

VARIATIONAL ITERATION METHOD FOR CAUCHY PROBLEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATION.

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Abstract

In this paper, we consider the numerical treatment of the nonlinear parabolic-hyperbolic partial differential equation of the form (P-H PDE)

$$u_t - \Delta u_{tt} - \Delta - Fu = 0,$$

where Fu is the nonlinear term, and $\Delta \in \mathbb{R}^1$ is a Laplace operator. Here, the variational iteration method (VIM) was employed to examine the convergence of solution of the nonlinear P-H PDE. Results obtained showed that there is a rapid rate of convergence of the approximate solution to the exact solution as the number of iterations increases. All computational framework of the research were performed using Maple 18 software.

Keywords: Variational iteration method, Nonlinear PDE, Lagrange Multiplier, Linear differential operator

1. Introduction

Most physical problems that are expressed using more than one variable involve partial derivatives. They are even more significant in modeling real life situations such as shock waves (Burgers equation), gas dynamics (gas dynamic equation), steady state distribution of heat in a two – dimensional plain (Poisson equation), steady state problems involving incompressible fluid in a two – dimensional plain (Poisson equation), the effect of gravitational force on the potential energy of a point in a two – dimensional region (Poisson equation) etc. The parabolic partial differential equation (P-PDE) is very significant in the study of gas diffusion, which is generally called the diffusion equation.

Most conventional analytic solvers (such as the integral transform method (ITM), method of characteristic (MOC), separation of variable methods, and the change of variable methods, etc) for partial differential equation over the years proved complex and difficult to handle, and does not have a precise and concise solution that can effectively and sufficiently interpret the external and internal variables of the model in consideration. This defect has attributed to the use of numerical methods by researchers in recent years for the approximation of the analytic solutions of partial differential equations . Popular numerical schemes developed and implemented over the years for partial differential equations include; the finite difference method, the finite element method, the cranck – Nicolsone method, the Bender - Schimdt method etc.

The variational iteration method (IVM) was first proposed by the Chinese mathematician, J.H. He in 1998. The method has proved effective in solving both linear and nonlinear problems in differential equations, integral equations, boundary value problems, initial value problems, integro-differential problems etc. For instance, He (2000) seeks the numerical solution of autonomous ordinary differential systems via the variational iteration method. He (1998) also adopted the variational iteration method for the treatment of nonlinear problems and applications respectively. Similarly, Saberi – Nadjafi et al (2008) employed the VIM for the numerical solution of system of integro – differential equations. Martinfar et. al. (2010) equally applied the VIM on systems of integro – differential equation. The convergence analysis of VIM has been reported in Mamadu and Njoseh (2016) for nonlinear integro-differential equations where some sufficient conditions were derived.

This research reveals the numerical application of the variational iteration method to nonlinear partial differential equations. At this point, we consider the special case of nonlinear parabolic-hyperbolic equation (PDE) of the form (Duangpithak, 2012)

$$u_t - \Delta u_{tt} - \Delta - Fu = 0, \tag{1}$$

where Fu is the nonlinear term, and $\Delta \in \mathbb{R}^1$ is a Laplace operator.

2. Basic ideas of VIM

Let the nonlinear equation be given as

$$Lu + Ru + Nu = r(x) \tag{2}$$

with some prescribed auxiliary conditions, u is an unknown function, L is a linear differential operator of the highest order, R is also a linear differential operator of order less than L, N is a nonlinear term and $r(x)$ is the source term.

The variational iteration method involves the construction of a correction functional for (2) given as

$$u_{k+1} x = u_k x + \int_0^x \lambda s[Lu_k s + Ru_k s + N\tilde{u}_k s - r(s)]ds, r \geq 0 \tag{3}$$

where λs is called the general langrange multiplier obtained optimally via the variational theorem, and $\tilde{u}_k s = 0$, called the restricted variable.

3. VIM for Nonlinear PDE

The variational iteration scheme for (1) becomes:

$$u_{k+1} x, t = u_k x, t + \int_0^t \lambda\, x, s[(u_k)_s - \Delta (u_k)_{ss} - \Delta - G(u_k)]\, ds, k \geq 0 \tag{4}$$

For convenience sake, we can rewrite (4) as:

$$u_{k+1} x, t = u_k x, t + \int_0^t \lambda\, x, s \left[\frac{\partial}{\partial s} - \Delta \frac{\partial^2}{\partial s^2} - \Delta\, u_k x, s - G(u_k x, s)\right] ds \tag{5}$$

Equation (5) can be transformed to the form;

$$u_{k+1} x, t = u_k x, t + \int_0^t \lambda\, x, s[u_k s - k_1 u_k]\, ds, \tag{6}$$

Thus, taking the variation δ on both sides of (6) to get;

$$\delta\, u_{k+1} x, t = \delta u_k x, t + \delta \int_0^t \lambda\, x, s[u_k s - k_1 u_k]\, ds.$$

$$\Rightarrow \quad \lambda' x, s + k_1 \lambda x, s = 0 \tag{7}$$

$$1 + \lambda x, s|_{s=t} = 0 \tag{8}$$

Solving the linear differential equation (1.7) and using the initial condition (8), we obtain

$$\lambda x, s = -e^{-k_1 (s-t)},\ k_1 \neq 0. \tag{9}$$

Thus, the variational scheme for the nonlinear parabolic – hyperbolic Partial differential equation (1) is given as

$$u_{k+1}x,t = u_k x,t - \int_0^t e^{-k_1(s-t)}\left[\frac{\partial}{\partial s} - \Delta\frac{\partial^2}{\partial s^2} - \Delta\, u_k x,s - G u_k x,s\right] ds, k \geq 0. \qquad (10)$$

4. Numerical Applications

Example 4.1 (Duangpithak, 2012):

Consider

$$u_t - u_{xx}u_{tt} - u_{xx} = u_{xx} + u_{tt} - u_t + u\,, \qquad (11)$$

with initial conditions

$$u(x,0) = e^x,$$

and analytic solution given as $u(x,t) = e^{x+t}$.

Using the variational iterative method (10), we have,

$$u_{k+1}x,t = u_k x,t - \int_0^t e^{-(s-t)}\left[\frac{\partial}{\partial s}u_k x,s - \frac{\partial^2}{\partial x^2}u_k x,s\frac{\partial^2}{\partial \xi^2}u_k x,s - \frac{\partial^2}{\partial x^2}u_k x,s - \frac{\partial^2}{\partial x^2}u_k x,s + \frac{\partial^2}{\partial s^2}u_k x,s - \frac{\partial}{\partial s}u_k x,s + u_k x,s\right] ds, k \geq 0 \qquad (12)$$

where $k_1 = 1$ by mutual comparison with the exact.

Our initial approximation is given as $u_0 x,0 = e^x$.

Thus, executing (12) with the help of Maple 18 software for $k \geq 0$, we obtain the following approximations for the problem (4.1):

$$u_1 := -e^x - e^{t+2x} + e^{2x} + 2e^{t+x}$$

$$u_2 := e^x + 8e^{t+2x} - 3e^{2x} - 4e^{t+x} - 9e^{3t+4x} - 6e^{2t+3x} + 33e^{2t+4x} + 14e^{t+3x} - 40e^{t+4x} - 5e^{2t+2x} + 4e^{2t+x} - 8e^{3x} + 16e^{4x}$$

$$\begin{aligned}u_3 := {} & -e^x - 36864e^{7x} + 2240e^{5x} + 366864e^{3t+7x} - 624e^{5t+4x} + 8484e^{5t+5x} \\ & - 202908e^{4t+7x} - 1656e^{6t+5x} - 630e^{6t+6x} + 56808e^{5t+7x} - 6156e^{6t+7x} \\ & + 504e^{4t+3x} + 11066e^{2t+5x} - 13400e^{4t+5x} + 180608e^{t+7x} - 9584e^{t+5x} \\ & + 2850e^{3t+5x} - 120e^{5t+3x} - 358352e^{2t+7x} - 372736e^{t+8x} + 65536e^{8x} \\ & + 11520e^{t+6x} - 960e^{6x} - 48e^{5t+2x} + 48e^{4t+2x} + 886848e^{2t+8x} \\ & - 1140528e^{3t+8x} + 44616e^{3t+6x} - 34488e^{2t+6x} - 366390e^{5t+8x} \\ & + 851832e^{4t+8x} + 7650e^{5t+6x} - 27708e^{4t+6x} - 7371e^{7t+8x} + 82809e^{6t+8x} \\ & + 16e^{3t+x} - 19e^{3t+2x} - 638e^{3t+3x} + 2449e^{4t+4x} + 48e^{3x} - 256e^{4x} \\ & + 54e^{2t+2x} - 20e^{2t+x} + 560e^{t+4x} + 854e^{2t+4x} - 134e^{t+3x} - 2983e^{3t+4x} \\ & + 340e^{2t+3x} + 13e^{2x} - 48e^{t+2x} + 6e^{t+x}\end{aligned}$$

⋮

Example 4.2 (Duangpithak, 2012):

Consider

$$u_t - u_{x_1x_1} - u_{x_2x_2}u_{tt} - u_{x_1x_1} - u_{x_2x_2} = u_{x_1} - u_{tt} + u \qquad (13)$$

with initial conditions

$$u(x_1, x_2, 0) = e^{x_1+x_2},$$

The exact solution is given as

$$u(x, t) = e^{x_1+x_2+t}.$$

By VIM , we have,

$$u_{k+1}(x, t) = u_k(x, t) - \int_0^t e^{-2(s-t)} \left[\frac{\partial}{\partial s} u_k(x, s) - \frac{\partial^2}{\partial x_1^2} u_k(x, s) - \frac{\partial^2}{\partial x_2^2} u_k(x, s) \frac{\partial^2}{\partial \xi^2} u_k(x, s) - \frac{\partial^2}{\partial x_1^2} u_k(x, s) - \frac{\partial^2}{\partial x_2^2} u_k(x, s) - \frac{\partial}{\partial x_1} u_k(x, s) + \frac{\partial^2}{\partial s^2} u_k(x, s) + 2u_k(x, s)\right] ds, k \geq 0 \quad (14)$$

The initial approximation is given as $u_0(x_1, x_2, 0) = e^{x_1+x_2}$.

Executing (14) with the help of Maple 18 software for $k \geq 0$, we obtain the following approximations for the problem (4.2):

$$u_1 := -e^{x_1+x_2} - 4e^{t+2x_1+2x_2} + 4e^{2x_1+2x_2} + 2e^{t+x_1+x_2}$$

$$\begin{aligned} u_2 := {} & e^{x_1+x_2} + 20e^{t+2x_1+2x_2} - 12e^{2x_1+2x_2} - 2e^{t+x_1+x_2} + 1024e^{4x_1+4x_2} \\ & + 112e^{3t+3x_1+3x_2} - 352e^{2t+3x_1+3x_2} + 2576e^{2t+4x_1+4x_2} + 368e^{t+3x_1+3x_2} \\ & - 784e^{3t+4x_1+4x_2} - 4e^{3t+2x_1+2x_2} - 2816e^{t+4x_1+4x_2} + 2e^{2t+x_1+x_2} \\ & - 4e^{2t+2x_1+2x_2} - 128e^{3x_1+3x_2} \end{aligned}$$

$$\begin{aligned} u_3 := {} & -e^{x_1+x_2} - 465884160e^{6t+7x_1+7x_2} + 1424e^{4t+3x_1+3x_2} \\ & - 3315309056e^{4t+7x_1+7x_2} + 3821065728e^{3t+7x_1+7x_2} + 5701056e^{2t+5x_1+5x_2} \\ & + 81750400e^{4t+6x_1+6x_2} - 2594766848e^{2t+7x_1+7x_2} - 150994944e^{7x_1+7x_2} \\ & - 129344e^{6t+5x_1+5x_2} - 141120e^{5t+5x_1+5x_2} - 983040e^{6x_1+6x_2} \\ & + 1689429504e^{5t+7x_1+7x_2} - 67829120e^{3t+6x_1+6x_2} - 51585280e^{5t+6x_1+6x_2} \\ & + 3443390720e^{6t+8x_1+8x_2} - 409975552e^{7t+8x_1+8x_2} + 1073741824e^{8x_1+8x_2} \\ & - 6794772480e^{t+8x_1+8x_2} + 23511541760e^{4t+8x_1+8x_2} - 1908736e^{t+6x_1+6x_2} \\ & + 6912e^{4t+4x_1+4x_2} - 12166350336e^{5t+8x_1+8x_2} + 2238016e^{4t+5x_1+5x_2} \\ & - 48e^{5t+3x_1+3x_2} - 21792e^{5t+4x_1+4x_2} + 18237423616e^{2t+8x_1+8x_2} \\ & + 16256128e^{6t+6x_1+6x_2} - 1963136e^{7t+6x_1+6x_2} + 963248128e^{t+7x_1+7x_2} \\ & - 26894999552e^{3t+8x_1+8x_2} + 6416e^{6t+4x_1+4x_2} + 80e^{7t+4x_1+4x_2} \\ & + 26262784e^{2t+6x_1+6x_2} + 768e^{3x_1+3x_2} + 92e^{2t+2x_1+2x_2} + 67008e^{t+4x_1+4x_2} \\ & + 53211648e^{7t+7x_1+7x_2} + 4e^{3t+2x_1+2x_2} - 2896e^{t+3x_1+3x_2} \\ & + 59728e^{3t+4x_1+4x_2} - 101968e^{2t+4x_1+4x_2} - 3744e^{3t+3x_1+3x_2} \\ & + 4576e^{2t+3x_1+3x_2} - 16384e^{4x_1+4x_2} - 80e^{6t+3x_1+3x_2} + 13440e^{7t+5x_1+5x_2} \\ & - 5355520e^{3t+5x_1+5x_2} + 573440e^{5x_1+5x_2} - 2899968e^{t+5x_1+5x_2} \\ & - 16e^{4t+2x_1+2x_2} + 52e^{2x_1+2x_2} + 2e^{t+x_1+x_2} - 132e^{t+2x_1+2x_2} \end{aligned}$$

⋮

5. Tables of Results and Graphical simulations

Table 1: Comparison of results between the exact solution and VIM Example 4.1

x	$Exact$	VIM, u_2	Error
0.00	1.0000000	1.0000000	0.0000e+00
0.10	1.1051709	1.1051709	2.0000e-09
0.20	1.2214028	1.2214028	2.0000e-09
0.30	1.3498588	1.3498589	7.2000e-08
0.40	1.4918247	1.4918248	6.2000e-08
0.50	1.6487213	1.6487214	1.2900e-07
0.60	1.8221188	1.8221187	1.0000e-07
0.70	2.0137527	2.0137527	7.0000e-09
0.80	2.2255409	2.2255409	2.8000e-08
0.90	2.4596031	2.4596031	1.1000e-08
1.00	2.7182818	2.7182819	7.2000e-08

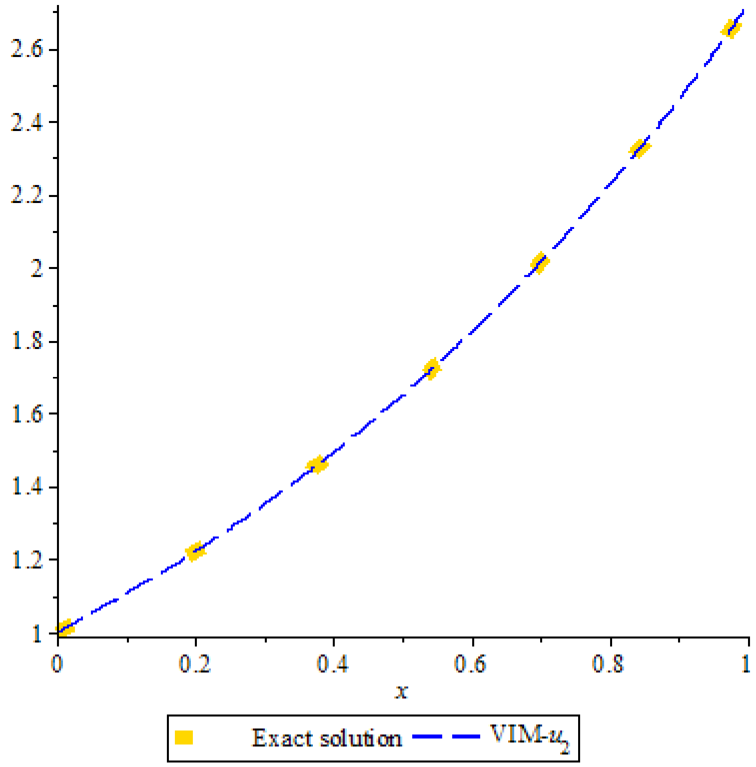

Figure 1. Graphical simulation of the exact solution versus VIM for Example 4.1

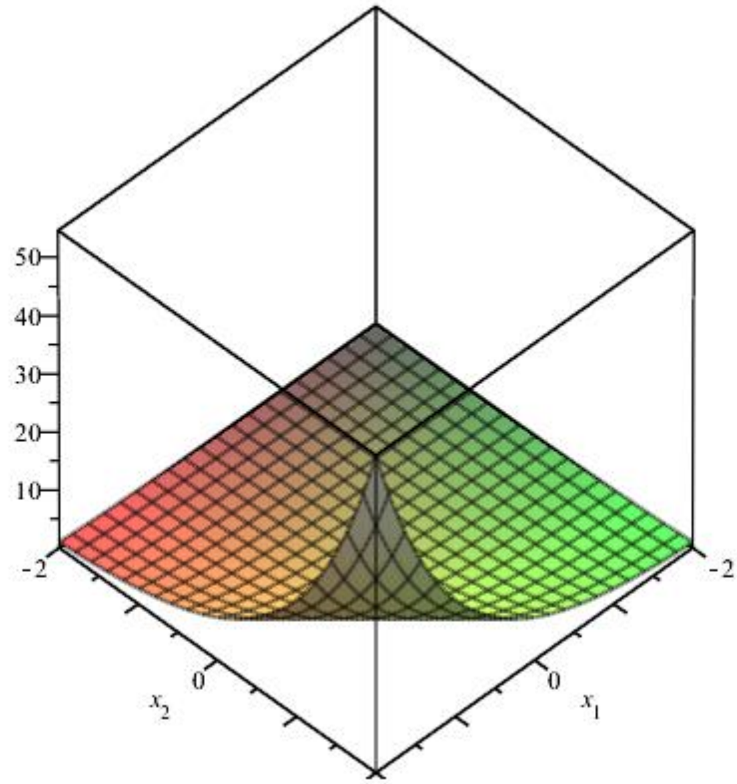

Figure 2a. Graphical simulation of the exact solution at t=0 for Example 4.2

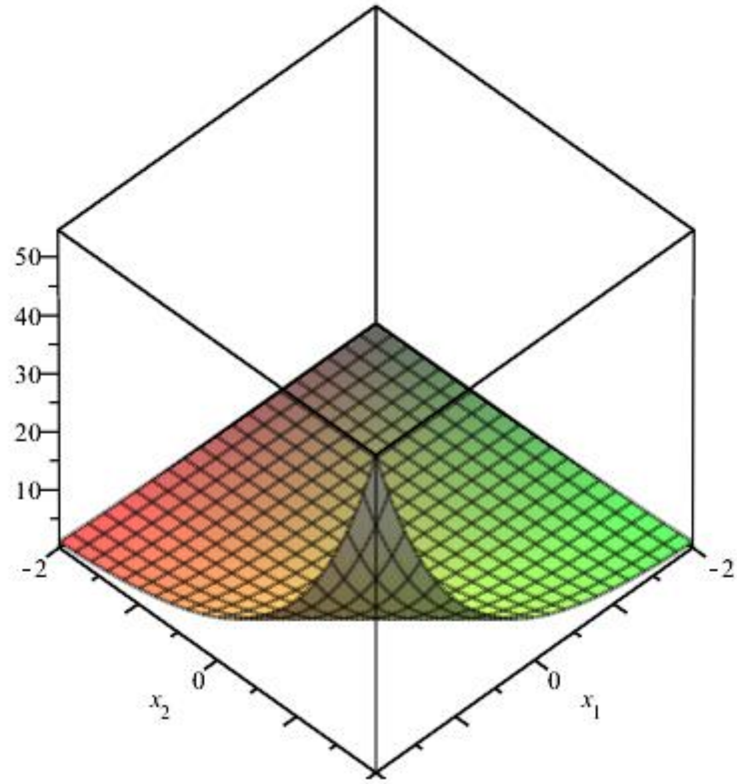

Figure 2b. Graphical simulation of the approximate solution u_3 at t=0 for Example 4.2 with VIM

6. Discussion of Results

We have successively applied the variational iteration method (VIM) to the various forms of (1.1). Numerical evidences from VIM were compared with the exact solution for accuracy and convergence. In **Example 4.1**, it is observed from table 1 and figure 1 that VIM converges rapidly to the exact solution with a maximum error of order 10^{-9}. This is also evident in the graphical simulation of the problem as shown in figure 1.

Similarly, In the **Examples 4.2,** we attained an absolute convergence at the initial value of t. Here, the VIM converges absolutely to the exact solution as shown in the the figures 2a and 2b respectively.

7. Conclusion

We have successively applied the variational iteration method (MVIM) to the various forms of (1.1) .Our results have shown that the variational iterative method (VIM) encourages rapid convergence for the Cauchy problem of nonlinear parabolic-hyperbolic partial differential equation. We have applied VIM for the problem (1.1) without any form of discretization or perturbation.

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