On K^*Quasi n- Normal Operator

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Abstract

In this paper, we will give new type of quasi normal operator is called K^* quasi n- normal operator and as well as, study the direct sum of these operators and some properties of this concept have been given.

1.1Introduction

An bounded linear operator $T: H \to K$ is said to be quasi normal operator if T commute with T^*T the first given this concept by A.Brown in 1953 [2], also in 1976 A.Bala was gave some properties of quasi normal operator [1]. In 2011 O.Ahmed is introduced K quasi normal operator with some basic properties [4]. In 2015 Laith K. Shakir and Saad S.Marai the n quasi normal operator [3]

2.1Definitions

1. An bounded linear operator $T: H \to K$ is called quasi normal operator if satisfy $T(T^*T) = (T^*T)T$
2. An bounded linear operator $T: H \to K$ is called K quasi normal operator if satisfy $T^K(T^*T) = (T^*T)T^K$
3. An bounded linear operator $T: H \to K$ is called n quasi normal operator if satisfy $T(T^*T)^n = (T^*T)^n T$

Now, we introduce the definition of K^* quasi n- normal operator

3.1Definition

Let $T: H \to K$ is a bounded liner operator then T is said to be K^* quasi n- normal operator if $(T^*)^K(T^*T^n) = (T^*T^n)(T^*)^K$

3.2Remark

1. Clearly that if T is self adjoint, $n = 1$ and $K = 1$ then T is quasi normal operator
2. Clearly that if T is self adjoint and $n = 1$then T is K- quasi normal operator

3.3Propositions

Let $T: H \to K$ is K^* quasi n- normal operator then

1. T^m is also K^* quasi n- normal operator where $m \geq 1$

2. λT is K^* quasi n- normal operator where $\lambda \in R$
3. If T^{-1} is exist then T^{-1} is K^* quasi n- normal operator

Proof

1. Let T be a K^*quasi n- normal operator

We prove that T^m is K^*quasi n- normal operator

By using mathematical induction

Since T is K^*quasi n- normal operator the result is true for $A = 1$

$$\left(T^*\right)^K \left(T^* T^n\right) = \left(T^* T^n\right)\left(T^*\right)^K \qquad (1)$$

Now we assume that the result is true for $m = A$

$$\left(\left(T^*\right)^K \left(T^* T^n\right)\right)^A = \left(\left(T^* T^n\right)\left(T^*\right)^K\right)^A \qquad (2)$$

Let us prove the result for $m = A+1$

That is $\left(\left(T^*\right)^K \left(T^* T^n\right)\right)^{A+1} = \left(\left(T^* T^n\right)\left(T^*\right)^K\right)^{A+1}$

$$\left(\left(T^*\right)^K \left(T^* T^n\right)\right)^{A+1} = \left(\left(T^*\right)^K \left(T^* T^n\right)\right)^A \left(T^*\right)^K \left(T^* T^n\right)$$

$$= \left(\left(T^* T^n\right)\left(T^*\right)^K\right)^A \left(T^* T^n\right)\left(T^*\right)^K$$

$$= \left(\left(T^* T^n\right)\left(T^*\right)^K\right)^{A+1}$$

Thus the result is true for $m = A+1$

There for T^m is also K^* quasi n- normal operator for each m where $m \geq 1$

2.

$$\left(\left(\lambda T\right)^*\right)^K \left(\lambda T\right)^* \left(\lambda T\right)^n = \left(\lambda T^*\right)^K \left(\lambda T^*\right)\left(\lambda^n T^n\right)$$

$$= \lambda^K \left(T^*\right)^K \left(\lambda T^*\right)\left(\lambda^n T^n\right)$$

$$= \lambda^K \lambda \lambda^n \left(T^*\right)^K \left(T^*\right)\left(T^n\right)$$

$$= \lambda^K \lambda \lambda^n \left(T^* T^n\right)\left(T^*\right)^K$$

$$= \lambda T^* \lambda^n T^n \left(\lambda^K T^*\right)^K$$

$$= (\lambda T)^{*}(\lambda T)^{n}\left((\lambda T)^{*}\right)^{K}$$

$$\therefore \left((\lambda T)^{*}\right)^{K}(\lambda T)^{*}(\lambda T)^{n} = (\lambda T)^{*}(\lambda T)^{n}\left((\lambda T)^{*}\right)^{K}$$

Hence the λT is K^{*} quasi n- normal operator

3.

$$\left((T^{-1})^{*}\right)^{K}(T^{-1})^{*}(T^{-1})^{n} = \left((T^{*})^{-1}\right)^{K}(T^{*})^{-1}(T^{n})^{-1}$$

$$= \left((T^{*})^{K}\right)^{-1}(T^{*}T^{n})^{-1}$$

$$= \left((T^{*})^{K}(T^{*}T^{n})\right)^{-1}$$

$$= \left(T^{*}T^{n}(T^{*})^{K}\right)^{-1}$$

$$= (T^{*}T^{n})^{-1}\left((T^{*})^{K}\right)^{-1}$$

$$= (T^{*})^{-1}(T^{n})^{-1}\left((T^{*})^{-1}\right)^{K}$$

$$= (T^{-1})^{*}(T^{-1})^{n}\left((T^{-1})^{*}\right)^{K}$$

$$\therefore \left((T^{-1})^{*}\right)^{K}(T^{-1})^{*}(T^{-1})^{n} = (T^{-1})^{*}(T^{-1})^{n}\left((T^{-1})^{*}\right)^{K}$$

Hence the T^{-1} is K^{*} quasi n- normal operator

3.4Remark

Let T_1 and T_2 be two K^{*} quasi n- normal operators then $T_1 + T_2$ is not necessary K^{*} quasi n- normal operator, to illustrate that consider the following example

3.5Example

$$T_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

$$T_1 + T_2 = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$$

If $n = 1$ and $K = 1$

$$\left((T_1 + T_2)^{*}\right)^{1}\left((T_1 + T_2)^{*}(T_1 + T_2)^{1}\right) = \left(\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}3 & 2\\ 4 & 2\end{bmatrix}\right)\left(\begin{bmatrix}13 & 16\\ 16 & 20\end{bmatrix}\right)$$

$$=\begin{bmatrix}71 & 88\\ 84 & 104\end{bmatrix} \qquad (1)$$

$$\left((T_1+T_2)^*(T_1+T_2)^1\right)\left((T_1+T_2)^*\right)^1=\left(\begin{bmatrix}3 & 2\\ 4 & 2\end{bmatrix}\begin{bmatrix}3 & 4\\ 2 & 2\end{bmatrix}\right)\left(\begin{bmatrix}3 & 2\\ 4 & 2\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}13 & 16\\ 16 & 20\end{bmatrix}\right)\left(\begin{bmatrix}3 & 2\\ 4 & 2\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}103 & 56\\ 128 & 72\end{bmatrix}\right) \qquad (2)$$

$(1)\neq(2)$

Then T_1+T_2 is not K^* quasi n- normal operator

Now, the following theorem given the condition to make remark (3-4) is true

3.6Theorem

Let T_1 and T_2 be two K^* quasi n- normal operator on a Hilbert space H such that $T_1^*T_2^*=T_1T_2=T_1^*T_2=0$ then T_1+T_2 K^* quasi n- normal operator

Proof

$$\left((T_1+T_2)^*\right)^K\left((T_1+T_2)^*(T_1+T_2)^n\right)=\left(T_1^*+T_2^*\right)^K\left(\left(T_1^*+T_2^*\right)(T_1+T_2)^n\right)$$

$$=\left(\left(T_1^*\right)^K+K\left(T_1^*\right)^{K-1}T_2^*+\ldots+\left(T_2^*\right)^K\right)\left(\left(T_1^*+T_2^*\right)\left(T_1^n+nT_1^{n-1}T_2+\ldots+T_2^n\right)\right)$$

$$=\left(\left(T_1^*\right)^K+\left(T_2^*\right)^K\right)\left(\left(T_1^*+T_2^*\right)\left(T_1^n+T_2^n\right)\right)$$

$$=\left(\left(T_1^*\right)^K+\left(T_2^*\right)^K\right)\left(T_1^*T_1^n+T_1^*T_2^n+T_2^*T_1^n+T_2^*T_2^n\right)$$

$$=\left(\left(T_1^*\right)^K+\left(T_2^*\right)^K\right)\left(T_1^*T_1^n+T_2^*T_2^n\right)$$

$$=\left(\left(T_1^*\right)^K T_1^*T_1^n+\left(T_1^*\right)^K T_2^*T_2^n+\left(T_2^*\right)^K T_1^*T_1^n+\left(T_2^*\right)^K T_2^*T_2^n\right)$$

$$=\left(\left(T_1^*\right)^K T_1^*T_1^n+\left(T_2^*\right)^K T_2^*T_2^n\right)$$

Since T_1 and T_2 are K^* quasi n- normal operator

$$=\left(T_1^*T_1^n\left(T_1^*\right)^K\right)+\left(T_2^*T_2^n\left(T_2^*\right)^K\right)$$

Hence $T_1 + T_2$ is K^* quasi n- normal operator

3.7Remark

Let T_1 and T_2 be two K^* quasi n- normal operators then T_1T_2 is not necessary K^* quasi n- normal operator, to explain that see the following example

3.8Example

$$T_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$T_1T_2 = \begin{bmatrix} 5 & 6 \\ 5 & 2 \end{bmatrix}$$

If $n = 1$ and $K = 1$

$$\left((T_1T_2)^*\right)^1\left((T_1T_2)^*(T_1T_2)^1\right) = \left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 5 & 2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 50 & 40 \\ 40 & 40 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 450 & 400 \\ 380 & 320 \end{bmatrix} \qquad (1)$$

$$\left((T_1T_2)^*(T_1T_2)^1\right)\left((T_1T_2)^*\right)^1 = \left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 5 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 50 & 40 \\ 40 & 40 \end{bmatrix}\right)\left(\begin{bmatrix} 5 & 5 \\ 6 & 2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 490 & 330 \\ 440 & 289 \end{bmatrix}\right) \qquad (2)$$

$$(1) \neq (2)$$

Then T_1T_2 is not K^* quasi n- normal operator

Next, the following theorem given the condition to make remark (3.7) is true

3.9Theorem

Let $T_1, T_2 : H \to H$ are K^* quasi n normal operators then T_1T_2 is K^* quasi n- normal operator if the following conditions are satisfied $T_1T_2 = T_2T_1$, $T_2^*T_1 = T_1 T_2^*$

Proof

$$\left((T_1T_2)^*\right)^K\left((T_1T_2)^*(T_1T_2)^n\right)=\left((T_2T_1)^*\right)^K\left((T_2T_1)^*(T_1T_2)^n\right)$$

$$=\left((T_1^*T_2^*)\right)^K\left((T_1^*T_2^*)(T_1T_2)^n\right)$$

$$=\left((T_1^*)^K(T_2^*)^K\right)\left((T_1^*T_2^*)(T_1^nT_2^n)\right)$$

$$=(T_1^*)^K\left((T_2^*)^KT_1^*\right)(T_2^*T_1^n)T_2^n$$

$$=(T_1^*)^K\left(T_1^*(T_2^*)^K\right)(T_1^nT_2^*)T_2^n$$

$$=\left((T_1^*)^KT_1^*\right)\left((T_2^*)^KT_1^n\right)(T_2^*T_2^n)$$

$$=\left((T_1^*)^KT_1^*\right)\left(T_1^n(T_2^*)^K\right)(T_2^*T_2^n)$$

$$=\left((T_1^*)^KT_1^*T_1^n\right)\left((T_2^*)^KT_2^*T_2^*\right)$$

$$=\left(T_1^*T_1^n(T_1^*)^K\right)\left(T_2^*T_2^n(T_2^*)^K\right)$$

$$=(T_1^*T_1^n)\left((T_1^*)^KT_2^*\right)\left(T_2^n(T_2^*)^K\right)$$

$$=(T_1^*T_1^n)\left(T_2^*(T_1^*)^K\right)\left(T_2^n(T_2^*)^K\right)$$

$$=T_1^*(T_1^nT_2^*)\left((T_1^*)^KT_2^n\right)(T_2^*)^K$$

$$=T_1^*(T_2^*T_1^n)\left(T_2^n(T_1^*)^K\right)(T_2^*)^K$$

$$=(T_1^*T_2^*)(T_1^nT_2^n)\left((T_1^*)^K(T_2^*)^K\right)$$

$$=(T_2T_1)^*(T_1T_2)^n\left((T_1^*)(T_2^*)\right)^K$$

$$=(T_2T_1)^*(T_1T_2)^n\left((T_2T_1)^*\right)^K$$

$$=(T_1T_2)^*(T_1T_2)^n\left((T_1T_2)^*\right)^K$$

$$\therefore\left((T_1T_2)^*\right)^K(T_1T_2)^*(T_1T_2)^n=(T_1T_2)^*(T_1T_2)^n\left((T_1T_2)^*\right)^K$$

Hence the product T_1T_2 is K^* quasi n normal operator

3.10Theorem

Let $T:H\rightarrow H$ be K^* quasi n- normal operator such that $T^nT^*=T^*T^n$ Then T^* is K^* quasi n- normal operator

Proof

$$\left((T^*)^*\right)^K\left((T^*)^*(T^*)^n\right)=\left((T^*)^K\right)^*\left((T^*)^*(T^n)^*\right)$$

$$=\left((T^*)^K\right)^*\left(T^nT^*\right)^*$$

$$=\left((T^nT^*)(T^*)^K\right)^*$$

$$=\left((T^*)^K(T^nT^*)\right)^*$$

$$=\left((T^*)^K(T^*T^n)\right)^*$$

$$=\left((T^*T^n)(T^*)^K\right)^*$$

$$=\left((T^*)^K\right)^*\left(T^*T^n\right)^*$$

$$=\left((T^*T^n)(T^*)^K\right)^*$$

$$=\left((T^*)^K(T^*T^n)\right)^*$$

$$=\left(T^*T^n\right)\left((T^*)^K\right)^*$$

$$=\left((T^*)^*(T^*)^n\right)\left((T^*)^*\right)^K$$

$$\therefore\left((T^*)^*\right)^K\left((T^*)^*(T^*)^n\right)=\left((T^*)^*(T^*)^n\right)\left((T^*)^*\right)^K$$

Hence the T^* is K^* quasi n- normal operator

3.11Proposition

Let $T_1,T_2,...T_n$ are K^* quasi n- normal operators from a Hilbert space H then $T_1\oplus T_2\oplus...\oplus T_n$ is also K^* quasi n- normal operator

Proof

$$\left((T_1\oplus T_2\oplus...\oplus T_n)^*\right)^K\left((T_1\oplus T_2\oplus...\oplus T_n)^*(T_1\oplus T_2\oplus...\oplus T_n)^n\right)$$

$$=\left(T_1^*\oplus T_2^*\oplus...\oplus T_n^*\right)^K\left((T_1^*\oplus T_2^*\oplus...\oplus T_n^*)(T_1^n\oplus T_2^n\oplus...\oplus T_n^n)\right)$$

$$=\left((T_1^*)^K\oplus(T_2^*)^K\oplus...\oplus(T_n^*)^K\right)\left(T_1^*T_1^n\oplus T_2^*T_2^n\oplus...\oplus T_n^*T_n^n\right)$$

$$=\left((T_1^*)^KT_1^*T_1^n\oplus(T_2^*)^KT_2^*T_2^n\oplus...\oplus(T_n^*)^KT_n^*T_n^n\right)$$

$$=\left(T_1^*T_1^n(T_1^*)^K \oplus T_2^*T_2^n(T_2^*)^K \oplus \ldots \oplus T_n^*T_n^n(T_n^*)^K\right)$$

$$=\left(T_1^*T_1^n \oplus T_2^*T_2^n \oplus \ldots \oplus T_n^*T_n^n\right)\left((T_1^*)^K \oplus (T_2^*)^K \oplus \ldots \oplus (T_n^*)^K\right)$$

$$=\left((T_1^* \oplus T_2^* \oplus \ldots \oplus T_n^*)(T_1^n \oplus T_2^n \oplus \ldots \oplus T_n^n)\right)\left((T_1^*)^K \oplus (T_2^*)^K \oplus \ldots \oplus (T_n^*)^K\right)$$

$$=\left((T_1 \oplus T_2 \oplus \ldots \oplus T_n)^*(T_1 \oplus T_2 \oplus \ldots \oplus T_n)^n\right)\left((T_1^* \oplus T_2^* \oplus \ldots \oplus T_n^*)^K\right)$$

$$=\left((T_1 \oplus T_2 \oplus \ldots \oplus T_n)^*(T_1 \oplus T_2 \oplus \ldots \oplus T_n)^n\right)\left((T_1 \oplus T_2 \oplus \ldots \oplus T_n)^*\right)^K$$

Then $T_1 \oplus T_2 \oplus \ldots \oplus T_n$ is K^* quasi n- normal operator

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