \leq t \leq 0.0112$	$100.31 \leq RE \leq 115.21$
	30	$0.0001 \leq t \leq 0.0095$	$100.31 \leq RE \leq 112.78$
	35	$0.0001 \leq t \leq 0.0083$	$100.09 \leq RE \leq 111.02$
	40	$0.0001 \leq t \leq 0.0073$	$100.26 \leq RE \leq 109.68$
	45	$0.0001 \leq t \leq 0.0065$	$100.44 \leq RE \leq 108.64$
	50	$0.0001 \leq t \leq 0.0059$	$100.36 \leq RE \leq 107.80$
	100	$0.0001 \leq t \leq 0.0030$	$100.42 \leq RE \leq 103.95$
30	5	$0.0001 \leq t \leq 0.0361$	$100.01 \leq RE \leq 164.85$
	10	$0.0001 \leq t \leq 0.0205$	$100.13 \leq RE \leq 135.59$
	15	$0.0001 \leq t \leq 0.0146$	$100.22 \leq RE \leq 124.59$
	20	$0.0001 \leq t \leq 0.0114$	$100.18 \leq RE \leq 118.79$
	25	$0.0001 \leq t \leq 0.0093$	$100.48 \leq RE \leq 115.21$
	30	$0.0001 \leq t \leq 0.0079$	$100.40 \leq RE \leq 112.78$
	35	$0.0001 \leq t \leq 0.0069$	$100.18 \leq RE \leq 111.02$
	40	$0.0001 \leq t \leq 0.0061$	$100.17 \leq RE \leq 109.68$
	45	$0.0001 \leq t \leq 0.0054$	$100.53 \leq RE \leq 108.64$
	50	$0.0001 \leq t \leq 0.0049$	$100.45 \leq RE \leq 107.79$
	100	$0.0001 \leq t \leq 0.0025$	$100.42 \leq RE \leq 103.95$
50	5	$0.0001 \leq t \leq 0.0216$	$100.27 \leq RE \leq 164.85$
	10	$0.0001 \leq t \leq 0.0123$	$100.13 \leq RE \leq 135.58$
	15	$0.0001 \leq t \leq 0.0087$	$100.67 \leq RE \leq 124.59$
	20	$0.0001 \leq t \leq 0.0068$	$100.50 \leq RE \leq 118.79$
	25	$0.0001 \leq t \leq 0.0056$	$100.31 \leq RE \leq 115.21$
	30	$0.0001 \leq t \leq 0.0047$	$100.74 \leq RE \leq 112.78$
	35	$0.0001 \leq t \leq 0.0041$	$100.53 \leq RE \leq 111.02$
	40	$0.0001 \leq t \leq 0.0036$	$100.70 \leq RE \leq 109.68$
	45	$0.0001 \leq t \leq 0.0032$	$100.88 \leq RE \leq 108.64$
	50	$0.0001 \leq t \leq 0.0029$	$100.80 \leq RE \leq 107.79$
	100	$0.0001 \leq t \leq 0.0015$	$100.42 \leq RE \leq 103.94$
100	5	$0.0001 \leq t \leq 0.0108$	$100.27 \leq RE \leq 164.84$
	10	$0.0001 \leq t \leq 0.0061$	$100.79 \leq RE \leq 135.58$
	15	$0.0001 \leq t \leq 0.0043$	$101.43 \leq RE \leq 124.59$
	20	$0.0001 \leq t \leq 0.0034$	$100.50 \leq RE \leq 118.79$
	25	$0.0001 \leq t \leq 0.0028$	$100.31 \leq RE \leq 115.20$
	30	$0.0001 \leq t \leq 0.0023$	$101.59 \leq RE \leq 112.76$
	35	$0.0001 \leq t \leq 0.0020$	$101.39 \leq RE \leq 111.02$
	40	$0.0001 \leq t \leq 0.0018$	$100.70 \leq RE \leq 109.68$
	45	$0.0001 \leq t \leq 0.0016$	$100.88 \leq RE \leq 108.64$
	50	$0.0001 \leq t \leq 0.0014$	$101.67 \leq RE \leq 107.78$
	100	$0.0001 \leq t \leq 0.0007$	$101.28 \leq RE \leq 103.94$

7. Results and Discussion

We write program in R to search for values of t and the relative efficiency of the proposed estimator of exponential parameter θ (mean) as compared to the MLE estimator. It appears that the values of t for data in Examples 1 and 2 are positive for relative efficiency to be more the proposed estimator compared to the MLE estimator. In the search of values of t, we restrict ourselves to positive values of t nearing 0 for higher relative efficiency for the proposed estimator. Theoretically, since the proposed estimate is unbiased as $t \rightarrow 0$, we wish to achieve efficiency as well as nearing unbiased estimate by choosing values of t nearing 0. For example, when $\theta = 0.5$ and the sample size $n = 5$, the relative efficiency of the proposed estimate ranges from 100.01 to 164.85 when t ranges from 0.0001 to 2.16 with an increment of 0.0001. This means that the by choosing a value of $t = 2.16$ in the estimator $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$, the relative efficiency of the estimator can be increased approximately

167% compared to the estimator $\hat{\theta} = \bar{x}$ when $\theta = 0.5$. However, when $\theta = 0.5$ and the sample size $n = 10$, the relative efficiency ranges from 100.00 to 135.59 when t ranges from 0.0001 to 1.232 with an increment of 0.0001. From the reported results, it appears that for a fixed parameter, lower sample size provides better efficiency for the proposed estimate. It also follows that relative efficiency of the proposed estimator is not sensitive to the values of the parameter θ, rather it is sensitive to the sample size n and the values of t.

8. Concluding remarks

We proposed a new estimator, $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$, $t \neq 0$, for estimating the unknown exponential parameter θ using mgf. Some statistical properties of the new estimator such as Expected value, Bias, MSE, Variance and RE have been studied. As $t \to 0$, the new estimator $\tilde{\theta}$ is unbiased, and MSE and Variance are identical to the variance of the MLE $\hat{\theta}$. By searching values of t nearing 0, we can have the higher relative efficiency of the proposed estimator $\tilde{\theta}$ compared to the ML estimator, $\hat{\theta} = \bar{x}$. The new estimator has been justified using two real-life examples, where the new estimator $\tilde{\theta}$ and the competitor estimator $\hat{\theta} = \bar{x}$ provide approximately similar fit, but the new estimate provide higher efficiency in the estimation of the parameter. In a broader search of relative efficiency, with varying values of the parameter θ, sample size n and t, it appears that the proposed estimator has much higher relative efficiency as compared to the MLE for smaller sample size. We write program in R to search for the range of t and range of relative efficiency (RE) of the proposed estimate as compared to MLE, which will provide a guide to implement the new method. Given the facts of the study and success in real-life application of the proposed estimator, we conclude that the proposed new estimator is more efficient than usual MLE for values of t nearing 0, and therefore, we recommend the proposed estimator for fitting exponential model to real-life data and the estimation of exponential parameter with higher efficiency.

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APPENDIX

PROOF OF THEOREM 3.1. The expected value of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$E(\tilde{\theta}) = \frac{1 - E(e^{-t\bar{x}})}{t} = \frac{1}{t}\left[1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right]; t \neq 0$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} E(\tilde{\theta}) = \lim_{t\to 0} \frac{(-)(-n)\left(1 + \frac{t}{n}\theta\right)^{-n-1}\left(\frac{1}{n}\theta\right)}{1} = \theta$$

PROOF OF THEOREM 3.2. The bias of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$B(\tilde{\theta}) = E(\tilde{\theta}) - \theta = \frac{1 - E(e^{-t\bar{x}})}{t} - \theta = \frac{\left[1 - \left(1 + \frac{t}{n}\theta\right)^{-n}\right]}{t} - \theta; t \neq 0$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} B(\tilde{\theta}) = \lim_{t\to 0} \frac{n\left(1 + \frac{t}{n}\theta\right)^{-n-1}\left(\frac{1}{n}\theta\right)}{1} - \theta = \theta - \theta = 0$$

PROOF OF THEOREM 3.3. Note that $M_{\sum_{i=1}^{n} x_i}(t) = E\left(e^{\sum_{i=1}^{n} x_i t}\right) = (1 - \theta t)^{-n}$

It also follows that

$$M_{\bar{x}}(t) = M_{\sum_{i=1}^{n} x_i}(t/n) = \left(1 - \frac{\theta t}{n}\right)^{-n}$$

$$M_{\bar{x}}(-t) = \left(1 + \frac{\theta t}{n}\right)^{-n}$$

$$M_{\bar{x}}(-kt) = \left(1 + \frac{\theta k t}{n}\right)^{-n}$$

Now, the variance of $\tilde{\theta} = \frac{1-e^{-t\bar{x}}}{t}$ is

$$V(\tilde{\theta}) = V\left(\frac{1 - e^{-t\bar{x}}}{t}\right) = \frac{1}{t^2}V(e^{-t\bar{x}})$$

$$\begin{aligned} V(e^{-t\bar{x}}) &= E(e^{-t\bar{x}})^2 - [E(e^{-t\bar{x}})]^2 \\ &= E(e^{-2t\bar{x}}) - [E(e^{-t\bar{x}})]^2 \\ &= M_{\bar{x}}(-2t) - [M_{\bar{x}}(-t)]^2 \\ &= \left(1 + \frac{2t\theta}{n}\right)^{-n} - \left[\left(1 + \frac{t\theta}{n}\right)^{-n}\right]^2 \\ &= \left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n} \end{aligned}$$

Then, $V(\tilde{\theta}) = \frac{1}{t^2}\left[\left(1 + \frac{2t\theta}{n}\right)^{-n} - \left(1 + \frac{t\theta}{n}\right)^{-2n}\right]; t \neq 0$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} V(\tilde{\theta}) = \lim_{t\to 0} \frac{(-n)\left(1+\frac{2t\theta}{n}\right)^{-n-1}\left(\frac{2\theta}{n}\right)-(-2n)\left(1+\frac{t\theta}{n}\right)^{-2n-1}\left(\frac{\theta}{n}\right)}{2t}$$

$$= \lim_{t\to 0} \frac{-\theta\left(1+\frac{2t\theta}{n}\right)^{-n-1}+\theta\left(1+\frac{t\theta}{n}\right)^{-2n-1}}{t}$$

$$= \lim_{t\to 0} \frac{\theta(n+1)\left(1+\frac{2t\theta}{n}\right)^{-n-2}\left(\frac{2\theta}{n}\right)-(2n+1)\theta\left(1+\frac{t\theta}{n}\right)^{-2n-2}\left(\frac{\theta}{n}\right)}{1}$$

$$= \theta(n+1)(1+0)\left(\frac{2\theta}{n}\right) - (2n+1)\theta(1+0)\left(\frac{\theta}{n}\right)$$

$$= \frac{2\theta^2(n+1)}{n} - \frac{\theta^2(2n+1)}{n}$$

$$= \frac{2\theta^2-\theta^2}{n} = \frac{\theta^2}{n} = V(\hat{\theta})$$

PROOF OF THEOREM 3.4. The MSE of $\tilde{\theta} = \frac{t\bar{x}}{e^t-1}$ is

$$MSE(\tilde{\theta}) = V(\tilde{\theta}) + [B(\tilde{\theta})]^2$$

$$= \frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2$$

Taking limit as $t \to 0$ and applying the L' Hospital Rule, we have

$$\lim_{t\to 0} MSE(\tilde{\theta}) = \lim_{t\to 0} V(\tilde{\theta}) + \left[\lim_{t\to 0} B(\tilde{\theta})\right]^2$$

$$= \frac{\theta^2}{n} + 0$$

$$= \frac{\theta^2}{n}$$

$$= V(\hat{\theta})$$

PROOF OF THEOREM 3.5. The relative efficiency of $\tilde{\theta} = \frac{t\bar{x}}{e^t-1}$ with respect to $\hat{\theta}$ is given by

$$RE = \frac{V(\hat{\theta})}{MSE(\tilde{\theta})} \times 100$$

$$= \frac{\theta^2/n}{\frac{1}{t^2}\left[\left(1+\frac{2t\theta}{n}\right)^{-n} - \left(1+\frac{t\theta}{n}\right)^{-2n}\right] + \left[\frac{1}{t}\left\{1-\left(1+\frac{t}{n}\theta\right)^{-n}\right\} - \theta\right]^2} \times 100$$

R Code to search for values of t towards efficiency of $\tilde{\theta}$

```
n=length(x);# number of observation in Example 1
mean=mean(x)#observed value of mean=99.82
theta=100;
nu=1;# number of parameter estimated
for (t in seq(0.00001,.001,.0001)){
a<-(1/t^2)*((1+(2*t*theta/n))^(-n)-(1+(t*theta/n))^(-2*n))+((1/t)*(1-(1+t*theta/n)^(-n))-theta)^2
b<-theta^2/n;
ifelse (a<b,{print(t);print(b/a*100)},print(0))
}
```



```
est1=round(mean(x),digits=3);# MLE estimator
est2=round((1-exp(-est1*0.00051))/0.00051,digits=3);#New estimator

#Assessing GOF for using est1;
u=c();
for (i in 1: 7){u[i]=round(-est1*log(1-i*.125),digits=3)}
print(u)
o=c();
o[1]=sum(x<u[1]);
o[2]=sum(x>=u[1] & x<u[2])
o[3]=sum(x>=u[2] & x<u[3])
o[4]=sum(x>=u[3] & x<u[4])
o[5]=sum(x>=u[4] & x<u[5])
o[6]=sum(x>=u[5] & x<u[6])
o[7]=sum(x>=u[6] & x<u[7])
o[8]=sum(x>=u[7])
print(o)
e=rep(1,8)*.125*n;
chi2.1=round(sum((o-e)^2/e), digits=3);
pval.1=pchisq(chi2.1, df=6,nc=0,lower.tail=F)
like1=((1/est1)^n)*(exp(-n))
aic1=-(2*log(like1))+2*nu
bic1=-2*log(like1)+nu*log(n)

# Assessing GOF for using est2;
u=c();
for (i in 1: 7){u[i]=round(-est2*log(1-i*.125),digits=3)}
print(u)
o=c();
o[1]=sum(x<u[1]);
o[2]=sum(x>=u[1] & x<u[2])
o[3]=sum(x>=u[2] & x<u[3])
o[4]=sum(x>=u[3] & x<u[4])
o[5]=sum(x>=u[4] & x<u[5])
o[6]=sum(x>=u[5] & x<u[6])
o[7]=sum(x>=u[6] & x<u[7])
o[8]=sum(x>=u[7])
print(o)
chi2.2=round(sum((o-e)^2/e), digits=3);
pval.2=pchisq(chi2.2, df=6,nc=0,lower.tail=F)
like2=((1/est2)^n)* exp((-n*est1/est2))
aic2=-(2*log(like2))+2*nu
bic2=-2*log(like2)+nu*log(n)
print(c(chi2.1, pval.1, chi2.2, pval.2,qchisq(0.05, df=6,nc=0,lower.tail=F)))
print(c(aic1, bic1))
print(c(aic2, bic2))
```