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SOME RESULTS ON OPERATORS CONSISTENT IN INVERTIBILITY

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Abstract

In this paper, we investigate the conditions under which some classes of operators in a complex Hilbert space H are said to be consistent in invertibility.

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1. INTRODUCTION

In this paper, Hilbert spaces or subspaces will be denoted by capital letters, H and K respectively and T, S, A, B etc. denotes bounded linear operators where an operator means a bounded linear transformation, (H) will denote the Banach algebra of bounded linear operators on H. $B(H, K)$ denotes the set of bounded linear transformations from H to K, which is equipped with the (induced uniform) norm. If $T \in B(H)$, then T^* denotes the adjoint while $\mathrm{Ker}(T)$ denotes the kernel of T. For an operator T, we also denote by (T) the spectrum of T.

An operator $T \in (H)$ is said to be:

- *Invertible* if it has zero kernel
- *Quasi-invertible* if it is injective and has a dense range
- *Positive* if $T \geq 0$
- *Projection* if $T^2 = T$
- *Normal* if $T^*T = TT^*$
- *Quasinormal* if $T^*TT = TT^*T$

- *Consistent in invertibility (C.I)* if both TS and ST are either invertible or non-invertible together.

2. RESULTS

Theorem 2.1

Let $T \in B(H)$. If Ker$T$ = 0 = KerT^*, then T is a C.I operator.

Proof

If Ker$T$ = 0, we have that T is invertible, it follows that T^* is also invertible.

Since TT^* is a product of invertible operators it has to be invertible too. We also have that $(TT^*)^*$ is invertible.

But $(TT^*)^* = T^*T$. Thus both TT^* and T^*T are invertible together. Hence T is a C.I operator.

Corollary 2.2

Let $T \in B(H)$ be quasi-invertible. Then T is a C.I operator.

Proof

If T is quasi-invertible, it follows that it is injective and has a dense range. As a consequence of being injective, we have that Ker$T$ = 0 therefore T is a C.I operator.

Corollary 2.2

Let $T^* \in B(H)$ be such that $0 \notin W(T^*)$. Then T^* is a C.I. operator.

Proof

Recall that $\sigma(T^*) \subseteq W(T^*)$

Therefore $0 \notin W(T^*) \Rightarrow 0 \notin \sigma(T^*) \Rightarrow 0$ in not an eigenvalue of $T^* \Rightarrow T^*$ is invertible $\Rightarrow T^*$ is a C.I. operator.

Theorem 2.3

Let $A, B \in B(H)$ be normal operators and $AB^* = B^*A$, then $A+iB$ is a C.I. operator.

Proof

$AB^* = B^*A \Rightarrow (AB^*)^* = (B^*A)^*$ i.e. $BA^* = A^*B$. It is enough to show that $A+iB$ is normal.

$(A+iB)^* = A^* - iB^*$

$$(A+iB)^*(A+iB) = (A^* - iB^*)(A+iB)$$
$$= A^*A + iA^*B - iB^*A + B^*B$$
$$= (A^*A + B^*B) + i(A^*B - B^*A)$$
$$= (A^*A + B^*B) + i(BA^* - AB^*) \text{...............(i)}$$

$$(A+iB)(A+iB)^* = (A+iB)(A^* - iB^*)$$
$$= AA^* - iAB^* + iBA^* + BB^*$$
$$= (AA^* + BB^*) + i(BA^* - AB^*) \text{.............(ii)}$$

From (i) and (ii) above it follows that $A+iB$ is normal, hence a C.I. operator.

Theorem 2.4

Let $A, B, X \in B(H)$ satisfy the operator equation $AXB = X$ where X is a quasi-invertible operator. Further, let A and B be quasinormal operators, then A and B^* are C.I. operators.

Proof

Since A is quasinormal, we have $A^*AA - AA^*A = 0$. By the hypothesis that $AXB = X$ it follows that:

$$AA^*AXB = AA^*X$$
$$A^*AAXB = AA^*X$$
$$A^*AX = AA^*X \quad \text{since } AXB = X$$
$$A^*A = AA^* \quad \text{since } X \text{ has a dense range}$$

Therefore, A is a normal operator, hence consistent in invertibility.

It can similarly be shown that B^* is consistent in invertibility.

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