On Quasi-invertibility and Quasi-similarity of Operators in Hilbert Space.

Kikete D.W.*, Luketero S.W., and Waihenya S.K

School of Mathematics, University of Nairobi, P.O. Box 30197, 00100 NAIROBI, KENYA.

* E-mail of the corresponding author: kiketedn@gmail.com

Abstract

In this paper we show that if two operators A and B are quasi-invertible then AB and BA are also quasi-similar. We also show that if two operators S and T are isometric ST is consistent in invertibility under further hypothesis.

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INTRODUCTION

Let H be a complex Hilbert space and $B(H)$ denote the Banach algebra of all bounded linear operators on H. The term operator is meant to imply boundedness and linearity. An operator $X \in B(H)$ is said to be a quasiaffinity if X is both one-to-one and has a dense range. Then two operators A and B are said to be similar if there exists an invertible operator S such that $AS = SB$, while A and B are said to be quasisimilar if there exists quasiaffinities X and Y such that $AX = XB$ and $BY = YA$.

For an operator $B \in B(H)$, we say that B is consistent in invertibility (CI) if for each $A \in B(H)$, AB and BA are invertible or non-invertible together. We show that if two operators A and B are quasi-invertible then AB and BA are quasisimilar. We also show that if two operators S and T are isometric ST is consistent in invertibility under further hypothesis.

THEOREM 1

Let $A, B \in B(H)$ be quasi-invertible operators then AB and BA are quasisimilar operators.

Proof

Consider the equations

$(BA)B = B(AB)$ and $(AB)A = A(BA)$

Let $H = BA$ and $K = AB$. Then we have that: $HB = BK$ and $KA = AH$.

Thus H and K are quasi-similar. Hence AB and BA are quasi-similar operators.

Corollary 1

Let $A, B \in B(H)$ be quasi-invertible operators then $\sigma(AB) = \sigma(BA)$ under any of the following conditions.

i. AB and BA are hyponormal

ii. AB is dominant and $(BA)^*$ is M-hyponormal

iii. AB and BA are p – hyponormal with U and V unitary in the polar decomposition $AB = U|AB|$ and $BA = V|BA|$

Thus, we also have $\sigma_e(AB) = \sigma_e(BA)$

Corollary 2

If A is a quasi-invertible operator, then we have that $\sigma_e(AA^*) = \sigma_e(A^*A)$

Proof

First note that if A is quasi-invertible, then A^* is also quasi-invertible. Hence by theorem 1 above, we have that AA^* and A^*A are quasi-similar. But $AA^* \geq 0$ and $A^*A \geq 0$. Hence by corollary 1, we have $\sigma_e(AA^*) = \sigma_e(A^*A)$.

An operator $B \in B(H)$ is a ***CI* operator** if for each $A \in B(H)$, AB and BA are invertible or non-invertible together. Thus B is a *CI* operator if and only if $\sigma(AB) = \sigma(BA)$. It has been shown by Halmos P.R. that if B is invertible then for any $A \in B(H)$ we have $AB = B^{-1}(BA)B$. Thus AB and BA are similar operators and hence $\sigma(AB) = \sigma(BA)$.

Corollary 3

Let $B \in B(H)$ be quasi-invertible, then B is a *CI* operator.

Proof

By corollary 2, we have that $\sigma(B^*B) = \sigma(BB^*)$. Hence *B* is a *CI* operator.

Corollary 4

Let $B \in B(H)$ be such that $0 \notin W(B)$, i.e. 0 does not belong to the numerical range of *B*, then *B* is a *CI* operator.

Proof

First note that if $0 \notin W(B)$ then both *B* and *B** are quasi-invertible. Hence by corollary 3 above both *B* and *B** are *CI* operators.

THEOREM 2

If *B* is an M – hyponormal operator satisfying the operator equation $BX = XB^*$ where *X* is a quasi-invertible operator, then *B* is a *CI* operator.

Proof

Since *B* is M – hyponormal, $BX = XB^*$ implies $B^*X = XB$.

Using the operator equation above, we have that:

$$B^*BX = B^*XB^* = XBB^*$$

And $$BB^*X = BXB = XB^*B$$

Hence BB^* and B^*B are quasi-similar. Thus $\sigma(B^*B) = \sigma(BB^*)$ and *B* is a *CI* operator.

Corollary 5

If an M – hyponormal operator *B* is quasisimilar to its adjoint, then *B* is a *CI* operator.

THEOREM 3

Let *S* and *T* are isometric operators. Then the operator *ST* is a *CI* operator if and only if both *S* and *T* are unitary.

Proof

Both *S* and *T* are isometric implies that *ST* is also an isometry. Hence

$$(ST)^*ST = I$$

$$T^*S^*ST = I$$

$T^*(S^*S)T = I$

But S and T are unitary, hence $S^*S = I = SS^*$ and $T^*T = I = TT^*$ thus

$T^*(S^*S)T = S^*S$ and $T^*(SS^*)T = S^*S$

Since S is unitary.

Hence SS^* is similar to S^*S and therefore $\sigma(S^*S) = \sigma(SS^*)$. Similarly, $\sigma(T^*T) = \sigma(TT^*)$. From which it follows that both S and T are *CI* operators. Also S^* and T^* are *CI* operators.

Corollary 6

If A and B are normal operators and $AB^* = B^*A$, then $A+iB$ is a *CI* operator.

Proof

$AB^* = B^*A$ implies $(AB^*)^* = (B^*A)^*$ i.e. $BA^* = A^*B$

It is sufficient to show that $A+iB$ is also normal

$$(A+iB)^* = A^* - iB^*$$

$$\begin{aligned}(A+iB)^*(A+iB) &= (A^* - iB^*)(A+iB) \\ &= A^*A + iA^*B - iB^*A + B^*B \\ &= A^*A + i(A^*B - B^*A) + A^*B\end{aligned}$$

Also

$$\begin{aligned}(A+iB)(A+iB)^* &= (A+iB)(A^* - iB^*) \\ &= A^*A - iAB^* + iBA^* + BB^* \\ &= A^*A + i(BA^* - AB^*) + BB^* \\ &= A^*A + i(A^*B - B^*A) + A^*B \quad \text{by normality of both } A \text{ and } B.\end{aligned}$$

Hence $(A+iB)(A+iB)^* = (A+iB)^*(A+iB)$. Thus $(A+iB)$ is a *CI* operator.

THEOREM 4

Let $A, B, X \in B(H)$ satisfy the operator equation $AXB = X$, where X is a quasi-invertible operator. Further, let A and B be quasi-normal operators, then A and B^* are *CI* operators.

Proof

Since A is quasi-normal, we have $[A^*A, A] = 0$ i.e. $A^*AA - AA^*A = 0$

By $AXB = X$ it follows that

$$AA^*AXB = AA^*X$$

$$A^*AAXB = AA^*X$$

$$A^*AX = A^*AX$$

Thus $A^*AX - A^*AX = 0$ implies $(A^*A - A^*A)X = 0$. Therefore, $A^*A - A^*A = 0$ since the operator X has dense range. Hence A is a CI operator.

Further, if B is quasi-normal, then $[BB^*, B] = 0$ and therefore $BB^*B = BBB^*$. By the hypothesis that $AXB = X$, it follows that:

$$AXBB^*B = XB^*B$$

$$AXBBB^* = XB^*B$$

$$XBB^* = XB^*B$$

Thus $X(BB^* - B^*B) = 0$. Hence $BB^* = B^*B$ since X has dense range. Therefore, B^* is a CI operator.

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