The Trembling Hand Approach to Automata in Iterated Games

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Abstract

We consider two state automata playing infinitely iterated two players, two strategies game, where each move can be mis-implemented (or mis-perceived) with a small error probability, and compute the payoff matrix by means of a perturbation approach.

Keywords: Alternating Prisoner's Dilemma, Perturbed Payoff, Repeated Games, Simultaneous Prisoner's Dilemma.

1. Introduction

The “trembling hand” approach plays an ever-increasing role in game-theoretic considerations, especially in Selten's notion of a perfect equilibrium (Selten 1975, Selten and Hammerstein 1984, Van Damme 1988, Boyd 1991, etc.). In many situations, it is indeed necessary to take into account errors occurring with some small probability and affecting the implementation of a strategy. In this note, we investigate the effect of such errors in a special, but important setup that of finite automata engaged in playing iterated games. Early accounts of such situations can be found in Rapoport and Chammah (1967) or in Aumann (1981). For more recent work, inspired by Axelrod's well-known computer tournaments with the Prisoner's Dilemma game (PD game) (Axelrod 1982) we refer to Rubinstein 1986, Abreu and Rubinstein 1988, Banks and Sundaram 1990, Binmore and Samuelson 1992 etc. In these papers, the automata were assumed to work faultlessly. But since the automata are usually meant to be abstractions of strategies implemented by agents of bounded rationality (like humans or other animals), it seems reasonable to allow for mistakes (see, e.g. May 1987, Axelrod and Dion 1988, Miller 1989, Lindgren 1991, Nowak and Sigmund 1993). The aim of this note is to compute the payoff matrix, for a small error probability ϵ, in the simplest case, when we assume that a game between two players having two strategies each is repeated endlessly. As payoff for the iterated game, we use the limit in the mean; as strategies, we allow all possible two state automata. In PD game the two players have the same two options denoted by C to cooperate and by D to defect. In each round of simultaneous model of this game the two players take their choice as the same time. The moves of the game are hidden and it is appropriate to model the situation as a simultaneous-move game (even if both players didn't take their decisions at the same time). But in the alternating model of this game one of the two players choose his option in a round and the other player reply with his option in another round. This means that in each round of the alternating model there is a single option for one player. This player is called leader (or donor) and the other is called recipient.

2. Automata Repeated Game

We denote the two states by C and D; these are also the two strategies which can be played in each round. The payoff matrix, for each round, is given by

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix} \tag{1}$$

Our finite automata corresponds to transition rules which specify the state in the current round according to the outcome in the previous round, i.e. as a function of the own move and the co-player's move .There are four possible outcomes, which are expressed by the player as *R*,*S*,*T* or *P*, and which we number 1,2,3 and 4. Each transition rule is given by a 4-tuple (u_1, u_2, u_3, u_4), where u_k is 1 or 0 depending on whether the automaton plays C or D after outcome k $(k = 1,2,3,4)$. For instance, (1,0,1,0) is the familiar Tit For Tat strategy (it repeats what the co-player did in the previous round. See Fig.1); (0,0,0,0) always plays *D*, etc. We shall denote these 16 strategies by S_i, where i is the integer given, in binary notation, by $u_1u_2u_3u_4$ $(i = 0,1 \ldots,15)$. Hence $S_0 = (0,0,0,0)$, $S_9 = (1,0,0,1)$, $S_{10} = (1,0,1,0)$ etc.

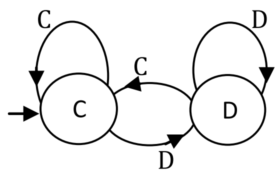

Figure 1 : Tit For Tat

3. The Perturbed Payoff matrix of Simultaneous PD game

If $\epsilon > 0$ denotes the probability of a mistake in implementing a move, then S_i is changed into $S_i(\epsilon)$, which is given by the 4-tuple obtained from (u_1, u_2, u_3, u_4) by replacing 0 with ϵ and 1 with $1-\epsilon$. The problem now is to compute the payoff for strategy $S_i(\epsilon)$ against $S_j(\epsilon)$.

More generally, let us consider strategies $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\mathbf{q} = (q_1, q_2, q_3, q_4)$, where the p_k and q_k are the probabilities to play C after outcome k $(k = 1,2,3,4)$. The transition matrix between the four states R, S, T and P, from one round to the next, are given by the stochastic matrix

$$H = \begin{pmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{pmatrix} \tag{2}$$

(we note the interchange of 2 and 3, due to the fact that one player's S is the other player's T). If the matrix H is irreducible [as is always the case when $0 < p_k, q_k < 1$ for all k , and in particular if $\mathbf{p}$ and $\mathbf{q}$ correspond to strategies S_i], H has a unique left eigenvector $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ to the eigenvalue 1 such that $0 < \eta_k$ for $k = 1,2,3,4$ and $\sum \eta_k = 1$. These η_k denote the relative frequencies of the states k of the corresponding Markov chain. They specify the limit in the mean payoff for strategy $\mathbf{p}$ against strategy $\mathbf{q}$ which is

$$\eta_1 R + \eta_2 S + \eta_3 T + \eta_4 P \tag{3}$$

The $\mathbf{q}$ player's payoff is obtained by interchanging η_2 and η_3 .

If $\mathbf{p} = S_i(\epsilon)$ and $\mathbf{q} = S_j(\epsilon)$, then we can write

$$H(\epsilon) = H + \epsilon H_1 + \epsilon^2 H_2 \tag{4}$$

where H is a stochastic matrix, H_1 and H_2 have row sum 0. Actually, H has exactly one entry 1 in each row, H_1 has -2 at this position, 0 in the mirror position and 1's at the other two places (position 2 is said to be the mirror position of 3, and 1 of 4); the matrix H_2 has a 1 wherever H_1 has a -2 or a 0, and a -1 wherever H_1 has a 1. The matrix H_2 is of rank 1, because if we denote its first column by α, then its other columns are $-\alpha, -\alpha, \alpha$ (in this order).

We may view $H(\epsilon)$ as a perturbation of the matrix H and treat accordingly the question of finding the left eigenvector $\eta(\epsilon)$ of $H(\epsilon)$ as a perturbation problem. Thus we set

$$\eta(\epsilon) = \eta + \epsilon x + \epsilon^2 y + \cdots \tag{5}$$

Where the stochastic vector η is a solution of the unperturbed problem

$$\eta H = \eta \tag{6}$$

Whereas the components of the vectors x and y must sum up to 0. Writing $H = I + H_0$ (where I is the identity matrix) and using (4) and (5), the eigenvalue equation

$$\eta(\epsilon)H(\epsilon) = \eta(\epsilon) \tag{7}$$

implies, upon comparing powers in ϵ , the three equations

$$\eta H_0 = 0 , \tag{8}$$

$$x H_0 + \eta H_1 = 0 , \tag{9}$$

$$y H_0 + x H_1 + \eta H_2 = 0 . \tag{10}$$

Of course (8) is just (6). In most cases ,the matrix H is not irreducible, so that η is not uniquely determined by (8). In Nowak, Sigmund and El-Sedy (1995), they have used a direct method to compare η, but they stress that even if η is known, x need not be fully specified by (9). The system (8), (9), (10), however, allows the computation of η and x (they omit the computation of y, for which, in general, we have to expand (5) by one further term). They are specified by the equation

$$(\eta, x, y)\begin{pmatrix} H_0 & H_1 & H_2 \\ 0 & H_0 & H_1 \\ 0 & 0 & H_0 \end{pmatrix} = (0,0,0) \tag{11}$$

subject to the condition that the components of η sum up to 1 while those of x and y sum up to 0 to keep $\sum_{i=1}^{4} \eta_i(\epsilon) = 1 = \sum_{i=1}^{4} \eta_i$. The results are encapsulated in tables 1 and 2. They yield the limiting term η and the perturbation term x, and hence, up to order ϵ, the payoff for the $S_i(\epsilon)$-strategy playing against the $S_j(\epsilon)$-strategy, for $0 \le i,j \le 15$ (We omitted the terms below the diagonal, since the (i,j)-term is obtained from the (j,i)-term by simply interchanging η_2 and η_3).

Let us check this, for instance, for $S_8(\epsilon) = (1-\epsilon, \epsilon, \epsilon, \epsilon)$ against $S_{11}(\epsilon) = (1-\epsilon, \epsilon, 1-\epsilon, 1-\epsilon)$. In this case

$$H(\epsilon) = \begin{pmatrix} (1-\epsilon)^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & \epsilon^2 \\ \epsilon(1-\epsilon) & \epsilon^2 & (1-\epsilon)^2 & \epsilon(1-\epsilon) \\ \epsilon^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & (1-\epsilon)^2 \\ \epsilon(1-\epsilon) & \epsilon^2 & (1-\epsilon)^2 & \epsilon(1-\epsilon) \end{pmatrix},$$

which implies

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, H_1 = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & -2 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

Equation (8) immediately yields $\eta_2 = 0$ and $\eta_3 = \eta_4$. Hence η is of the form $(1-2u, 0, u, u)$, for some suitable u, and $sH_1 = (-2+5u, 1-u, 1-3u, -u)$. On the other hand, $xH_0 = (0, -x_2, x_2 - x_3 + x_4, x_3 - x_4)$. The first component of (9) yields $u = 2/5$ and hence $\eta = (1/5,0,2/5,2/5)$, which in turn implies $x = (-1-2v, 3/5, 2/5+v, v)$ for some suitable v. Hence $\eta H_2 = (1/5, -1/5, -1/5, 1/5)$ and $xH_1 = (13/5+5v, -3/5-v, -9/5-3v, -1/5-v)$; since the first component of yH_0 is 0, equation (10) yields $v = -14/25$, and hence $x = (1/25)(3,15,-4,-14)$.

Up to order ϵ^2, the payoff for S_8 against S_{11} is therefore

$$F\big(S_i(\epsilon), S_j(\epsilon)\big) = \eta \begin{bmatrix} R \\ S \\ T \\ P \end{bmatrix} + \epsilon x \begin{bmatrix} R \\ S \\ T \\ P \end{bmatrix}$$

Then we have,

$F\big(S_8(\epsilon), S_{11}(\epsilon)\big) = \frac{1}{25}[(5+3\epsilon)R + 15\epsilon S + (10-4\epsilon)T + (10-14\epsilon)P].$

Table 1 has been discussed in Nowak, Sigmund and El-Sedy (1995), especially in the context of the PD game and the Chicken game. Concerning table 2, we only note that some payoff values are not at all affected by the error term (i.e. $x = 0$, for instance when S_3 plays against S_6, while sometimes the perturbation is five times as large as the error size, for instance when S_9 plays against S_{11}).

We also remark that there are $4^4 = 256$ different matrices H_0, one for every entry of the 16×16 payoff matrix. Wherever there is a 1 in $H = I + H_0$, there is a -2 in H_1. This specifies the position of the 0- entry, and hence also that of the two 1- entries in every row of H_1. Since H_2 has as entry 1 wherever H_1 has as entry -2 or 0, there are only half as many possibilities for H_2 as for H_1 in every row, and hence altogether only 16 different possible matrices H_2 .

So far, we have considered the effect of errors in the implementation also of a given strategy. We can also analyze the effect of errors in perception–in misunderstanding the other's *C* or *D*, for example (see, e.g., Axelrod and Dion 1988 and Miller 1989). This type of errors can sometimes lead to quite different results. Let us denote by ϵ the probability of mistaking the other player's previous move, and by $\lambda\epsilon$ the probability of mistaking one's own previous move (usually, λ should be smaller than 1). The perturbation of the Tit For Tat strategy S_{10} is $(1-\epsilon, \epsilon, 1-\epsilon, \epsilon)$, just as with mistakes in implementation. The perturbation of S_9 is $(1-(\lambda+1)\epsilon, (\lambda+1)\epsilon, (\lambda+1)\epsilon, 1-(\lambda+1)\epsilon)$; that of S_{11} is $(1-\epsilon, (\lambda+1)\epsilon, 1, 1-\lambda\epsilon)$, while that of S_0 is (0,0,0,0), i.e. no perturbation at all. In general the strategy (u_1, u_2, u_3, u_4) turns into

$$(1-(\lambda+1)\epsilon)(u_1, u_2, u_3, u_4) + \epsilon(u_2, u_1, u_4, u_3) + \lambda\epsilon(u_3, u_4, u_1, u_2) + \lambda\epsilon^2(v, -v, -v, v) \qquad (12)$$

where $v = u_1 + u_4 - u_2 - u_3$. Again, one can use the same perturbation method as before to find the payoff values. For S_1 against S_4, for instance, one obtains $\eta = (0, 1/2, 0, 1/2)$ and $x = (\lambda/2, -1-3\lambda/4, \lambda/2, 1-\lambda/4)$, which reduce neither for $\lambda = 1$ nor for $\lambda = 0$ to the corresponding perturbation term for mistakes in implementation.

Similarly, one can consider the joint effect of errors in implementation and perception; allow for different propensities to mis-implement (or mis-perceive) a *C* or a *D*; investigate repeated games where the players move alternatingly, rather than simultaneously (see Nowak and Sigmund 1994); compute the higher order terms of the perturbation of the payoff etc. In all these contexts, analogous perturbation arguments allow to compute the payoff matrix.

4. The unperturbed payoff matrix of randomly alternating PD game

Let us now turn to the randomly alternating PD game. We assume here that in every round, each player has a 50% chance of being the leader this mean that, in each round chance decides which of the two players is the leader (or donor) and which is the recipient. The leader then chooses between two options *C* and *D*. Option *C* yields a points to the donor and b points to the recipient, whereas option *D* yields c points to the donor and d points to the recipient. We shall assume that in a single round, playing *C* rather than *D* entails a cost to the donor which is smaller than the benefit that this action brings to the recipient. Since the cost is $c - a$ and the benefit $b - d$, this means that

$$0 < c - a < b - d$$

Let us consider two rounds for which the players are donors in turn. If both play *C*, both earn $a + b$, if one plays *C* and the other *D* while leader, the co-operator earns $a + b$ and the defector earns $c + b$. We have $R = a + b$, $T = c + b$, $P = c + d$ and $S = a + d$, then we have $T + S = R + P$. The outcome of one round of the repeated alternating PD game is completely specified by the payoff obtained by one of the players ; this can be a, b, c or d. We denote these outcomes by 1 to 4(in this order), noting that one player's a (or c) is the other player's b (resp. d). We restrict our attention to players whose strategy is determined by the outcome of the previous round only, i.e., given by a quadruple $\mathbf{p} = (p_1, p_2, p_3, p_4)$, where p_i denotes the propensity to play *C* after outcome i. If a $\mathbf{p}$ player is matched against a $\mathbf{q}$ player, then the transition probability from one round to the next is given by

$$M = \frac{1}{2}\begin{pmatrix} p_1 & q_2 & (1-p_1) & (1-q_2) \\ p_2 & q_1 & (1-p_2) & (1-q_1) \\ p_3 & q_4 & (1-p_3) & (1-q_4) \\ p_4 & q_3 & (1-p_4) & (1-q_3) \end{pmatrix}$$

The stationary distribution of transition matrix M is the eigenvector η where

$\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \quad ; \sum_{i=1}^{4} \eta_i = 1$

which correspond to the left eigenvalue of M. Hence η is given by $\eta M = \eta$, then η is given by the vector $\eta = (\sigma, \sigma', \frac{1}{2} - \sigma, \frac{1}{2} - \sigma')$ where σ and σ' are given by the relations (after straightforward computations) $\sigma = \frac{(p_3+p_4)(2+q_3-q_1)-(q_3+q_4)(p_4-p_2)}{2[(2+p_3-p_1)(2+q_3-q_1)-(p_4-p_2)(q_4-q_2)]}$, $\sigma' = \frac{(q_3+q_4)(2+p_3-p_1)-(p_3+p_4)(q_4-q_2)}{2[(2+p_3-p_1)(2+q_3-q_1)-(p_4-p_2)(q_4-q_2)]}$

Now if F is the payoff for the **p**-player, then F is given by

$$F = \eta \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \left(\sigma, \sigma', \tfrac{1}{2} - \sigma, \tfrac{1}{2} - \sigma'\right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \sigma(a-c) + \sigma'(b-d) + \tfrac{1}{2}(c+d).$$

For instance, if $\mathbf{p} = (0,0,1,0)$ and $\mathbf{q} = (0,1,0,1)$ then, $\sigma = \frac{1}{6}$ and $\sigma' = \frac{3}{6}$, $\eta = (\frac{1}{6}, \frac{3}{6}, \frac{2}{6}, 0)$. The payoff for **p**-player will be

$$F = \left(\tfrac{1}{6}, \tfrac{3}{6}, \tfrac{2}{6}, 0\right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{a}{6} + \frac{3b}{6} + \frac{2c}{6}.$$

5. The perturbed payoff matrix of randomly alternating PD game

5.1. The payoff matrix corresponding to the error in implementation

Consider that the game has an error in implementation in each move, then there is a probability for a mistake in each strategy. If $\epsilon > 0$ denotes the probability of mistake in implementation, then S_i becomes $S_i(\epsilon)$. If S_i is the quadruple (u_1, u_2, u_3, u_4) of zeros and ones, then $S_i(\epsilon)$ is the resulting quadruple after replacing 0 by ϵ and 1 by $1-\epsilon$. Suppose for instance, $S_0(\epsilon) = (\epsilon, \epsilon, \epsilon, \epsilon)$ play against $S_9(\epsilon) = (1-\epsilon, \epsilon, \epsilon, 1-\epsilon)$ then we have

$$M(\epsilon) = \frac{1}{2}\begin{pmatrix} \epsilon & \epsilon & 1-\epsilon & 1-\epsilon \\ \epsilon & 1-\epsilon & 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon & 1-\epsilon & \epsilon \\ \epsilon & \epsilon & 1-\epsilon & 1-\epsilon \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{\epsilon}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

which can be written as

$$M(\epsilon) = M + \epsilon M_1 , \qquad (13)$$

where $M = I + M_0$ (I is the identity matrix),

$$M_0 = \frac{1}{2}\begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ and } M_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Substituting from (13) and (5) in (7) after replacing the matrix H by the matrix M and comparing powers in ϵ we get the following three equations

$$\eta M_0 = 0 , \qquad (14)$$

$$x M_0 + \eta M_1 = 0 , \qquad (15)$$

$$y M_0 + x M_1 = 0. \qquad (16)$$

According to the last three equations we get $\eta = (0,1/2,1/2,0)$ and $x = (1/2, -3/2, -1/2, 3/2)$.

The payoff is given by the relation $F\left(s_i(\epsilon), s_j(\epsilon)\right) = \eta \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \epsilon x \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, hence

$F\left(s_0(\epsilon), s_9(\epsilon)\right) = \left(\frac{\epsilon}{2}\right) a + \frac{1}{2}(1-3\epsilon)b + \frac{1}{2}(1-\epsilon)c + \left(\frac{3\epsilon}{2}\right) d.$

The vectors η and x are shown in table 3 and table 4 respectively. If we take the numerical values $a = 2, b = 1, c = 3, d = -2$ $(R = 3, S = 0, T = 4, P = 1)$ and $\epsilon = 0.001$ then, $F\left(s_0(\epsilon), s_9(\epsilon)\right) = 1.995.$ Table 5 represents the payoff matrix due to the error in implementation of the 16 strategies.

5.2-The payoff matrix corresponding to the error in perception

As in section 3 let us consider there is an error in perception for the randomly alternating model, then if we have strategy $S_0 = (0,0,0,0)$ play against strategy $S_9 = (1-(1+\lambda)\epsilon, (1+\lambda)\epsilon, (1+\lambda)\epsilon, 1-(1+\lambda)\epsilon)$ then the corresponding transition matrix is

$$M(\epsilon) = \frac{1}{2}\begin{pmatrix} 0 & (1+\lambda)\epsilon & 1 & 1-(1+\lambda)\epsilon \\ 0 & 1-(1+\lambda)\epsilon & 1 & (1+\lambda)\epsilon \\ 0 & 1-(1+\lambda)\epsilon & 1 & (1+\lambda)\epsilon \\ 0 & (1+\lambda)\epsilon & 1 & 1-(1+\lambda)\epsilon \end{pmatrix}.$$

Again this matrix can be written as $M(\epsilon) = M + \epsilon M_1$, where $M = I + M_0$ This implies to

$$M = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_0 = \frac{1}{2}\begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \text{ and}$$

$$M_1 = \frac{1}{2}(1+\lambda)\begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Using (14), (15) and (16) we obtain $\eta = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ and $x = (0, -(1+\lambda), 0, (1+\lambda))$.

The payoff function according to these vectors will be

$$F\left(s_0(\epsilon), s_9(\epsilon)\right) = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \epsilon(0, -(1+\lambda), 0, (1+\lambda))\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

$$F\left(s_0(\epsilon), s_9(\epsilon)\right) = \left(\frac{1}{2} - \epsilon(1+\lambda)\right) b + \frac{1}{2}c + \epsilon(1+\lambda)d\ .$$

Table 6 and table 7 represents the vectors η and x respectively. Substituting by $a = 2, b = 1, c = 3, d = -2$ $(R = 3, S = 0, T = 4, P = 1)$, $\epsilon = 0.001$ and $\lambda = 0.01$, we get $F\left(s_0(\epsilon), s_9(\epsilon)\right) = 1.997$. Table 8 represents the payoff matrix due to the error in perception of the 16 strategies.

Note that for a large set of payoff values, no pure strategy is evolutionarily stable; every pure strategy can be invaded, and even outcompeted by another pure strategy. A strategy S_i can be invaded by a strategy S_j if the equilibrium point e_i is unstable in the one dimensional subsystem of the game dynamical equation obtained by setting $x_k = 0$ for all $k \neq i, j$, i.e. by its restrictions to the edge e_ie_j of the state simplex; this occurs exactly if the following two conditions are satisfied

i. $c_{ji} \geq c_{ii}$

ii. If $c_{ji} = c_{ii}$, then $c_{jj} > c_{ij}$

The strategy S_i is outcompeted by S_j if both $c_{ji} \geq c_{ii}$ and $c_{ji} \geq c_{ij}$ with at least one inequality being strict. For example the payoff matrix between the S_0-player and S_1-player is given by

S_0 S_1

$$\begin{matrix} S_0 \\ S_1 \end{matrix} \begin{pmatrix} 0.501 & 1.2488 \\ 0.25175 & 0.83356 \end{pmatrix}$$

Then, $c_{0,0} - c_{1,0} = 0.24925 > 0$ and $c_{0,1} - c_{1,1} = 0.41524 > 0$. This means that, the strategy S_1 is outcompeted by the strategy S_0.

Writing $S_i << S_j$ if S_i is outcompeted by S_j, then from the payoff matrix of table 5 for any choice of payoff values for the randomly alternating PD, we get that

$S_0 <<$ —

$S_1 << S_0, S_8, S_{12}, S_{14}$

$S_2 << S_0, S_8, S_{12}$

$S_3 << S_0, S_1, S_2, S_6, S_8$

$S_4 << S_1, S_{12}, S_{14}$

$S_5 << S_0, S_1, S_2, S_8$

$S_6 << S_0, S_2, S_8$

$S_7 << S_0, S_1, S_2, S_3, S_5, S_6, S_8, S_9, S_{10}, S_{11}$

$S_8 << S_0$

$S_9 << S_0, S_1, S_2, S_8$

$S_{10} << S_0, S_1, S_2, S_8$

$S_{11} << S_0, S_1, S_2, S_3, S_5, S_6, S_8, S_9, S_{10}$

$S_{12} << S_3, S_5, S_6, S_9, S_{10}, S_{14}$

$S_{13} << S_0, S_1, S_2, S_3, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{14}$

$S_{14} <<$ —

$S_{15} << S_0, S_1, S_2, S_3, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{13}, S_{14}$.

It is clear that all strategies except S_0 and S_{14} are outcompeted by at least one other strategies. We note that S_0 can outcompete the greatest number of rival strategies, while S_4 is the least able at invading or outcompeting a homogeneous population.

From the payoff matrix of table 8 for any choice of payoff values for the randomly alternating PD, we obtain

$S_0 <<$ —

$S_1 << S_0, S_2, S_8, S_{12}, S_{14}$

$S_2 << S_0, S_8$

$S_3 << S_0, S_1, S_2, S_8$

$S_4 << S_{12}, S_{14}$

$S_5 << S_0, S_1, S_2, S_8$

$S_6 << S_0, S_1, S_2, S_8, S_{14}$

$S_7 << S_0, S_1, S_2, S_3, S_5, S_6, S_8, S_9, S_{10}$

$S_8 << S_0$

$S_9 << S_0, S_1, S_2, S_8$

$S_{10} << S_0, S_1, S_2, S_8$

$S_{11} << S_0, S_1, S_2, S_3, S_5, S_6, S_8, S_9, S_{10}$

$S_{12} << S_7, S_{11}, S_{13}, S_{14}, S_{15}$

$S_{13} << S_0, S_1, S_2, S_3, S_5, S_7, S_8, S_9, S_{10}, S_{11}$

$S_{14} <<$ —

$S_{15} << S_0, S_2, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}$.

Similar to table 5 it is clear that all strategies except S_0 and S_{14} are invaded by at least one other strategies. Also, we see that S_0 can outcompete the greatest number of rival strategies, while S_4 cannot invade or outcompete any other strategy.

References

Aumann, R. (1981), Survey of repeated games, in Essays in Game Theory and Mathematical Economy in honor of Oskar Morgenstern, *Wissenschaftsverlag, Bibliographisches Institut*, Mannheim.

Axelrod, R. (1984) *The Evolution of Cooperation*, Basic Books, New York (reprinted 1989 in Penguin, Harmondsworth).

Axelrod, R. and Dion, D. (1988), The Further Evolution of Cooperation, *Science* **242**, 1385-90.

Abreu, D. and Rubinstein, A. (1988), The Structure of Nash Equilibria in Repeated Games with Finite Automata, *Econometrica* **56**, 1259-82.

Banks, J.S. and Sundaram, R.K. (1990)b, Repeated Games, Finite Automata and Complexity, *Games and Economic Behavior* **2**, 97-117.

Binmore, K.G. and Samuelson, L. (1992), Evolutionary Stability in Repeated Games Played by Finite Automata, *Journal of Economic Theory* **57**, 278-305.

Boyd, R. (1989), Mistakes Allow Evolutionary Stability in Repeated Prisoner's Dilemma Game, *J. theor. Biol*. **136**, 47-59.

El-Sedy, E. (2003), The Adaptive Dynamics for the Randomly Alternating Prisoner's Dilemma Game, *Rev. Unión Mat. Argent*. **44**, No. **1**, 99-108.

Lindgren, K. (1991) Evolutionary Phenomena in Simple Dynamics, in Artificial life II (ed. C.G. Langton et al), *Santa Fe Institute for Studies in the Science of Complexity*, Vol. **X**, 295-312.

May, R.M. (1987) More Evolution of Cooperation, *Nature* **327**, 15-17.

Miller, J.H. (1989) The Coevolution of Automata in the Repeated Prisoner's Dilemma, working paper of the *Santa Fe Institute Economics Research Program.*

Nowak, M. and Sigmund, K. (1993), Chaos and the Evolution of Cooperation, *Proc. Natl. Acad. Science USA* **90**: 5091-5094.

Nowak, M. and Sigmund, K. (1994), The Alternating Prisoner's Dilemma. *J. theor. Biol*. **168**: 219-226.

Nowak, M. and Sigmund, K. and El-Sedy, E. (1995), Automata, Repeated Games, and Noise. *J. Math. Biol.* **33**: 703-722.

Rapoport, A. and Chammah, A. (1995), *The Prisoner's Dilemma*, Univ. of Michigan Press, Ann Arbor.

Rubinstein, A. (1986), Finite Automata Play the Repeated Prisoner's Dilemma, *Journal of Economic Theory* **39**, 83-96.

Selten, R. (1975), Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games, *Int. J. Game Theory* **4**, 25-55.

Selten, R. and Hammerstein, P. (1984), Gaps in Harley's Argument on Evolutionarily Stable Learning Rules and in the Logic of Tit for Tat, *Behavioral and Brains Sciences* **7**, 115-6.

Van Damme, E. (1988) *Stability and Perfection of Nash Equilibria*, Springer, Heidelberg-Berlin.

Table 1: The vectors η of the 16 strategies due to the error in implementation for simultaneous PD, where the vector η for S_i against S_j is $(\eta_1, \eta_2, \eta_3, \eta_4)$, with $\eta_i = n_i(n_1 + n_2 + n_3 + n_4)^{-1}$, and (n_1, n_2, n_3, n_4) is the element in the *i*-th row and *j*-th column of this table.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,2)	(0,0,1,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,0)	(0,0,2,1)	(0,0,1,0)
s_1		(1,0,0,1)	(0,1,1,1)	(1,0,0,1)	(0,2,1,2)	(1,0,1,1)	(0,0,1,0)	(1,0,2,1)	(0,1,0,1)	(1,0,1,1)	(0,1,1,1)	(1,0,1,1)	(0,1,2,1)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)
s_2			(0,1,1,2)	(0,1,1,0)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(0,1,1,1)	(0,1,1,1)	(0,1,1,0)	(1,0,1,2)	(1,0,1,0)	(2,0,2,1)	(1,0,1,0)
s_3				(1,1,1,1)	(0,1,0,1)	(1,0,0,1)	(1,1,1,1)	(1,0,0,1)	(0,1,0,1)	(1,1,1,1)	(0,1,1,0)	(0,1,1,0)	(1,1,1,1)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
s_4					(0,1,1,2)	(0,0,1,0)	(0,0,1,2)	(0,0,1,0)	(0,1,0,2)	(0,1,2,2)	(0,0,0,1)	(0,0,1,1)	(0,1,3,2)	(0,0,1,0)	(0,0,2,1)	(0,0,1,0)
s_5						(1,1,1,1)	(1,1,1,1)	(1,0,1,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,1)	(1,0,1,1)	(0,1,1,0)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)
s_6							(0,0,0,1)	(1,0,1,1)	(0,2,0,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,0)	(1,1,1,1)	(2,1,2,0)	(2,0,2,1)	(1,0,1,0)
s_7								(1,0,0,1)	(0,1,0,0)	(0,1,0,0)	(1,1,1,0)	(1,1,1,0)	(0,1,2,1)	(2,1,2,0)	(1,0,1,0)	(1,0,1,0)
s_8									(0,0,0,1)	(1,0,2,2)	(0,0,0,1)	(1,0,2,2)	(1,0,2,3)	(1,0,2,0)	(1,0,2,1)	(1,0,2,0)
s_9										(1,0,0,0)	(1,1,1,1)	(1,0,0,0)	(1,1,1,1)	(2,0,1,0)	(1,0,2,0)	(1,0,1,0)
s_{10}											(1,1,1,1)	(1,1,1,0)	(1,0,0,1)	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)
s_{11}												(2,1,1,1)	(2,1,0,0)	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)
s_{12}													(1,1,1,1)	(2,1,3,0)	(3,0,2,1)	(1,0,1,0)
s_{13}														(2,1,1,0)	(2,0,1,0)	(2,0,1,0)
s_{14}															(1,0,0,0)	(1,0,0,0)
s_{15}																(1,0,0,0)

Table 2: The vectors x of the 16 strategies due to the error in implementation for Simultaneous PD.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	$(0,1,1,-2)$	$\frac{1}{4}(2,-3,2,-1)$	$(0,1,2,-3)$	$\frac{1}{2}(1,1,-1,-1)$	$\frac{1}{3}(1,2,-1,-2)$	$(1,0,-3,2)$	$\frac{1}{2}(1,1,-1,-1)$	$(1,0,-3,2)$	$(0,1,1,-2)$	$\frac{1}{2}(1,1,-1,-1)$	$(0,1,2,-3)$	$\frac{1}{4}(2,2,-1,-3)$	$\frac{1}{2}(1,1,-1,-1)$	$(1,0,-2,1)$	$\frac{1}{9}(6,3,-8,-1)$	$(1,0,-2,1)$
s_1		$\frac{1}{2}(-3,2,2,-1)$	$\frac{1}{9}(6,-2,-5,1)$	$(-2,1,1,-1)$	$\frac{1}{25}(15,-14,3,-4)$	$\frac{1}{9}(-4,6,-1,-1)$	$(1,2,-5,2)$	$\frac{1}{4}(1,2,-2,-1)$	$\frac{1}{2}(1,-2,2,-1)$	$\frac{1}{3}(-1,2,-1,0)$	$\frac{1}{9}(6,-4,-1,-1)$	$\frac{1}{9}(-5,6,1,-2)$	$\frac{1}{8}(6,-1,-6,1)$	$(2,0,-3,1)$	$(1,1,-3,1)$	$(2,0,-3,1)$
s_2			$\frac{1}{2}(1,0,0,-1)$	$(1,-2,-1,2)$	$(1,1,1,-3)$	$\frac{1}{9}(-4,6,-1,-1)$	$(2,1,2,-5)$	$\frac{1}{9}(-2,6,1,-5)$	$(0,2,1,-3)$	$\frac{1}{3}(2,-1,0,-1)$	$\frac{1}{9}(6,-4,-1,-1)$	$\frac{1}{2}(2,-2,-1,1)$	$\frac{1}{8}(-1,6,1,-6)$	$(-1,1,-1,1)$	$\frac{1}{25}(-16,15,-6,7)$	$\frac{1}{4}(-3,2,-1,2)$
s_3				$\frac{1}{16}(1,-1,-1,1)$	$\frac{1}{2}(2,-2,1,-1)$	$(-1,1,1,-1)$	$(0,0,0,0)$	$(-1,1,2,-2)$	$\frac{1}{2}(1,-1,2,-2)$	$(0,0,0,0)$	$(1,-1,-1,1)$	$(2,-2,-1,1)$	$\frac{1}{8}(1,-1,-1,1)$	$\frac{1}{2}(-1,1,-2,2)$	$\frac{1}{2}(-2,2,-1,1)$	$\frac{1}{2}(-1,1,-1,1)$
s_4					$\frac{1}{2}(1,0,0,-1)$	$(1,1,-4,2)$	$\frac{1}{9}(9,6,-2,-13)$	$(1,0,-3,2)$	$\frac{1}{9}(3,-2,9,-10)$	$\frac{1}{5}(3,0,-1,-2)$	$(1,1,2,-4)$	$\frac{1}{2}(2,1,-1,-2)$	$\frac{1}{9}(6,1,-6,-1)$	$\frac{1}{2}(2,1,-5,2)$	$\frac{1}{3}(3,1,-4,0)$	$(1,0,-2,1)$
s_5						$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{9}(-1,6,-1,-4)$	$(1,-4,1,2)$	$\frac{1}{16}(-3,1,5,-3)$	$(0,0,0,0)$	$\frac{1}{9}(-1,6,-1,-4)$	$(1,-1,-1,1)$	$(2,1,-4,1)$	$(2,1,-4,1)$	$(2,1,-4,1)$
s_6							$(2,1,1,-4)$	$\frac{1}{3}(0,2,-1,-1)$	$\frac{1}{9}(6,-14,9,-1)$	$(1,2,-4,1)$	$(0,0,0,0)$	$\frac{1}{3}(-1,-1,0,2)$	$(0,0,0,0)$	$\frac{1}{5}(-2,0,-1,3)$	$\frac{1}{5}(-1,3,-2,0)$	$\frac{1}{2}(-1,1,-1,1)$
s_7								$\frac{1}{2}(-1,2,2,-3)$	$(1,-3,1,1)$	$(2,-5,2,1)$	$\frac{1}{9}(-1,-1,-4,6)$	$\frac{1}{9}(1,-5,-2,6)$	$\frac{1}{8}(1,-6,-1,6)$	$\frac{1}{25}(-4,3,-14,15)$	$\frac{1}{2}(-1,2,-2,1)$	$\frac{1}{4}(-1,2,-3,2)$
s_8									$\frac{1}{2}(1,2,2,-5)$	$\frac{1}{5}(0,3,-2,-1)$	$(1,1,2,-4)$	$\frac{1}{25}(3,15,-4,-14)$	$\frac{1}{3}(1,6,-1,-6)$	$\frac{1}{9}(-2,3,-10,9)$	$\frac{1}{2}(0,1,-1,0)$	$\frac{1}{9}(-1,3,-8,6)$
s_9										$(-4,1,1,2)$	$(0,0,0,0)$	$(-5,1,2,2)$	$(0,0,0,0)$	$\frac{1}{9}(-14,6,-1,9)$	$\frac{1}{9}(-1,9,-14,6)$	$\frac{1}{2}(-1,1,-1,1)$
s_{10}											$(0,0,0,0)$	$\frac{1}{9}(-1,-4,-1,6)$	$\frac{1}{2}(-3,2,2,-1)$	$(-4,1,2,1)$	$(-4,1,2,1)$	$(-3,1,2,0)$
s_{11}												$\frac{1}{2}(-1,0,0,1)$	$\frac{1}{4}(0,-1,3,-2)$	$(-3,1,1,1)$	$(-3,1,2,0)$	$(-3,1,2,0)$
s_{12}													$(0,0,0,0)$	$\frac{1}{9}(-1,1,-6,6)$	$\frac{1}{9}(-6,6,-1,1)$	$\frac{1}{2}(-1,1,-1,1)$
s_{13}														$\frac{1}{2}(-1,0,0,1)$	$\frac{1}{9}(-10,9,-2,3)$	$\frac{1}{9}(-8,6,-1,3)$
s_{14}															$\frac{1}{2}(-5,2,2,1)$	$(-2,1,1,0)$
s_{15}																$(-2,1,1,0)$

Table 3: The vectors η of the 16 strategies due to the error in implementation for the randomly alternating PD, where the vector η for S_i against S_j is $(\eta_1, \eta_2, \eta_3, \eta_4)$, with $\eta_i = n_i(n_1 + n_2 + n_3 + n_4)^{-1}$, and (n_1, n_2, n_3, n_4) is the element in the i-th row and j-th column of this table.

NOTE: we omitted the terms below the diagonal since the (i, j) -term is obtained from the (j, i) -term by interchanging η_1 and η_2 also η_3 and η_4.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	(0,0,1,1)	(0,1,2,1)	(0,1,3,2)	(0,2,3,1)	(0,0,1,1)	(0,1,2,1)	(0,1,3,2)	(0,2,3,1)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)
s_1		(1,1,2,2)	(1,1,2,2)	(1,3,4,2)	(2,1,3,4)	(1,2,3,2)	(2,3,5,4)	(1,4,5,2)	(1,0,1,2)	(0,1,1,0)	(1,2,3,2)	(0,1,1,0)	(1,1,2,2)	(0,1,1,0)	(2,1,3,4)	(0,1,1,0)
s_2			(1,1,2,2)	(3,5,6,4)	(2,1,4,5)	(2,3,4,3)	(3,4,6,5)	(1,2,2,1)	(1,0,2,3)	(1,2,2,1)	(2,3,4,3)	(2,5,4,1)	(1,1,2,2)	(1,3,2,0)	(1,2,2,1)	(1,3,2,0)
s_3				(1,1,1,1)	(4,2,3,5)	(1,1,1,1)	(1,1,1,1)	(4,6,5,3)	(2,0,1,3)	(1,1,1,1)	(1,1,1,1)	(2,4,3,1)	(1,1,1,1)	(1,3,2,0)	(3,5,4,2)	(1,3,2,0)
s_4					(0,0,1,1)	(1,2,3,2)	(1,2,4,3)	(1,2,2,1)	(0,0,1,1)	(1,2,2,1)	(1,2,3,2)	(2,4,3,1)	(0,0,1,1)	(1,2,1,0)	(1,2,2,1)	(1,2,1,0)
s_5						(1,1,1,1)	(1,1,1,1)	(3,4,3,2)	(1,0,1,2)	(1,1,1,1)	(1,1,1,1)	(2,3,2,1)	(1,1,1,1)	(1,2,1,0)	(2,3,2,1)	(1,2,1,0)
s_6							(1,1,1,1)	(5,6,4,3)	(1,0,2,3)	(1,1,1,1)	(1,1,1,1)	(4,5,3,2)	(1,1,1,1)	(2,3,1,0)	(3,4,2,1)	(2,3,1,0)
s_7								(2,2,1,1)	(2,0,1,3)	(2,1,1,2)	(4,3,2,3)	(2,2,1,1)	(2,2,1,1)	(2,3,1,0)	(4,5,2,1)	(2,3,1,0)
s_8									(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)
s_9										(1,1,1,1)	(1,1,1,1)	(0,1,1,0)	(1,1,1,1)	(0,1,1,0)	(1,2,2,1)	(0,1,1,0)
s_{10}											(1,1,1,1)	(2,3,2,1)	(1,1,1,1)	(1,2,1,0)	(2,3,2,1)	(1,2,1,0)
s_{11}												(2,2,1,1)	(2,2,1,1)	(1,2,1,0)	(3,4,2,1)	(1,2,1,0)
s_{12}													(1,1,1,1)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)
s_{13}														(1,1,0,0)	(1,1,0,0)	(1,1,0,0)
s_{14}															(1,1,0,0)	(1,1,0,0)
s_{15}																(1,1,0,0)

Table 4: The error vectors x in implementation for the randomly alternating PD of the 16 strategies.

NOTE: we omitted the terms below the diagonal since the (i,j) -term is obtained from the (j,i) -term by interchanging η_1 and η_2 also η_3 and η_4.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_0	$\frac{1}{2}(1,1,-1,-1)$	$\frac{1}{4}(2,-1,-2,1)$	$\frac{1}{18}(9,2,-9,-2)$	$\frac{1}{18}(9,-5,-9,5)$	$\frac{1}{4}(2,3,-2,-3)$	$\frac{1}{2}(1,0,-1,0)$	$\frac{1}{18}(9,5,-9,-5)$	$\frac{1}{18}(9,-2,-9,2)$
s_1		$\frac{1}{9}(1,1,-1,-1)$	$\frac{1}{9}(1,1,-1,-1)$	$\frac{1}{5}(2,-1,-2,1)$	$\frac{1}{25}(7,-1,-7,1)$	$\frac{1}{4}(1,0,-1,0)$	$\frac{1}{49}(8,5,-8,-5)$	$\frac{1}{18}(7,-2,-7,2)$
s_2			$\frac{1}{9}(1,1,-1,-1)$	$\frac{1}{27}(3,-2,-3,2)$	$\frac{1}{18}(2,7,-2,-7)$	$\frac{1}{9}(1,0,-1,0)$	$\frac{1}{27}(3,2,-3,-2)$	$\frac{1}{9}(1,-1,-1,1)$
s_3				$(0,0,0,0)$	$\frac{1}{49}(-5,8,5,-8)$	$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{27}(2,-3,-2,3)$
s_4					$(1,1,-1,-1)$	$\frac{1}{4}(1,0,-1,0)$	$\frac{1}{5}(2,1,-2,-1)$	$\frac{1}{9}(1,-1,,-1,1)$
s_5						$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{9}(0,-1,0,1)$
s_6							$(0,0,0,0)$	$\frac{1}{27}(-2,-3,2,3)$
s_7								$\frac{1}{6}(-1,-1,1,1)$

	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	$\frac{1}{2}(1,2,-1,-2)$	$\frac{1}{2}(1,-3,-1,3)$	$\frac{1}{2}(1,0,-1,0)$	$\frac{1}{4}(2,-3,-2,3)$	$\frac{1}{2}(1,3,-1,-3)$	$\frac{1}{2}(1,-2,-1,2)$	$\frac{1}{4}(2,1,-2,-1)$	$\frac{1}{2}(1,-1,-1,1)$
s_1	$\frac{1}{2}(-1,2,1,-2)$	$(2,-3,-2,3)$	$\frac{1}{4}(1,0,-1,0)$	$(1,-1,-1,1)$	$\frac{1}{3}(0,1,0,-1)$	$(1,-1,-1,1)$	$\frac{1}{25}(-1,7,1,-7)$	$\frac{1}{4}(3,-2,-3,2)$
s_2	$\frac{1}{9}(1,9,-1,-9)$	$\frac{1}{9}(1,-4,-1,4)$	$\frac{1}{9}(1,0,-1,0)$	$\frac{1}{18}(2,-7,-2,7)$	$\frac{1}{9}(1,4,-1,-4)$	$\frac{1}{9}(1,-9,-1,9)$	$\frac{1}{9}(1,-1,-1,1)$	$\frac{1}{18}(2,-9,-2,9)$
s_3	$\frac{1}{9}(-4,9,4,-9)$	$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{5}(1,-2,-1,2)$	$(0,0,0,0)$	$\frac{1}{9}(5,-9,-5,9)$	$\frac{1}{49}(5,-8,-5,8)$	$\frac{1}{18}(5,-9,-5,9)$
s_4	$(1,1,,-1,-1)$	$\frac{1}{3}(0,-1,0,1)$	$\frac{1}{4}(1,0,-1,0)$	$\frac{1}{25}(-1,-7,1,7)$	$(2,3,-2,-3)$	$\frac{1}{2}(-1,-2,2,1)$	$\frac{1}{9}(1,-1,-1,1)$	$\frac{1}{4}(-1,-2,1,2)$
s_5	$(0,1,0,-1)$	$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{4}(0,-1,0,1)$	$(0,0,0,0)$	$(0,-1,0,1)$	$\frac{1}{4}(0,-1,0,1)$	$\frac{1}{2}(0,-1,0,1)$
s_6	$\frac{1}{9}(4,9,-4,-9)$	$(0,0,0,0)$	$(0,0,0,0)$	$\frac{1}{49}(-5,-8,5,8)$	$(0,0,0,0)$	$\frac{1}{9}(-4,-9,4,9)$	$\frac{1}{5}(-1,-2,1,2)$	$\frac{1}{18}(-5,-9,5,9)$
s_7	$\frac{1}{9}(-1,9,1,-9)$	$\frac{1}{9}(-1,4,1,-4)$	$\frac{1}{9}(-1,0,1,0)$	$\frac{1}{9}(-1,-1,1,1)$	$\frac{1}{9}(-1,-4,1,4)$	$\frac{1}{9}(-1,-9,1,9)$	$\frac{1}{18}(-2,-7,2,7)$	$\frac{1}{18}(-2,-9,2,9)$
s_8	$(1,1,,-1,-1)$	$(1,-2,-1,2)$	$(1,0,-1,0)$	$(1,-1,-1,1)$	$(1,2,-1,-2)$	$(1,-1,-1,1)$	$\frac{1}{2}(2,1,-2,-1)$	$\frac{1}{2}(2,-1,-2,1)$
s_9		$(0,0,0,0)$	$(0,0,0,0)$	$(3,-2,-3,2)$	$(0,0,0,0)$	$(2,-1,-2,1)$	$\frac{1}{3}(1,0,-1,0)$	$\frac{1}{2}(3,-1,-3,1)$
s_{10}			$(0,0,0,0)$	$\frac{1}{4}(0,-1,0,1)$	$(0,0,0,0)$	$(0,-1,0,1)$	$\frac{1}{4}(0,-1,0,1)$	$\frac{1}{2}(0,-1,0,1)$
s_{11}				$\frac{1}{9}(-1,-1,1,1)$	$\frac{1}{3}(0,-1,0,1)$	$\frac{1}{2}(1,-2,-1,2)$	$\frac{1}{25}(1,-7,-1,7)$	$\frac{1}{4}(1,-2,-1,2)$
s_{12}					$(-1,-1,1,1)$	$(-2,-1,2,1)$	$(-3,-2,3,2)$	$\frac{1}{2}(-3,-1,3,1)$
s_{13}						$(-1,-1,1,1)$	$(-1,-1,1,1)$	$\frac{1}{2}(-2,-1,2,1)$
s_{14}							$(-1,-1,1,1)$	$\frac{1}{4}(-3,-2,3,2)$
s_{15}								$\frac{1}{2}(-1,-1,1,1)$

Table 5: The payoff matrix due to the error in implementation of the 16 strategies for the randomly alternating PD game for the values

$a = 2, b = 1, c = 3, d = -2$ $(R = 3, S = 0, T = 4, P = 1)$ with $\epsilon = 0.001$

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	0.501	1.2488	0.99983	1.4987	0.50175	1.2495	1.0003	1.4992	0.5025	1.995	1.2495	1.9973	0.504	1.9965	1.2503	1.998
s_1	0.25175	0.83356	0.83356	1.299	0.59992	1.1248	0.99957	1.4167	0.252	1.989	1.1248	1.996	0.83433	1.996	0.60088	1.9978
s_2	0.33472	0.83356	0.83356	1.3997	0.58439	1.0832	1.0001	1.3329	0.33622	1.3319	1.0832	1.5821	0.83456	1.8302	1.3329	1.8317
s_3	0.16844	0.5014	0.72263	1	0.64345	1	1	1.2774	0.17011	1	1	1.4986	1	1.8298	1.3566	1.8316
s_4	0.50075	0.9996	0.91661	1.2138	0.502	1.1248	1.0002	1.3329	0.502	1.3323	1.1248	1.4992	0.507	1.75	1.3329	1.7488
s_5	0.2515	0.62575	0.75033	1	0.62575	1	1	1.2497	0.253	1	1	1.3743	1	1.747	1.3743	1.7485
s_6	0.33456	0.71467	0.77804	1.0003	0.6018	1	1	1.2220	0.33589	1	1	1.2853	1	1.6641	1.399	1.6654
s_7	0.16828	0.41794	0.66711	0.83383	0.66711	0.91678	0.99983	1.1663	0.16978	0.66811	0.91678	1.1664	1.1654	1.6638	1.4156	1.6653
s_8	0.5005	1.2475	0.99933	1.4977	0.502	1.249	1.0003	1.4987	0.502	1.993	1.246	1.996	0.505	1.996	1.2505	1.9975
s_9	0.003	0.009	0.66744	1	0.667	1	1	1.3326	0.005	1	1	1.991	1	1.995	1.333	1.997
s_{10}	0.2515	0.62575	0.75033	1	0.62575	1	1	1.2497	0.253	1	1	1.3743	1	1.747	1.3743	1.7485
s_{11}	0.00225	0.004	0.58406	0.701	0.70016	0.87525	0.99986	1.1664	0.004	0.011	0.87525	1.1664	1.1657	1.7465	1.3991	1.7483
s_{12}	0.5	1.2497	1.2494	1	0.503	1	1	1.1677	0.501	1	1	1.167	0.998	1.499	1.497	1.5
s_{13}	0.0025	0.004	0.50133	0.50267	0.7495	0.751	0.99967	1.0007	0.004	0.007	0.751	0.7525	1.495	1.498	1.498	1.4995
s_{14}	0.25125	0.9996	0.66711	0.78618	0.66711	0.87525	0.9998	1.0834	0.2525	0.66767	0.87525	1.004	1.498	1.498	1.4995	1.4993
s_{15}	0.002	0.00275	0.50083	0.50133	0.74975	0.75025	0.99967	0.99983	0.0035	0.005	0.7505	0.75125	1.496	1.4975	1.4983	1.499

Table 6: The vectors η for the randomly alternating PD of the 16 strategies due to the error in perception, where the vector η for η_i against η_j is $(\eta_1, \eta_2, \eta_3, \eta_4)$, with $\eta_i = n_i(n_1 + n_2 + n_3 + n_4)^{-1}$, and (n_1, n_2, n_3, n_4) is the element in the i-th row and j-th column of this table.

NOTE: we omitted the terms below the diagonal since the (i, j) -term is obtained from the (j, i) -term by interchanging η_1 and η_2 also η_3 and η_4.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	(0,0,1,1)	(0,1,2,1)	(0,1,3,2)	(0,2,3,1)	(0,0,1,1)	(0,1,2,1)	(0,1,3,2)	(0,2,3,1)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)
s_1		(1,1,2,2)	(1,1,2,2)	(1,3,4,2)	(2,1,3,4)	(1,2,3,2)	(2,3,5,4)	(1,4,5,2)	(1,0,1,2)	(0,1,1,0)	(1,2,3,2)	(0,1,1,0)	(1,1,2,2)	(0,1,1,0)	(2,1,3,4)	(0,1,1,0)
s_2			(1,1,2,2)	(3,5,6,4)	(2,1,4,5)	(2,3,4,3)	(3,4,6,5)	(1,2,2,1)	(1,0,2,3)	(1,2,2,1)	(2,3,4,3)	(2,5,4,1)	(1,1,2,2)	(1,3,2,0)	(1,2,2,1)	(1,3,2,0)
s_3				(1,1,1,1)	(4,2,3,5)	(1,1,1,1)	(1,1,1,1)	(4,6,5,3)	(2,0,1,3)	(1,1,1,1)	(1,1,1,1)	(2,4,3,1)	(1,1,1,1)	(1,3,2,0)	(3,5,4,2)	(1,3,2,0)
s_4					(0,0,1,1)	(1,2,3,2)	(1,2,4,3)	(1,2,2,1)	(0,0,1,1)	(1,2,2,1)	(1,2,3,2)	(2,4,3,1)	(0,0,1,1)	(1,2,1,0)	(1,2,2,1)	(1,2,1,0)
s_5						(1,1,1,1)	(1,1,1,1)	(3,4,3,2)	(1,0,1,2)	(1,1,1,1)	(1,1,1,1)	(2,3,2,1)	(1,1,1,1)	(1,2,1,0)	(2,3,2,1)	(1,2,1,0)
s_6							(1,1,1,1)	(5,6,4,3)	(1,0,2,3)	(1,1,1,1)	(1,1,1,1)	(4,5,3,2)	(1,1,1,1)	(2,3,1,0)	(3,4,2,1)	(2,3,1,0)
s_7								(2,2,1,1)	(2,0,1,3)	(2,1,1,2)	(4,3,2,3)	(2,2,1,1)	(2,2,1,1)	(2,3,1,0)	(4,5,2,1)	(2,3,1,0)
s_8									(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)	(0,0,1,1)	(0,1,1,0)	(0,1,2,1)	(0,1,1,0)
s_9										(1,1,1,1)	(1,1,1,1)	(0,1,1,0)	(1,1,1,1)	(0,1,1,0)	(1,2,2,1)	(0,1,1,0)
s_{10}											(1,1,1,1)	(2,3,2,1)	(1,1,1,1)	(1,2,1,0)	(2,3,2,1)	(1,2,1,0)
s_{11}												(2,2,1,1)	(2,2,1,1)	(1,2,1,0)	(3,4,2,1)	(1,2,1,0)
s_{12}													(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)
s_{13}														(1,1,0,0)	(1,1,0,0)	(1,1,0,0)
s_{14}															(1,1,0,0)	(1,1,0,0)
s_{15}																(1,1,0,0)

Table 7: The error vectors x in perception for the randomly alternating PD of the 16 strategies.

	s_0	s_1	s_2	s_3	s_4	s_5
s_0	$(0,0,0,0)$	$\frac{(2\lambda+1)}{8}(0,-1,0,1)$	$\frac{(\lambda-1)}{18}(0,-1,0,1)$	$\frac{\lambda}{9}(0,-1,0,1)$	$\frac{\lambda}{4}(0,1,0,-1)$	$(0,0,0,0)$
s_1		$\frac{\lambda}{9}(1,-2,-1,2)$	$\frac{\lambda}{18}(-1,-1,1,1)$	$\frac{1}{50}((5\lambda+6),-(5\lambda+2),-(5\lambda+6),(5\lambda+2))$	$\frac{1}{50}(-(7\lambda+1),-(\lambda+3),(7\lambda+1),(\lambda+3))$	$\frac{1}{16}(1,0,-1,0)$
s_2			$\frac{1}{18}(-(\lambda+2),-\lambda,(\lambda+2),\lambda)$	$\frac{1}{162}(-(9\lambda+6),-(3\lambda-2),(9\lambda+6),(3\lambda-2))$	$\frac{(2\lambda-1)}{36}(-1,1,1,-1)$	$\frac{(2\lambda+1)}{36}(-1,0,1,0)$
s_3				$(0,0,0,0)$	$\frac{1}{98}(-(3\lambda-2),-(5\lambda+6),(3\lambda-2),(5\lambda+6))$	$(0,0,0,0)$
s_4					$\frac{\lambda}{2}(1,1,-1,-1)$	$\frac{1}{16}(-1,0,1,0)$
s_5						$(0,0,0,0)$

	s_6	s_7	s_8	s_9	s_{10}
s_0	$\frac{(\lambda+1)}{9}(0,1,0,-1)$	$\frac{(\lambda+2)}{18}(0,1,0,-1)$	$\frac{\lambda}{2}(0,1,0,-1)$	$(\lambda+1)(0,-1,0,1)$	$(0,0,0,0)$
s_1	$\frac{1}{98}(-(5\lambda-1),(3\lambda+5),(5\lambda-1),-(3\lambda+5))$	$\frac{(2\lambda+3)}{36}(1,1,-1,-1)$	$\frac{(2\lambda+1)}{4}(-1,1,1,-1)$	$\frac{(\lambda+1)}{2}(3,-5,-3,5)$	$\frac{1}{16}(1,0,-1,0)$
s_2	$\frac{(3\lambda+1)}{162}(-3,-1,3,1)$	$\frac{1}{18}\left(-(\lambda+2),(\lambda+1),(\lambda+2),-(\lambda+1)\right)$	$\frac{1}{6}\left((\lambda+1),(3\lambda+1),-(\lambda+1),-(3\lambda+1)\right)$	$\frac{(\lambda+1)}{3}(-1,-5,1,5)$	$\frac{(2\lambda+1)}{36}(-1,0,1,0)$

s_3	$(0,0,0,0)$	$\frac{1}{162}\left((3\lambda-2),(9\lambda+6),-(3\lambda-2),-(9\lambda+6)\right)$	$\frac{1}{18}\left(-(5\lambda+2),(9\lambda+6),(5\lambda+2),-(9\lambda+6)\right)$	$(0,0,0,0)$	$(0,0,0,0)$
s_4	$\frac{1}{50}\left((5\lambda-1),(5\lambda+3),-(5\lambda-1),-(5\lambda+3)\right)$	$\frac{(\lambda+1)}{18}(-1,1,1,-1)$	$\frac{\lambda}{2}(1,1,-1,-1)$	$\frac{(\lambda+1)}{6}(-1,-1,1,1)$	$\frac{1}{16}(-1,0,1,0)$
s_5	$(0,0,0,0)$	$\frac{(2\lambda+1)}{36}(0,1,0,-1)$	$\frac{(2\lambda+1)}{4}(0,1,0,-1)$	$(0,0,0,0)$	$(0,0,0,0)$
s_6	$(0,0,0,0)$	$\frac{1}{54}\left(-\left(\lambda+\frac{5}{3}\right),(3\lambda+1),\left(\lambda+\frac{5}{3}\right),-(3\lambda+1)\right)$	$\frac{1}{18}\left((5\lambda+3),(9\lambda+3),-(5\lambda+3),-(9\lambda+3)\right)$	$(0,0,0,0)$	$(0,0,0,0)$
s_7		$\frac{\lambda}{18}(1,1,-1,-1)$	$\frac{1}{18}\left((\lambda+2),(9\lambda+6),-(\lambda+2),-(9\lambda+6)\right)$	$\frac{(\lambda+1)}{18}(1,5,-1,-5)$	$\frac{(2\lambda+1)}{36}(1,0,-1,0)$
s_8			$\frac{\lambda}{2}(1,1,-1,-1)$	$\frac{(\lambda+1)}{2}(1,-3,-1,3)$	$\frac{(2\lambda+1)}{4}(1,0,-1,0)$
s_9				$(0,0,0,0)$	$(0,0,0,0)$
s_{10}					$(0,0,0,0)$

Table 7 (cont.)

NOTE: we omitted the terms below the diagonal since the (i,j) -term is obtained from the (j,i) -term by interchanging $\eta_1$and η_2 also η_3 and η_4 .

	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	$\frac{(\lambda+1)}{4}(0,-1,0,1)$	$\lambda(0,1,0,-1)$	$\frac{(\lambda+1)}{2}(0,-1,0,1)$	$\frac{(2\lambda+1)}{8}(0,1,0,-1)$	$(0,0,0,0)$
s_1	$\frac{(\lambda+1)}{2}(1,-1,-1,1)$	$\frac{\lambda}{6}(-1,1,1,-1)$	$\frac{(\lambda+1)}{2}(1,-1,-1,1)$	$\frac{1}{50}\left(-(7\lambda+5),-(\lambda-5),(7\lambda+5),(\lambda-5)\right)$	$\frac{(\lambda+1)}{4}(1,0,-1,0)$
s_2	$\frac{(2\lambda+3)}{36}(-1,-1,1,1)$	$\frac{\lambda}{18}(-1,5,1,-5)$	$\frac{1}{18}\left(-(\lambda+2),-(9\lambda+6),(\lambda+2),(9\lambda+6)\right)$	$\frac{(\lambda+1)}{18}(-1,1,1,-1)$	$\frac{(\lambda+2)}{18}(-1,0,1,0)$
s_3	$\frac{1}{50}\left((5\lambda+2),-(5\lambda+6),-(5\lambda+2),(5\lambda+6)\right)$	$(0,0,0,0)$	$\frac{1}{18}\left((5\lambda+2),-3(3\lambda+2),-(5\lambda+2),3(3\lambda+2)\right)$	$\frac{1}{98}\left((3\lambda-2),(5\lambda+6),-(3\lambda-2),-(5\lambda+6)\right)$	$\frac{\lambda}{21}(2,1,-2,-1)$
s_4	$\frac{1}{50}\left(-(7\lambda+6),(\lambda-2),(7\lambda+6),-(\lambda-2)\right)$	$\frac{\lambda}{2}(3,5,-3,-5)$	$\frac{(3\lambda+2)}{50}(-3,-1,3,1)$	$\frac{(\lambda+1)}{18}(-1,1,1,-1)$	$\frac{(2\lambda+1)}{8}(-1,0,1,0)$
s_5	$\frac{1}{16}(0,-1,0,1)$	$(0,0,0,0)$	$\frac{(2\lambda+1)}{4}(0,-1,0,1)$	$\frac{1}{16}(0,1,0,-1)$	$(0,0,0,0)$

s_6	$\frac{1}{98}(-(3\lambda+5),-(1-5\lambda),(3\lambda+5),(1-5\lambda))$	(0,0,0,0)	$\frac{1}{18}(-(5\lambda+1),-(9\lambda+3),(5\lambda+1),(9\lambda+3))$	$\frac{(1-\lambda)}{10}(1,1,-1,-1)$	$\frac{(\lambda+1)}{9}(-1,0,1,0)$
s_7	$\frac{\lambda}{18}(-1,-1,1,1)$	$\frac{\lambda}{18}(1,-5,-1,5)$	$\frac{1}{18}((\lambda-1),-(9\lambda+3),-(\lambda-1),(9\lambda+3))$	$\frac{(2\lambda-1)}{36}(1,-1,-1,1)$	$\frac{(\lambda-1)}{18}(1,0,-1,0)$
s_8	$\frac{(\lambda+1)}{2}(1,-1,-1,1)$	$\frac{\lambda}{2}(1,3,-1,-3)$	$\frac{(\lambda+1)}{2}(1,-1,-1,1)$	$\frac{(2\lambda+1)}{4}(1,1,-1,-1)$	$\frac{(\lambda+1)}{2}(1,0,-1,0)$
s_9	$\frac{(\lambda+1)}{2}(5,-3,-5,3)$	(0,0,0,0)	$\frac{(\lambda+1)}{2}(3,-1,-3,1)$	$\frac{(\lambda+1)}{6}(1,1,-1,-1)$	$(\lambda+1)(1,0,-1,0)$
s_{10}	$\frac{\lambda}{16}(0,1,0,-1)$	(0,0,0,0)	$\frac{(2\lambda+1)}{4}(0,-1,0,1)$	$\frac{1}{16}(0,1,0,-1)$	(0,0,0,0)
s_{11}	$\frac{\lambda}{18}(1,1,-1,-1)$	$\frac{\lambda}{6}(1,-1,-1,1)$	$\frac{(2\lambda+1)}{4}(1,-1,-1,1)$	$\frac{1}{50}((7\lambda+1),(\lambda+3),-(7\lambda+1),-(\lambda+3))$	$\frac{(2\lambda+1)}{8}(1,0,-1,0)$
s_{12}		(0,0,0,0)	$\frac{\lambda}{2}(-3,-1,3,1)$	$\frac{\lambda}{2}(-5,-3,5,3)$	$\lambda(-1,0,1,0)$
s_{13}			$\frac{\lambda}{2}(-1,-1,1,1)$	$\frac{\lambda}{2}(-1,-1,1,1)$	$\frac{\lambda}{2}(-1,0,1,0)$
s_{14}				$\frac{\lambda}{2}(-1,-1,1,1)$	$\lambda(-1,0,1,0)$
s_{15}					(0,0,0,0)

Table 8:The payoff matrix due to the error in perception of the 16 strategies for the randomly alternating PD game for the values

$a = 2, b = 1, c = 3, d = -2$ $(R = 3, S = 0, T = 4, P = 1)$ with $\epsilon = 0.001$ and $\lambda = 0.01$

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_0	0.5	1.2496	1.0002	1.5	0.50001	1.25	1.0003	1.5003	0.50002	1.9970	1.25	1.9992	0.50003	1.9985	1.2504	2
s_1	0.25013	0.83333	0.83333	1.2998	0.59984	1.1249	1.0001	1.4168	0.25102	1.9909	1.1249	1.9980	0.83334	1.998	0.6004	0.59975
s_2	0.33328	0.83333	0.83344	1.1667	0.58322	1.0833	1	1.3336	0.33368	1.3286	1.0834	1.5832	0.83334	1.8324	1.3336	1.8334
s_3	0.16667	0.5004	0.72210	1	0.64265	1	1	1.2779	0.16780	1	1	1.4996	1	1.8322	1.3573	1.1
s_4	0.5	1	0.91678	1.2144	0.50001	1.1251	1.0002	1.3336	0.50001	1.3330	1.1251	1.5	0.50006	1.75	1.3336	1.7501
s_5	0.25	0.62519	0.74997	1	0.62481	1	1	1.2501	0.25077	1	1	1.3748	1	1.7492	1.3752	1.75
s_6	0.33322	0.71426	0.77773	1	0.59988	1	1	1.2223	0.33368	1	1	1.2857	1	0.71383	1.4002	1.6668
s_7	0.16656	0.41683	0.66628	0.83326	0.66644	0.91664	0.99989	1.1667	0.16757	0.66745	0.91664	1.1667	1.1667	1.6662	1.4168	1.6667
s_8	0.5	1.2490	1.0003	2.2493	0.50001	1.2497	1.0003	1.5	0.50001	1.995	1.2497	1.998	0.50004	1.998	1.2505	1.9995
s_9	0.00101	0.00707	0.66734	1	0.66633	1	1	1.3332	0.00303	1	1	1.9929	1	1.999	1.3337	1.999
s_{10}	0.25	0.62519	0.74992	1	0.62481	1	1	1.2501	0.25077	1	1	1.375	1	1.7492	1.3752	1.75
s_{11}	0.0002525	0.00202	0.58317	0.70024	0.69968	0.87506	0.87486	1.1667	0.00202	0.00909	0.87500	1.1667	1.1667	1.7490	1.4002	1.7499
s_{12}	0.49999	0.83333	0.83333	1	0.50002	1	1	1.1667	0.5	1	1	1.1667	1	1.5	1.5	1.5
s_{13}	0.000505	0.00202	0.49999	0.50068	0.74968	0.75026	1	1	0.00202	0.00505	0.75026	0.75102	1.5	1.5	1.5	1.5
s_{14}	0.24987	0.99960	0.66644	0.78559	0.66644	0.87494	1.0002	1.0832	0.25051	0.667	0.87494	1	1.4999	1.5	1.5	1.5
s_{15}	0	0.0007575	0.49967	0.5	0.74962	0.75	0.99966	0.99984	0.001515	0.00303	0.75	0.75038	1.5	1.5	1.5	1.5