A Comparison of Numerical Methods for Solving the Unforced Van Der Pol's Equation

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Abstract

Due to the advancements in the field of computational mathematics, numerical methods are most widely being utilized to solve the equations arising in the fields of applied medical sciences, engineering and technology. In this paper, the numerical solutions of an important equation of applied dynamics: namely, the Unforced Van der Pol's Equation (UFVDP) are obtained by reducing it to a system of two first order differential equations. The objective of this work is to investigate the efficiency of improved Heun's (IH) method against the classical Runge-Kutta (RK4) and Mid-point (MP) methods for UFVDP equation. For analysis of accuracy, the Poincare-Lindstedt method has been used as a comparison criterion and respective error bounds are obtained. The results show that the popular RK4 method retains its better accuracy than other methods used for comparison.

Keywords: Van der Pol, Runge-Kutta, Mid-point, Improved Heun's, Poincare-Lindstedt.

1. Introduction

Mathematics is referred as the language of science because almost every physical system and indeed any phenomenon in nature may be modelled using mathematical equations. Thus solving the mathematical models can help in efficient analysis and careful examination of the physical and dynamical properties and characteristics of the systems etc. Mathematical models arising from physical phenomenon may be linear or non-linear but only the linear and a few non-linear models can easily be solved using analytical methods. To deal with non-linear and stiff problems, typically most mathematicians rely on numerical methods that closely approximate the solutions [1]. Recent research in computational mathematics is not only devoted to the proposition of new numerical methods, but is also primarily concerned with the comparison of available and new class of methods on the basis of their accuracy, reliability, stability and consistency. In this paper, an important equation of applied dynamics is explored (Equation 1).

$$\frac{d^2x}{dt^2}+\varepsilon\left(x^2-1\right)\frac{dx}{dt}+x=f(t) \qquad (1)$$

It is better known as the Van der Pol's equation, named after a Dutch electrical engineer, "Balthazar Van der Pol" [2]. Equation 1 is a second order non-homogeneous non-linear ordinary differential equation that represents dynamical behaviour of the Van der Pol oscillator and demands at least two initial or boundary conditions for solution. In Equation 1,

$\varepsilon > 0$ is a real constant representing strength of damping of the oscillator. The right hand side in Equation 1 represents the forced behaviour of the oscillator and is given as: $f(t) = X_0 \cos t$

where X_0 is amplitude of motion. If $f(t) = 0$, then Equation 1 reduces to an autonomous homogeneous second order non-linear ordinary differential equation:

$$\frac{d^2x}{dt^2}+\varepsilon\left(x^2-1\right)\frac{dx}{dt}+x=0$$

$$\text{subject to} \quad x(t_0)=x_0 \text{ and } x'(t_0)=x'_0 \qquad (2)$$

Equation 2 is referred to as the unforced Van der Pol's equation (UFVDP). The solution, *x(t)* of Equation 2 is assumed to follow a damped oscillatory pattern, depending on the strength of damping ε.

2. Background

In 1927, while studying triode oscillations in electrical circuits, B. Van der Pol observed the convergence of all initial conditions to same periodic orbit of finite amplitude. This led him to propose Equation 2 for the oscillatory model. B. Van der Pol himself then studied forced case of his Equation 1 with $\varepsilon \geq 1$ and termed the phenomenon as relaxation oscillations. [2,3,4]. Equation 2 has also been used for modelling oscillatory phenomenon in Physics, Electronics, Biology, Neurology, Sociology, Chemical reactions, and Wind-induced motions of structures, Rheology and even in Economics. [2,5,6,7].

Since then many attempts have been made to solve Equation 2 analytically and numerically to investigate the presence of limit cycle [8]. Presently in research the theory of the existence, uniqueness and stability of solution to Equation 2 are well developed and have been discussed in detail in [1, 9]. The solution to Equations 2 has been obtained by a number of authors: for example, in [10] using Collocation method, in [11] using MATLAB Ode45 and Ode15s built-in functions, in [12] using damped Fourier series method; also, in [7] using restarted Adomain decomposition method, in [5] using Mid-point method, and in [6] using predictor-corrector Adam-Bashforth-Moulton method.

Due to advancements in the field of computational mathematics the accuracy and performance of any new numerical method or modification to existing are being investigated by applying them to some physical models whose theory is well developed. The Unforced Van der Pol's equation is in fact such a model. This model is considered a classical test problem and a dynamical model to test efficiency and reliability of new methods for solving non-linear differential equations. [2, 13]

In 2010, authors in [14] attempted to improve efficiency of existing Heun's method for solving initial value problems concerned with first order ordinary differential equations. The proposed Improved Heun's method was used in some first order initial value problems and the method provided better accuracy than the usual Heun's method. In this paper, the efficiency of improved Heun's (IH) method is tested against classical Runge-Kutta (Rk4) and mid-point (MP) methods for solving Equation 2.

Equation 2 is a non-linear equation in which degree of non-linearity depends on magnitude of the parameter ε. There is no complete analytic solution to Equation 2 and other similar non-linear equations [15, 16]. Thus in order to be at a position to compare the performance of IH against RK4 and MP methods, the Poincare-Lindstedt method (P-L) has been used as an approximate analytical solution. The P-L method is a perturbation method that is a consequence of successive improvement as presented by Lindstedt (1883) and elaborated upon by Poincare (1892) of his original Poisson's (1830) perturbation method. The P-L method solves Equation 2 in the form of a power series with respect to ε to obtain periodic solutions. [17] The P-L method has been used by many authors to solve Equation 2 [16, 18], to solve other non-linear equations like duffing Van der Pol's equation [7, 15, 16] and equation of conservative oscillator with cubic restoring force [17].

Authors in [7, 15] used perturbation method - P-L as a comparison criterion to test the accuracy of numerical solutions. Like-wise, in this work, the solutions to Equation 2 developed numerically using the RK4, IH and MP methods are compared in terms accuracy and performance with respect to the P-L method as an approximate analytic solution to Equation 2.

3. Method

In this section we present the UFVDP (Equation 2) split into a system of two first order ordinary differential equations and specific algorithms related to each of RK4, MP and IH methods for its solution in turn. Finally the solution to Equation 2 by Poincare-Lindstedt (P-L) method is also provided for the purpose of comparison of solutions. Using the substitution: $\frac{dx}{dy} = y$ the Equation 2, the second order initial value problem can be reduced to the following system of first order ordinary differential equations (Equation 3):

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \varepsilon\left(1-x^2\right)y - x \end{aligned}\right\} \text{subject to } x\left(t_0\right) = x_0 \text{ and } y\left(t_0\right) = y_0 \tag{3}$$

3.1. Algorithms used for solving Equation 3:

Consider a general system of two first order ordinary differential equations (Equation 4):

$$\left.\begin{aligned} \frac{dx}{dt} &= f\left(t,x,y\right) \\ \frac{dy}{dt} &= g\left(t,x,y\right) \end{aligned}\right\} \text{subject to } x\left(t_0\right) = x_0 \text{ and } y\left(t_0\right) = y_0 \tag{4}$$

3.1.1. Classical Runge-Kutta Method (RK4)

The classical Runge-Kutta method (RK4) for the system Equation 4 takes the form (Equation 5):

$$\left.\begin{array}{ll} x_{n+1}=x_n+K \text{ and } y_{n+1}=y_n+L & \\ \text{Where } K=\frac{1}{6}(k_1+2k_2+2k_3+k_4), L=\frac{1}{6}(l_1+2l_2+2l_3+l_4) \text{ and} & \\ k_1=\Delta t f(t_n,x_n,y_n) \quad | & l_1=\Delta t g(t_n,x_n,y_n) \\ k_2=\Delta t f(t_n+0.5\Delta t,x_n+0.5k_1,y_n+0.5l_1) \quad | & l_2=\Delta t g(t_n+0.5\Delta t,x_n+0.5k_1,y_n+0.5l_1) \\ k_3=\Delta t f(t_n+0.5\Delta t,x_n+0.5k_2,y_n+0.5l_2) \quad | & l_3=\Delta t g(t_n+0.5\Delta t,x_n+0.5k_2,y_n+0.5l_2) \\ k_4=\Delta t f(t_n+\Delta t,x_n+k_3,y_n+l_3) \quad | & l_4=\Delta t g(t_n+\Delta t,x_n+k_3,y_n+l_3) \end{array}\right\} \quad (5)$$

3.1.2 Mid-Point Method (MP)

The extension to the algorithm presented in [5] for solving Equation 4 can be written as (Equation 6):

$$\left.\begin{array}{l} x_{n+1}=x_n+\Delta t f\left(t_n+0.5\Delta t, x_n+0.5\Delta t f(t_n,x_n,y_n), y_n+0.5\Delta t g(t_n,x_n,y_n)\right) \\ y_{n+1}=y_n+\Delta t g\left(t_n+0.5\Delta t, x_n+0.5\Delta t f(t_n,x_n,y_n), y_n+0.5\Delta t g(t_n,x_n,y_n)\right) \end{array}\right\} \quad (6)$$

3.1.3. Improved Heun's Method (IH)

The Improved Heun's method (IH) as proposed in [14] can be extended to solve a system of two first order equations as (Equation 7):

$$\begin{array}{l} x_{n+1}=x_n+\Delta t f(t_n+0.5\Delta t,x_n+k_1,y_n+l_1) \text{ and } y_{n+1}=y_n+\Delta t g(t_n+0.5\Delta t,x_n+k_1,y_n+l_1) \\ \text{Where } k_1=0.5\Delta t\left[f(t_n,x_n,y_n)+f(t_{n+1},x^*_{n+1},y^*_{n+1})\right] \text{ and } l_1=0.5\Delta t\left[g(t_n,x_n,y_n)+g(t_{n+1},x^*_{n+1},y^*_{n+1})\right] \\ \text{Also } x^*_{n+1}=x_n+\Delta t f(t_n+0.5\Delta t,x_n+0.5\Delta t,y_n+0.5\Delta t) \text{ and } y^*_{n+1}=y_n+\Delta t g(t_n+0.5\Delta t,x_n+0.5\Delta t,y_n+0.5\Delta t) \end{array} \quad (7)$$

In Equations 5 - 7, Δt is the spacing between discrete values of the independent variable t and n is a whole number such that $n=\frac{t_{n+1}-t_0}{h}$ is the number of steps required to reach the desired value of solution.

3.2. Comparison Criterion: Poincare-Lindstedt Method (P-L)

The algorithm and detailed description and implementation of the P-L method can be found in [16-18]. However for the purpose of comparison the solution of Equation 2, with specific initial conditions defined by $x(0)=2, x'(0)=0$, using P-L method is quoted from [19] as shown in Equation 8:

$$x(t) = 2\cos\omega t + \varepsilon \sin^3 \omega t + \ldots$$
$$\text{where } \omega = 1 - \frac{1}{16}\varepsilon^2 + \ldots \tag{8}$$

Equation 8 provides very good approximate analytic solution when the parameter $\varepsilon > 0$ is small. Moreover, for weak nonlinearities Equation 2 has periodic behaviour with amplitude 2 [19].

For analysis of associated errors in the numerical methods: MP, IH and RK4, formula for absolute discrepancy with respect to the P-L method is used as (Equation 9):

$$A.E_{(METHOD)} = \left| x(t)_{(METHOD)} - x(t)_{(P\text{-}L)} \right| \tag{9}$$

4. Results and Discussion

The following example problem associated with Equation 2 will be considered for the purpose of conducting comparisons of solutions based on the RK4, MP and IH methods.

Example:

$$\frac{d^2x}{dt^2} + \varepsilon\left(x^2 - 1\right)\frac{dx}{dt} + x = 0$$

subject to

$$x(0) = 2, x'(0) = 0$$

$$\varepsilon = 0.01, 0.1, 1$$

h = 0.04 from t = 0 to t = 40. [10].

Using step-length of h = 0.04, the original transient domain from t = 0 to t = 40 is divided into 1000 interior points at which numerical solutions of example problem are obtained.

Figures 1-3 present visualizations to solution of the Van der Pol's Equation (Example Problem) using RK4, IH, and MP methods along with the P-L method for small values of the parameter: $\varepsilon = 0.01$, $\varepsilon = 0.1$ and $\varepsilon = 1$; respectively.

Solution of the test example by P-L method along with respective absolute errors involved in numerical methods MP, RK4 and IH at time steps 5, 10, 15, 20, 25, 30, 35 and 40 (selected out of 1000 total time steps); for three values: $\varepsilon = 0.01$, $\varepsilon = 0.1$ and $\varepsilon = 1$ of the parameter are depicted in Tables 1-3 respectively.

It can be observed that for all values of ε, solution to test problem using the MP and RK4 methods show relatively same graphical behaviour of the oscillatory pattern as is inferred by the P-L method (Figures 1-3) and is consistent with the solution obtained using Mid-point method [5] and Collocation method [10]. On the other hand, the Improved Heun's method for every case of example problem behaves differently than RK4 and MP methods. The Absolute errors presented in Tables 1-3 also show that the RK4 method heads the ascendency among discussed methods from the stand point of smaller discrepancies. IH method behaves less accurately even than the MP method. There are many critical illustrations and points of concern based on the solution behaviour by IH method given below:

- Solution to Test problem for weak non-linearities (small values of $\varepsilon > 0$) is assumed to be periodic with amplitude 2. [19] However, Figures 1-2 clearly show that in the periodic numerical solution to the same by IH method, the amplitude gradually decreases as the time steps advance.
- For the case $\varepsilon = 1$, though the amplitude of the solution curve of IH method (despite being less than 2) does not seem to decrease rapidly but every Van der Pol cycle is reached earlier than usual in fewer time steps. It can be seen in Figure-3 that IH method unlikely exhibits a few more oscillations than the other methods;
- To guarantee this fact further, the example problem was re-solved for $\varepsilon = 10$ from $t = 0$ to $t = 80$ with $h = 0.04$ (2000 time steps) by numerical methods and the results are presented in Figure-4. The IH method in the same time span exhibits two more cycles than do the MP and Rk4 methods. As a consequence the distance between every two adjacent troughs or crests in the curve of IH method (i.e. the wave-length of the solution curve) is smaller than those of MP and Rk4 methods; and
- Tables 1-3 all show that absolute errors involved in IH method go on increasing in order from 10^{-2} to 10^{0}; in fact the errors will eventually increase even more in order as values of t advance. This contradicts with author's [14] claim that their proposed method (IH) results local errors of order $O(h^3)$.

Since the theory of Classical Runge-Kutta method is well-developed with regards to stability and accuracy and alike solutions of the Van der Pol's equation are also confirmed by mid-point and Poincare-Lindstedt methods etc. So, the results produced by RK4 method are better when compared to Improved Heun's method in all manner of analysis for the case of Van der Pol's Equation and are relatively similar to the mid-point method.

5. Conclusion

In this paper, the numerical solutions of an important equation of applied dynamics: namely, the Unforced Van der Pol's Equation (UFVDP) are obtained by reducing it to a system of two first order differential equations. The objective of this work is to investigate the efficiency of improved Heun's (IH) method against the classical Runge-Kutta (RK4) and Mid-point (MP) methods for UFVDP equation. For analysis of accuracy, the Poincare-Lindstedt method has been used as a comparison criterion and respective error bounds are obtained. The solutions to the Van der Pol's equation were obtained using three numerical methods: the classical RK4, Improved Heun's, and Mid-point methods and the performance and efficiency of Improved Heun's method were tested against MP and RK4 methods for the solution of considered equation. The solutions to the test problems using different values of parameter involved confirms the better accuracy for the RK4 method over IH and MP methods. The solutions by the RK4 are relatively better as expected.

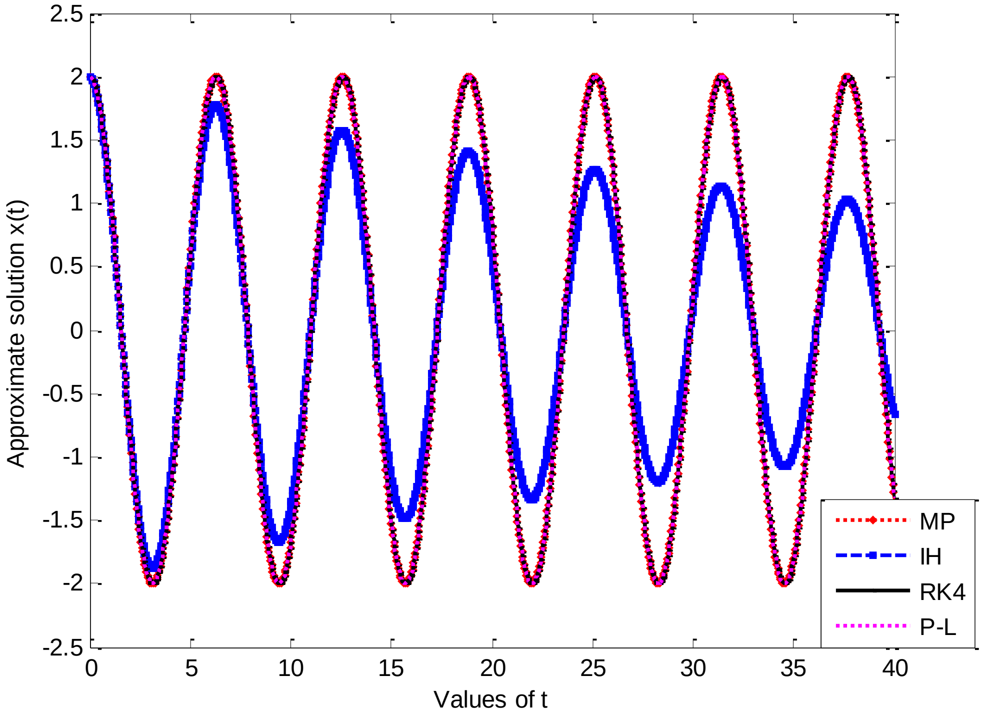

Figure-1: Comparison of Solution to Example by MP, IH, RK4 and P-L methods with $\varepsilon = 0.01$

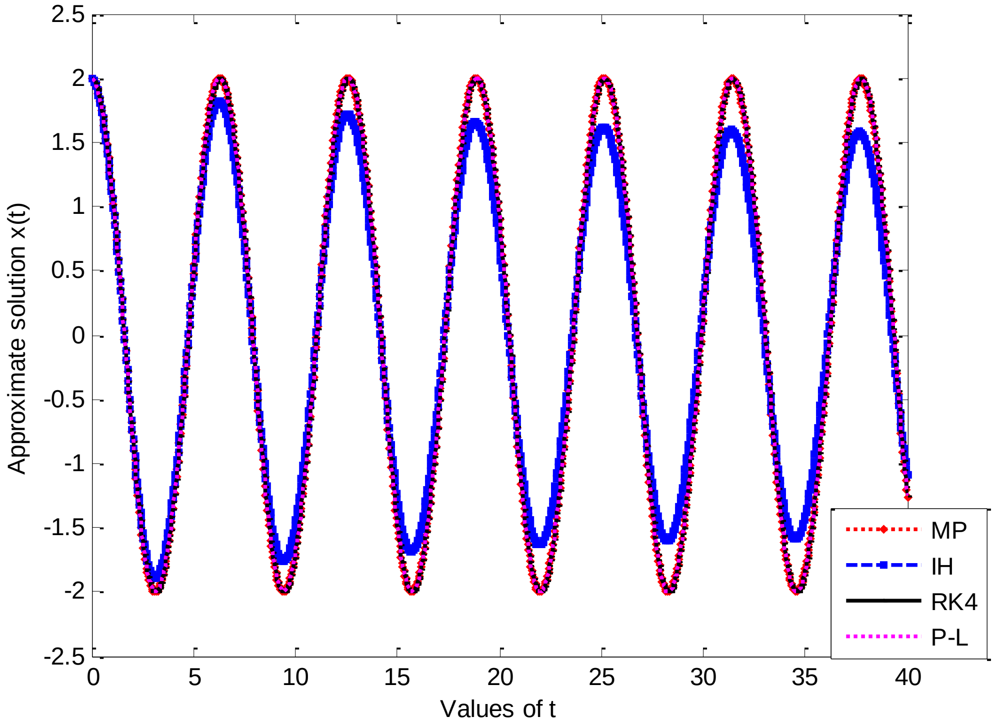

Figure-2: Comparison of solutions using MP, IH, RK4 and P-L methods with $\varepsilon = 0.1$.

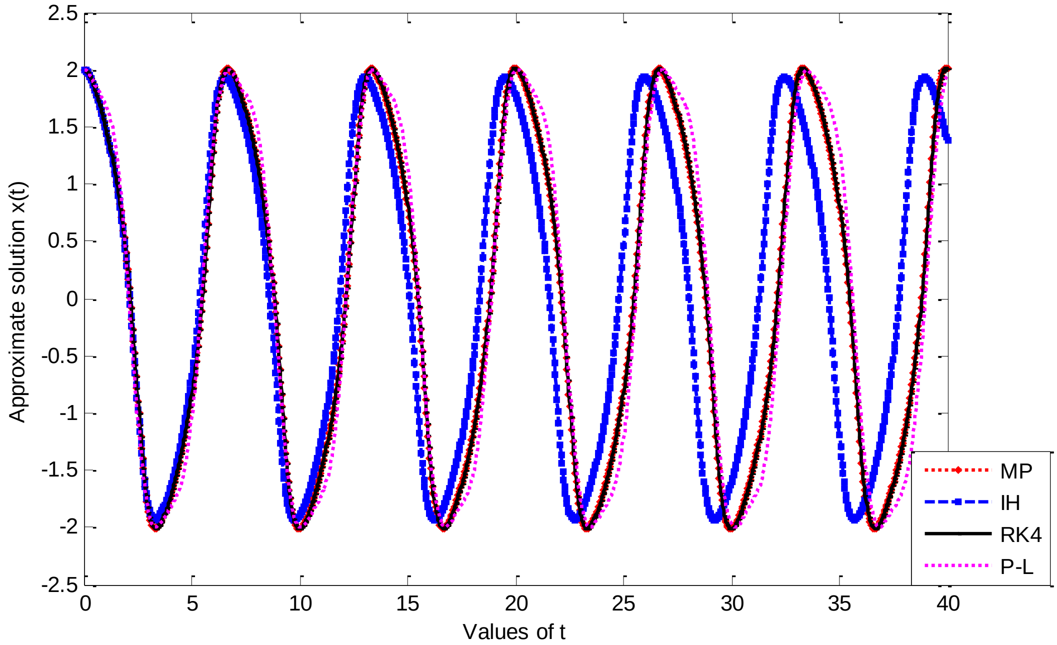

Figure-3: Comparison of solution using the MP, IH, RK4 and P-L methods with $\varepsilon = 1$.

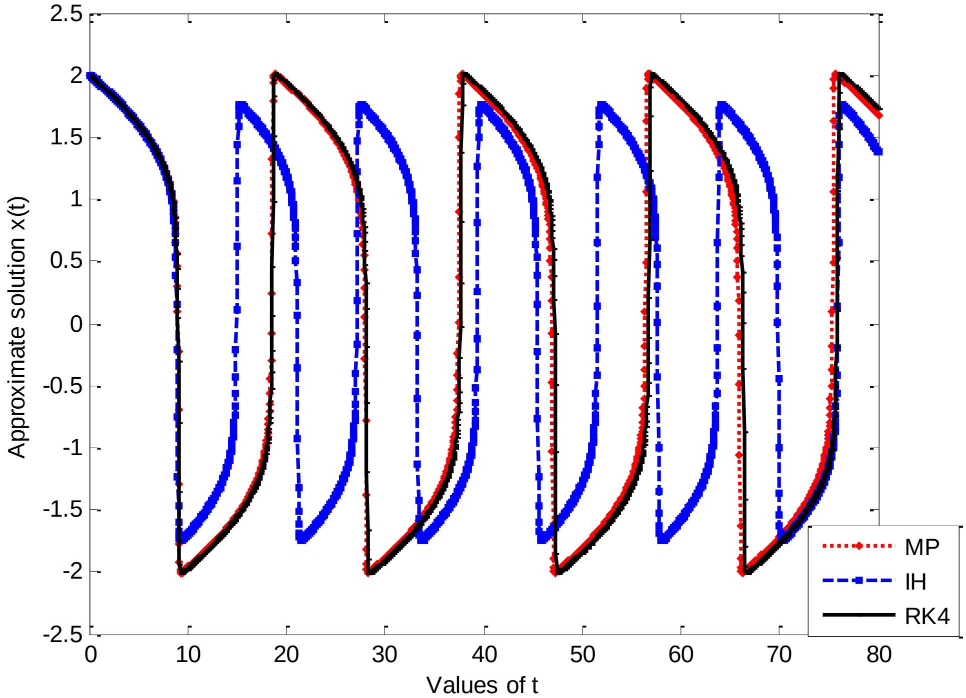

Figure-4: Comparison of solutions using the MP, IH and RK4 methods with $\varepsilon = 10$ from t = 0 to t = 80, with h = 0.04 and N = 2000.

Table-1: Comparison of absolute errors in the numerical solution using the MP, RK4 and IH methods with respect to P-L method at different time steps using Equation 9 for solution of Van der Pol's equation with $\varepsilon = 0.01$.

Time Values	**Solution by P-L method**	**Absolute Errors**		
"t"	$x(t)$	MP	RK4	IH
5	0.558446541697392	2.5380e-003	2.3330e-005	4.8278e-002
10	-1.679820671200842	2.8005e-003	8.8869e-006	2.9561e-001
15	-1.516503086495721	5.3363e-003	1.5712e-005	3.6765e-001
20	0.824000193924650	9.6129e-003	2.7063e-005	2.6535e-001
25	1.982340974085762	1.9903e-003	1.7456e-007	7.2502e-001
30	0.298486307119957	1.5838e-002	1.5166e-005	1.0948e-001
35	-1.808355641817753	7.7049e-003	3.6946e-006	8.4481e-001
40	-1.329363926969835	1.6172e-002	2.2900e-005	6.6082e-001

Table-2: Comparison of absolute errors in the numerical solutions using the MP, RK4 and IH methods with respect to P-L method at different time steps using Equation 9 for solution of Van der Pol's equation with $\varepsilon = 0.1$.

Time Values	**Solution by P-L method**	**Absolute Errors**		
"t"	$x(t)$	MP	RK4	IH
5	0.472908350352229	2.3031e-004	2.1735e-003	2.0005e-002
10	-1.700549225912158	3.6390e-003	8.1966e-004	2.2172e-001
15	-1.478707602744024	3.9136e-003	1.5573e-003	1.9951e-001
20	0.913728606400655	1.1875e-002	2.8027e-003	2.2047e-001
25	1.977704797615117	1.8369e-003	7.0541e-005	3.6696e-001
30	0.174149068652465	1.4794e-002	1.0225e-003	7.4748e-002
35	-1.832490895434053	7.8233e-003	3.1736e-004	4.2287e-001
40	-1.252040444383581	1.4919e-002	2.2396e-003	1.6168e-001

Table-3: Comparison of absolute errors in the numerical solutions by the MP, RK4 and IH methods with respect to P-L method at different time steps using Equation 9 for solution of Van der Pol's equation with $\varepsilon = 1.0$.

Time Values	**Solution by P-L method**	**Absolute Errors**		
"t"	$x(t)$	MP	RK4	IH
5	-1.048843965363322	2.1302e-001	2.1176e-001	3.6571e-001
10	-1.997399476861792	1.0028e-002	1.0942e-002	1.0212e-001
15	1.140859586006049	3.1677e-001	3.1042e-001	9.5017e-001
20	1.989114944826498	1.7882e-002	1.9035e-002	2.1540e-001
25	-1.225227005905389	4.1297e-001	4.0145e-001	1.6929e+000
30	-1.974447521693010	3.1973e-002	3.3463e-002	3.7568e-001
35	1.301954326027987	5.0164e-001	4.8488e-001	2.5006e+000
40	1.952746529159455	5.2955e-002	5.4878e-002	5.7341e-001

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