Random Fixed Points For Occasionally Weakly Compatible Mappings

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Abstract

In this paper, we obtain common random fixed point theorems for two pairs of occasionally weakly compatible self random mappings under contractive conditions involving two generalized altering distance functions in a complete separable metric space.

Keywords: Common random fixed point, Complete separable metric space, Generalized altering distance function, Occasionally weakly compatible mappings, Point of coincidence.

1. Introduction and preliminaries

During the last fifty years there have been so many exciting developments in the field of random operator theory. Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Spacek [21] and Hans [11], [12]. The survey article by Bharucha-Reid [7] in 1976 attracted the attention of several mathematician and gave wings to this theory. Itoh [13] extended Spacek's and Hans's theorem to multivalued contraction mappings.

The Banach contraction mapping principle is one of the pivotal results of analysis. There are a lot of the generalizations of the Banach contraction mapping principle in the literature (see e.g. [1], [2], [9], [10], [18], [19], [20]) and others. Khan et al. [16] addressed a new category of contractive fixed point problems for single self-map with the help of a control function that alters distance between two points in a metric space which they called an altering distance.

Definition 1.1 [16]
A function $\varphi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following conditions are satisfied:

1. $\varphi(t)=0 \Leftrightarrow t=0$,
2. φ is a continuous monotonically non-decreasing.

Khan et al. [16] proved the following result:

Theorem 1.2 [16]
Let (X,d) be a complete metric space, $\varphi:[0,\infty)\to[0,\infty)$ be an altering distance function, and $T:X\to X$ be a self-mapping which satisfies the following inequality:

$$\varphi(d(Tx,Ty))\le c\varphi(d(x,y)), \qquad (1)$$

for all $x,y\in X$ and for some $0<c<1$. Then T has a unique fixed point.

Remark 1.3
Letting $\varphi(t)=t$ in Theorem 1.2, we obtain the Banach contraction principle.
Alber and Guerre-Delabriere [2] introduced the notion of weakly contractive mappings in Hilbert spaces.

Definition 1.4 (Weakly contractive mapping).
Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be a weakly contractive if for $x,y\in X$

$$d(Tx,Ty)\le d(x,y)-\varphi(d(x,y)), \qquad (2)$$

where $\varphi:[0,\infty)\to[0,\infty)$ is an altering distance function.

Remark 1.5 If we put $\varphi(t)=kt$, where $0<k<1$, then (2) reduces to (1).

Rhoades [20] extended some of Alber's and Guerre-Delabriere's work and obtained the following result:

Theorem 1.6
Let (X,d) is a complete metric space and $T:X\to X$ is a weakly contractive mapping, Then T has a unique fixed point.

Afterward, Beg and Abbas [6] proved a generalization of the corresponding theorem of Rhoades [20] for a pair of mappings in which one is weakly contractive with respect to the other . This is further generalized by Azam and Shakeel [4] in convex metric spaces.

Definition 1.7
Let X be a metric space . A mapping $T:X\to X$ is called weakly contractive with respect to $f:X\to X$ if for $x,y\in X$

$$d(Tx,Ty)\le d(fx,fy)-\varphi(d(fx,fy)),$$

where $\varphi:[0,\infty)\to[0,\infty)$ is continuous and nondecreasing such that φ is positive on $(0,\infty),\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$.
In [9] Choudhury introduced the concept of a generalized altering distance function in three variables which can extend to n variables defined as follows:

Definition 1.8
Let Ψ_n denote the set of all functions ψ satisfying the following conditions:

1. ψ is continuous.
2. ψ is monotone increasing in all the variables
3. $\psi(t_1,t_2,t_3,\dots,t_n)=0$ if and only if $t_1=t_2=t_3=\dots=t_n=0$.

We define $\varphi(x)=\psi(x,x,x,\dots,x)$ for $x\in[0,\infty)$. Clearly, $\varphi(x)=0$ if and only if $x=0$.
Examples of ψ

1. $\psi(t_1,t_2,t_3,\dots,t_n)=k\max\{t_1,t_2,t_3,\dots,t_n\}$ for $k>0$.
2. $\psi(t_1,t_2,t_3,\dots,t_n)=t_1^{a_1}+t_2^{a_2}+t_3^{a_3}+\dots+t_n^{a_n}, a_1,a_2,\dots,a_n\ge 1$.

In addition, Choudhury [9] proved the following common fixed point theorem:

Theorem 1.9 [9]
Let (X,d) be a complete metric space and $S,T:X\to X$ are two self mappings such that the following inequality is satisfied:

$$\varphi_1(d(Sx,Ty))\le\psi_1(d(x,y),d(x,Sx),d(y,Ty))-\psi_2(d(x,y),d(x,Sx),d(y,Ty)),$$

for all $x,y\in X$, where ψ_1 and ψ_2 are generalized altering distance functions and $\varphi_1(x)=\psi_1(x,x,x)$. Then S and T have a common fixed point.

Branciari [8] established the following fixed point theorem which opened the way of the study the mappings satisfying a contractive condition of integral type.

Theorem 1.10 [8]

Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ a mapping such that, for each $x, y \in X$

$$\int_0^{d(fx,fy)} \phi(t)dt \le c\int_0^{d(x,y)} \phi(t)dt,$$

where $\phi:[0,\infty) \to [0,\infty]$ is a Lebesgue-measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$ such that for $\varepsilon > 0, \int_0^{\varepsilon}\phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that $x \in X, \lim_{n\to\infty} f^n x = z$.

Throughout this paper, Let (Ω,Σ) be a measurable space, (X,d) is a complete separable metric space and C a nonempty closed subset of X. A mapping $\xi:\Omega \to C$ is called measurable if $\xi^{-1}(B\bigcap C) \in \Sigma$ for every Borel subset B of X. A mapping $T:\Omega\times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(.,x):\Omega \to C$ is measurable. A measurable mapping $\xi:\Omega \to C$ is called a random fixed point of the random mapping $T:\Omega\times C \to C$ if $T(w,\xi(w)) = \xi(w)$ for each $w \in \Omega$.

Definition 1.11
A mapping $\eta(w):\Omega \to X$ is said to be a random coincidence point of random operators $S,T:\Omega\times X \to X$ if $\eta(w)$ is measurable and $S(w,\eta(w)) = T(w,\eta(w)), w \in \Omega$. A measurable mapping $\xi(w):\Omega \to X$ is said to be a point of coincidence of S and T if there exists a measurable mapping $\eta(w):\Omega \to X$ so that $\xi(w) = S(w,\eta(w)) = T(w,\eta(w)), w \in \Omega$.

Definition 1.12 [14]
Let X be a separable complete metric space. Random operators $S,T:\Omega\times X \to X$ are weakly compatible if $T(w,\xi(w)) = S(w,\xi(w))$, for some measurable mappings ξ, then $T(w,S(w,\xi(w))) = S(w,T(w,\xi(w)))$ for every $w \in \Omega$.

Quite recently, Al-Thagafi and Shahzad [3] introduced the concept of occasionally weakly compatible mappings .

Definition 1.13 [3]
Two self-mappings $S,T:\Omega\times X \to X$ are said to be occasionally weakly compatible (owc) if and only if there exists a coincidence point of S and T at which S and T commute.

Remark 1.14
The notion of occasionally weakly compatible is a proper generalization of weakly compatible. Every weakly compatible mappings with coincidence points are occasionally weakly compatible, but the converse is not true (for example see [3]).

Lemma 1.15 [15]
Let X be a nonempty set, and let f and g be owc self-mappings of X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

The following lemma shows that contractive conditions of integral type can be considered as contractive conditions involving as altering distance.

Lemma 1.16
Let $\phi:[0,\infty) \to [0,\infty)$ be as in Theorem 1.10. Define $\Phi(b) = \int_0^b \phi(t)dt, b \in [0,\infty)$. Then Φ is an altering distance.

Proof. $\Phi:[0,\infty)\to[0,\infty)$ is well-defined and increasing since ϕ is Lebesgue measurable, summable and positive. Moreover, $\Phi(0)=0$ and $\Phi(b)>0$ for every $b>0$. The continuity of Φ follows from the continuity of the Lebesgue integral. The proof of the lemma is completed.

Random fixed point theorems for weakly compatible random operators under generalized contractive conditions are studied by Beg [5]. Afterward, Nashine [17] presented a random version improvement of Theorem 1 in [9].
In continuation of these results, we obtain several common random fixed point theorems for two pairs of occasionally weakly compatible random self mappings satisfying contractive inequalites which involving generalized altering distance functions.

2 . Main Results

Theorem 2.1

Let C be a nonempty closed subset of a separable complete metric space (X,d). Let S,T,E and $F:\Omega\times C\to C$ be four self random mappings defined on C such that for $x,y\in C$, $w\in\Omega$, the following conditions are satisfied:

$$S(w,X)\subseteq F(w,X),\ \ T(w,X)\subseteq E(w,X). \tag{3}$$

$$\begin{aligned}\varphi_1(d(S(w,x),T(w,y)))\le{}&\psi_1(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y)))\\ &-\psi_2(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y))),\end{aligned} \tag{4}$$

where $\psi_1,\psi_2\in\Psi_3$ and $\varphi_1(x)=\psi_1(x,x,x)$. Then each of the pairs (S,E) and (T,F) has a unique point of coincidence. Moreover, if each of the pairs (S,E) and (T,F) is owc, then S,T,E and F have a unique common random fixed point.

Proof. We will prove that

$$\lim_{n\to\infty} d(\xi_n(w),\xi_{n+1}(w))=0.$$

Let the function $\eta_0:\Omega\to C$ be an arbitrary measurable function on Ω. By (3) there exists a function $\eta_1:\Omega\to C$ such that for $w\in\Omega$, $F(w,\eta_1(w))=S(w,\eta_0(w))$ and for this function $\eta_1:\Omega\to C$ we can choose another function $\eta_2:\Omega\to C$ such that for $w\in\Omega$, $E(w,\eta_2(w))=T(w,\eta_1(w))$ and so on. By using the method of induction we construct a sequence of measurable mappings $\{\xi_n(w)\}$ from Ω to C as following:

$$\begin{aligned}&\xi_{2n+1}(w)=F(w,\eta_{2n+1}(w))=S(w,\eta_{2n}(w)),\\ &\xi_{2n+2}(w)=E(w,\eta_{2n+2}(w))=T(w,\eta_{2n+1}(w)),w\in\Omega,n=0,1,2,..\end{aligned} \tag{5}$$

Let $a_n(w)=d(\xi_n(w),\xi_{n+1}(w))$

Putting $x=\eta_{2n}(w),y=\eta_{2n+1}(w)$ in (4), we obtain

$$\begin{aligned}\varphi_1(d(S(w,\eta_{2n}(w)),T(w,\eta_{2n+1}(w))))\le{}&\psi_1(d(E(w,\eta_{2n}(w)),F(w,\eta_{2n+1}(w)))\\ &,d(E(w,\eta_{2n}(w)),S(w,\eta_{2n}(w)))\\ &,d(F(w,\eta_{2n+1}(w)),T(w,\eta_{2n+1}(w))))\\ &-\psi_2(d(E(w,\eta_{2n}(w)),F(w,\eta_{2n+1}(w)))\\ &,d(E(w,\eta_{2n}(w)),S(w,\eta_{2n}(w)))\end{aligned}$$

$$,d(F(w,\eta_{2n+1}(w)),T(w,\eta_{2n+1}(w)))),$$

It follows by (5) that

$$\varphi_1(a_{2n+1}(w))\le\psi_1(a_{2n}(w),a_{2n}(w),a_{2n+1}(w))-\psi_2(a_{2n}(w),a_{2n}(w),a_{2n+1}(w)). \quad (6)$$

If $a_{2n}<a_{2n+1}$, then by using the property that ψ_1 is monotone increasing in all variables and $\psi_2(a_{2n},a_{2n},a_{2n+1})\neq 0$ whenever $a_{2n+1}(w)\neq 0$, we have in (6) that

$$\begin{aligned}\varphi_1(a_{2n+1}(w)) &< \psi_1(a_{2n+1}(w),a_{2n+1}(w),a_{2n+1}(w))-\psi_2(a_{2n}(w),a_{2n}(w),a_{2n+1}(w))\\ &= \varphi_1(a_{2n+1}(w))-\psi_2(a_{2n}(w),a_{2n}(w),a_{2n+1}(w))\\ &< \varphi_1(a_{2n+1}(w)).\end{aligned}$$

Thus, we have a contradiction, so that

$$a_{2n+1}(w)\le a_{2n}(w). \quad (7)$$

Again, putting $x=\eta_{2n}(w), y=\eta_{2n-1}(w)$ in (4), we obtain

$$\varphi_1(a_{2n})<\psi_1(a_{2n-1}(w),a_{2n-1}(w),a_{2n}(w))-\psi_2(a_{2n-1}(w),a_{2n-1}(w),a_{2n}(w)) \quad (8)$$

By the same argument we obtain

$$a_{2n}(w)\le a_{2n-1}(w). \quad (9)$$

From (7) and (9), we have

$$a_{n+1}\le a_n.$$

Hence, $\{a_n(w)\}$ is a decreasing sequence and bounded so is convergent, then there exists $a(w)\ge 0$ such that

$$\lim_{n\to\infty} a_n(w)=a(w), w\in\Omega. \quad (10)$$

form (6) and (7), we have

$$\varphi_1(a_{2n+1}(w))\le\varphi_1(a_{2n}(w))-\varphi_2(a_{2n+1}(w)), \quad (11)$$

where $\varphi_1(x)=\psi_1(x,x,x)$ and $\varphi_2(x)=\psi_2(x,x,x)$.
Similarly, form (8) and (9), we have

$$\varphi_1(a_{2n}(w))\le\varphi_1(a_{2n-1}(w))-\varphi_2(a_{2n}(w)). \quad (12)$$

Combining (11) and (12), we obtain

$$\varphi_1(a_{n+1}(w))\le\varphi_1(a_n(w))-\varphi_2(a_{n+1}(w)),$$

or equivalently

$$\varphi_2(a_{n+1}(w))\le\varphi_1(a_n(w))-\varphi_1(a_{n+1}(w)). \quad (13)$$

Summing up in (13), we obtain

$$\Sigma_{n=0}^{\infty}\varphi_2(a_{n+1}(w))\le\varphi_1(a_0(w))<\infty.$$

Hence

$$\lim_{n\to\infty}\varphi_2(a_n(w))=0. \quad (14)$$

Now, from (10),(14) and the continuity of φ_2, we obtain $\varphi_2(a(w))=0$, which implies that $a(w)=0, w\in\Omega$, that is

$$\lim_{n\to\infty} a_n(w)=d(\xi_n(w),\xi_{n+1}(w))=0. \quad (15)$$

Now, we will prove that for $w\in\Omega,\{\xi_n(w)\}$ is a Cauchy sequence in C. By (15), it is sufficient to prove that $\{\xi_{2n}(w)\}$ is a Cauchy sequence. We proceed by negation, suppose that $\{\xi_{2n}(w)\}$ is not a Cauchy sequence, then there exists $\varepsilon>0$ for which we can find two sequences of positive integers $\{m(i)\},\{n(i)\}$ such that for positive integer i we have

$$n(i) > m(i) > i, d(\xi_{2m(i)}(w), \xi_{2n(i)}(w)) \geq \varepsilon, d(\xi_{2m(i)}(w), \xi_{2n(i)-1}(w)) < \varepsilon. \quad (16)$$

Using (16) and the triangle inequality, we obtain

$$\varepsilon \leq d(\xi_{2m(i)}(w), \xi_{2n(i)}(w)) \leq d(\xi_{2m(i)}(w), \xi_{2n(i)-1}(w)) + d(\xi_{2n(i)-1}(w), \xi_{2n(i)}(w))$$
$$< \varepsilon + d(\xi_{2n(i)-1}(w), \xi_{2n(i)}(w)).$$

Letting $i \to \infty$ in the above inequality we have

$$\lim_{n\to\infty} d(\xi_{2m(i)}(w), \xi_{2n(i)}(w)) = \varepsilon, w \in \Omega. \quad (17)$$

In addition, we have

$$d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w)) \leq d(\xi_{2n(i)+1}(w), \xi_{2n(i)}(w)) + d(\xi_{2n(i)}(w), \xi_{2m(i)}(w)),$$

and

$$d(\xi_{2n(i)}(w), \xi_{2m(i)}(w)) \leq d(\xi_{2n(i)}(w), \xi_{2n(i)+1}(w)) + d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w)).$$

Letting $i \to \infty$ in the above two inequalities, using (15) and (17) we obtain

$$\lim_{n\to\infty} d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w)) \leq \varepsilon.$$

and

$$\varepsilon \leq \lim_{n\to\infty} d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w)).$$

or equivalently

$$\lim_{n\to\infty} d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w)) = \varepsilon. \quad (18)$$

In the same way, we have

$$d(\xi_{2n(i)}(w), \xi_{2m(i)-1}(w)) \leq d(\xi_{2n(i)}(w), \xi_{2m(i)}(w)) + d(\xi_{2m(i)}(w), \xi_{2m(i)-1}(w)).$$

and

$$d(\xi_{2n(i)}(w), \xi_{2m(i)}(w)) \leq d(\xi_{2n(i)}(w), \xi_{2m(i)-1}(w)) + d(\xi_{2m(i)-1}(w), \xi_{2m(i)}(w)).$$

Again, letting $i \to \infty$ in the above two inequalities, using (15) and (17) we obtain

$$\lim_{n\to\infty} d(\xi_{2n(i)}(w), \xi_{2m(i)-1}(w)) = \varepsilon. \quad (19)$$

Setting $x = \eta_{2n(i)}(w), y = \eta_{2m(i)-1}(w)$ in (4) for all $i = 1,2,\ldots$, we obtain

$$\varphi_1(d(S(w,\eta_{2n(i)}(w)), T(w,\eta_{2m(i)-1}(w)))) \leq \psi_1(d(E(w,\eta_{2n(i)}(w)), F(w,\eta_{2m(i)-1}(w)))$$
$$, d(E(w,\eta_{2n(i)}(w)), S(w,\eta_{2n(i)}(w)))$$
$$, d(F(w,\eta_{2m(i)-1}(w)), T(w,\eta_{2m(i)-1}(w))))$$
$$-\psi_2(d(E(w,\eta_{2n(i)}(w)), F(w,\eta_{2m(i)-1}(w)))$$
$$, d(E(w,\eta_{2n(i)}(w)), S(w,\eta_{2n(i)}(w)))$$
$$, d(F(w,\eta_{2m(i)-1}(w)), T(w,\eta_{2m(i)-1}(w)))).$$

It follows by (5) that

$$\varphi_1(d(\xi_{2n(i)+1}(w), \xi_{2m(i)}(w))) \leq \psi_1(d(\xi_{2n(i)}(w), \xi_{2m(i)-1}(w)))$$
$$, d(\xi_{2n(i)}(w), \xi_{2n(i)+1}(w)))$$
$$, d(\xi_{2m(i)-1}(w), \xi_{2m(i)}(w)))$$
$$-\psi_2(d(\xi_{2n(i)}(w), \xi_{2m(i)-1}(w)))$$
$$, d(\xi_{2n(i)}(w), \xi_{2n(i)+1}(w)))$$
$$, d(\xi_{2m(i)-1}(w), \xi_{2m(i)}(w))).$$

Letting $i \to \infty$ in the above inequality, using (15), (18), (19) and the continuity of ψ_1 and ψ_2, we have

$$\varphi_1(\varepsilon) \le \psi_1(\varepsilon,0,0) - \psi_2(\varepsilon,0,0) \le \varphi_1(\varepsilon) - \psi_2(\varepsilon,0,0),$$

which implies that $\psi_2(\varepsilon,0,0)=0$, this leads to a contradiction since $\varepsilon > 0$. It follows that $\{\xi_{2n}\}$ is a Cauchy sequence in C and hence $\{\xi_n\}$ is also a Cauchy sequence in the closed subset C of a separable complete metric space X, then there exists $\xi(w):\Omega \to C$ such that

$$\{\xi_n(w)\} \to \{\xi(w)\} \; as \; n \to \infty \; for \; w \in \Omega, \tag{20}$$

and consequently the subsequences $\{F(w,\eta_{2n+1}(w))\}, \{S(w,\eta_{2n}(w))\}$, $\{E(w,\eta_{2n+2}(w))\}$ and $\{T(w,\eta_{2n+1}(w))\}$ of $\{\xi_n(w)\}$, for $w \in \Omega$ also converge to $\{\xi(w)\}$. Also the closeness of C implies that $\xi(w)$ is a mapping from Ω to C.

Since $S(w,X) \subseteq F(w,X)$ then there exists $h(w) \in C$ such that

$$\xi(w) = F(w,h(w)),\; w \in \Omega. \tag{21}$$

Putting $x = \eta_{2n}(w), y = h(w)$ in (4), we obtain

$$\begin{aligned}
&\varphi_1(d(S(w,\eta_{2n}(w)), T(w,h(w)))) \le \psi_1(d(E(w,\eta_{2n}(w)), F(w,h(w)))\\
&,d(E(w,\eta_{2n}(w)), S(w,\eta_{2n}(w)))\\
&,d(F(w,h(w)), T(w,h(w))))\\
&-\psi_2(d(E(w,\eta_{2n}(w)),F(w,h(w)))\\
&,d(E(w,\eta_{2n}(w)), S(w,\eta_{2n}(w)))\\
&,d(F(w,h(w)), T(w,h(w)))).
\end{aligned}$$

Taking the limit on both sides of the above inequality as $n \to \infty$, and using (21) we obtain

$$\begin{aligned}
&\varphi_1(d(\xi(w), T(w,h(w)))) \le \psi_1(d(\xi(w),\xi(w)), d(\xi(w),\xi(w))\\
&,d(\xi(w), T(w,h(w))))\\
&-\psi_2(d(\xi(w),\xi(w)), d(\xi(w),\xi(w))\\
&,d(\xi(w), T(w,h(w)))).
\end{aligned}$$

It follows that

$$\begin{aligned}
\varphi_1(d(\xi(w),T(w,h(w)))) &\le \psi_1(0,0,d(\xi(w),T(w,h(w)))) - \psi_2(0,0,d(\xi(w),T(w,h(w))))\\
&< \varphi_1(d(\xi(w),T(w,h(w)))),
\end{aligned}$$

which is a contradiction. It follows that

$$\xi(w) = T(w,h(w)) \; for w \in \Omega. \tag{22}$$

From (21) and (22), we have

$$\xi(w) = F(w,h(w)) = T(w,h(w)).$$

Therefore $\xi(w)$ is a point of coincidence of F and T.

Again, since $\xi(w) = T(w,h(w)) \in T(w,X) \subseteq E(w,X)$, there exists $f(w) \in C$ such that

$$\xi(w) = E(w,f(w)) \; for w \in \Omega. \tag{23}$$

Putting $x = f(w), y = \eta_{2n+1}(w)$, by the same argument we can show that

$$S(w,f(w)) = \xi(w) \; for w \in \Omega. \tag{24}$$

From (23) and (24), we have

$$\xi(w) = E(w,f(w)) = S(w,f(w)).$$

Therefore $\xi(w)$ is a point of coincidence of E and S.
Finally, for the uniqueness of the point of coincidence.
Let $p(w):\Omega \to C$ be measurable mapping such that such that $E(w, p(w)) = S(w, p(w)),\ w \in \Omega$
Setting $x = p(w)$ and $y = h(w)$ in (4) we obtain

$$\begin{aligned}
&\varphi_1(d(S(w,p(w)),T(w,h(w))) \le \psi_1(d(E(w,p(w)),F(w,h(w)))\\
&,d(E(w,p(w)),S(w,p(w)))\\
&,d(F(w,h(w)),T(w,h(w))))\\
&-\psi_2(d(E(w,p(w)),F(w,h(w)))\\
&,d(E(w,p(w)),S(w,p(w)))\\
&,d(F(w,h(w)),T(w,h(w)))),
\end{aligned}$$

which yields

$$\begin{aligned}
&\varphi_1(d(S(w,p(w)),\xi(w))) \le \psi_1(d(S(w,p(w)),\xi(w))),0,0)\\
&-\psi_2(d(S(w,p(w)),\xi(w))),0,0)\\
&< \varphi_1((S(w,p(w)),\xi(w)))
\end{aligned}$$

It follows that $\xi(w) = S(w, p(w)) = E(w, p(w)),\ w \in \Omega.$
Similarly, we can show that $\xi(w)$ is the unique point of coincidence of F and T.
Hence, the pairs (E,S) and (F,T) have a unique point of coincidence $\xi(w)$.
If the pairs (E,S) is owc (respectively, (F,T) is owc), then by Lemma 1.15, $\xi(w)$ is the unique common random fixed point of E and S (respectively of F and T). The proof of the theorem is completed.

Remark 2.2
Theorem 2.1 remains true if one replace the following contractive condition in lieu of the existing one.

$$\begin{aligned}
\varphi_1(d(S(w,x),T(w,y))) \le{}& \psi_1(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y))\\
&,\frac{1}{2}[d(S(w,x),F(w,y))+d(E(w,x),T(w,y))])\\
&-\psi_2(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y))\\
&,\frac{1}{2}[d(S(w,x),F(w,y))+d(E(w,x),T(w,y))]),
\end{aligned}$$

where $\psi_1, \psi_2 \in \Psi_4$ and $\varphi_1(x) = \psi_1(x,x,x,x)$.

Remark 2.3
Theorem 2.1 is a random version improvement, extension and generalization of Choudhury [9] for pairs of owc random mappings using considering a generalized altering distance functions.

Theorem 2.4
Let C be a nonempty closed subset of a separable complete metric space (X,d). Let S,T,E and $F:\Omega\times C \to C$ be four self random mappings defined on C such that for $x, y \in C$, $w \in \Omega$,

$$S(w,X) \subseteq F(w,X),\ T(w,X) \subseteq E(w,X),$$

and satisfying one of the following conditions:

(I) $\varphi_1(\theta(d(S(w,x),T(w,y)))) \le \psi_1(\theta(d(E(w,x),F(w,y))),\theta(d(E(w,x),S(w,x))),\theta(d(F(w,y),T(w,y))))$

$$-\psi_2(\theta(d(E(w,x),F(w,y))),\theta(d(E(w,x),S(w,x))),\theta(d(F(w,y),T(w,y)))),$$

where $\theta \in \Psi_1$ and $\psi_1,\psi_2 \in \Psi_3$ with $\varphi_1(x)=\psi_1(x,x,x)$.

$$(II)\ \varphi_1(\theta(d(S(w,x),T(w,y))))\le \psi_1(\theta(d(E(w,x),F(w,y))),\theta(d(E(w,x),S(w,x))),\theta(d(F(w,y),T(w,y))),\frac{1}{2}[\theta(d(S(w,x),F(w,y)))+\theta(d(E(w,x),T(w,y)))])$$
$$-\psi_2(\theta(d(E(w,x),F(w,y))),\theta(d(E(w,x),S(w,x))),\theta(d(F(w,y),T(w,y))),\frac{1}{2}[\theta(d(S(w,x),F(w,y)))+\theta(d(E(w,x),T(w,y)))]),$$

where where $\theta \in \Psi_1$ and $\psi_1,\psi_2 \in \Psi_4$ with $\varphi_1(x)=\psi_1(x,x,x,x)$.

Then each of the pairs (S,E) and (T,F) has a unique point of coincidence. Moreover, if each of the pairs (S,E) and (T,F) is owc, then S,T,E and F have a unique common random fixed point.

Proof. Applying the same steps of the proof of Theorem 2.1 and Remark 2.2, then the claim of 2.4 follows simply.

Theorem 2.5

Let C be a nonempty closed subset of a separable complete metric space (X,d). Let S,T,E and $F:\Omega\times C\to C$ be four self random mappings defined on C such that for $x,y\in C$, $w\in\Omega$,

$$S(w,X)\subseteq F(w,X),\ T(w,X)\subseteq E(w,X),$$

and satisfying one of the following conditions:

$$(III)\ \int_0^{\varphi_1(d(S(w,x),T(w,y)))}\phi(t)dt \le \int_0^{\psi_1(M_1(x,y))}\phi(t)dt - \int_0^{\psi_2(M_1(x,y))}\phi(t)dt,$$

where $M_1(x,y)=(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y)))$ and $\psi_1,\psi_2\in\Psi_3$ with $\varphi_1(x)=\psi_1(x,x,x)$.

$$(IV)\ \int_0^{\varphi_1(d(S(w,x),T(w,y)))}\phi(t)dt \le \int_0^{\psi_1(M_2(x,y))}\phi(t)dt - \int_0^{\psi_2(M_2(x,y))}\phi(t)dt,$$

where $M_2(x,y)=(d(E(w,x),F(w,y)),d(E(w,x),S(w,x)),d(F(w,y),T(w,y)),\frac{1}{2}[d(S(w,x),F(w,y))+d(E(w,x),T(w,y))])$ and $\psi_1,\psi_2\in\Psi_4$ with $\varphi_1(x)=\psi_1(x,x,x,x)$.

The function $\phi:[0,\infty)\to[0,\infty]$ is a Lebesgue-measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$ such that for $\varepsilon>0, \int_0^{\varepsilon}\phi(t)dt>0,$.

Then each of the pairs (S,E) and (T,F) has a unique point of coincidence. Moreover, if each of the pairs (S,E) and (T,F) is owc, then S,T,E and F have a unique common random fixed point.

Proof. By lemma 1.16, the functions $\Phi:[0,\infty)\to R, \Phi(b)=\int_0^b\phi(t)dt$ is an altering distance function. Setting in the above two inequalities the following

$$\Phi(\varphi_1(d(S(w,x),T(w,y))))=\int_0^{\varphi_1(d(S(w,x),T(w,y)))}\phi(t)dt$$

$$\Phi(\psi_1(M_i(x,y))) = \int_0^{\psi_1(M_i(x,y))} \phi(t)dt, i = 1,2$$

$$\Phi(\psi_2(M_i(x,y))) = \int_0^{\psi_2(M_i(x,y))} \phi(t)dt, i = 1,2$$

then the proof of Theorem 2.5 follows by the same way of Theorem 2.1 and Remark 2.2.

Remark 2.6

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions are obtained from Theorem 2.1.
For example, we derive the following corollary of Theorem 2.1.

Corollary 2.7

Let C be a nonempty closed subset of a separable complete metric space (X,d). Let S,T,E and $F:\Omega\times C\to C$ be four self random mappings defined on C such that for $x,y\in C$ and $w\in\Omega$,

$$S(w,X)\subseteq F(w,X),\ T(w,X)\subseteq E(w,X), \tag{25}$$

and satisfying one of the following conditions:

$$(V)\,[d(S(w,x),T(w,y))]^r \le k_1[d(E(w,x),F(w,y))]^r + k_2[d(E(w,x),S(w,x))]^r + k_3[d(F(w,y),T(w,y))]^r, \tag{26}$$

where $0<k_1+k_2+k_3<1$ and $r>0$.

$$(VI)\,[d(S(w,x),T(w,y))]^r \le k_1[d(E(w,x),F(w,y))]^r + k_2[d(E(w,x),S(w,x))]^r + k_3[d(F(w,y),T(w,y))]^r + k_4[\frac{d(S(w,x),F(w,y))+d(E(w,x),T(w,y))}{2}], \tag{27}$$

where $0<k_1+k_2+k_3+k_4<1$ and $r>0$.

Then each of the pairs (S,E) and (T,F) has a unique point of coincidence. Moreover, if each of the pairs (S,E) and (T,F) is owc, then S,T,E and F have a unique common random fixed point.

Proof. Consider in case **(V)** the following

$$\psi_1(a,b,c) = k_1a^r + k_2b^r + k_3c^r,$$
$$\psi_2(a,b,c) = (1-k)[k_1a^r + k_2b^r + k_3c^r],$$

with $k = k_1+k_2+k_3$, and consider in case **(VI)** the following

$$\psi_1(a,b,c,d) = k_1a^r + k_2b^r + k_3c^r + k_4d^r,$$
$$\psi_2(a,b,c,d) = (1-k)[k_1a^r + k_2b^r + k_3c^r + k_4d^r],$$

with $k = k_1+k_2+k_3+k_4$. Then (26) and (27) can be obtained from (4) and the contractive condition in Remark 2.2, The corollary follows by applying Theorem 2.1 and Remark 2.2.

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