GAMMA-NORMAL VERSION OF THE LEE-CARTER MODEL FOR FORECASTING MORTALITY

Adegbilero-Iwari Oluwaseun Eniola[1*, 2] and Chukwu Angela Unna[2]

[1]Department of Community Medicine, Afe Babalola University, Ado-Ekiti, Nigeria

[2] Department of Statistics, University of Ibadan, Ibadan, Nigeria.

*E-mail of the corresponding author: seuneniola01@gmail.com

Abstract

The Lee-Carter model was primarily designed for the mortality pattern of the United States and has been used by many developed Countries as a gold standard to model their mortality pattern. Despite its merits, it is observed from literature that the Lee and Carter (1992) and some of its variants have problems capturing a good fit when mortality rate has an increasing trend for some ages while decreasing for others or a fluctuating increase and decrease especially with respect to mortality data from developing Countries. In this study, based on the contributions of Zografos and Balakrishnan (2009) and Lima *et al.* (2015), we propose a Gamma-normal version of the Lee-Carter model which addresses the distribution of the error term inherent in the Lee-Carter approach. It is hoped that this new model will give a better fit and account for peculiarities at different ages without any loss of generality.

Keywords: Lee-Carter model, mortality, Gamma-normal, Age, developing Countries.

1. Introduction

Mortality rates for many developed Countries declined dramatically during the past century. Improvements in standards of living, sanitary conditions, hygiene, and medicine led to rapidly decreasing infant and adult mortality rates. However, in many developing Countries, the story is not the same as mortality levels varies markedly across age groups and between regions within particular Countries. Bearing this in mind, it is thoughtful as well as needful to say that the assumptions used by mortality models with respect to developed Countries should be modified when it comes to applying them to developing Countries especially in situations when the mortality data set has a fluctuating increase and decrease pattern.

In 1992, Lee and Carter published a stochastic model for the long-run forecasts of the level and age pattern of mortality, based on a combination of statistical time series methods and a simple approach to dealing with the age distribution of mortality. Lee and Carter applied their model on U.S mortality data from the time period 1933-1987 and projections were made up to the year 2065. Although the Lee-Carter model was specifically designed for the United States mortality pattern, it has been modified and applied widely beyond the scope of its original creation.

Several researchers have worked on the development of the Lee-Carter model. To mention a few, see; Wilmoth (1993), Lee and Miller (2001), Li *et al.* (2002), Booth *et al.* (2002), Brouhns *et al.* (2002), Renshaw and Haberman (2006), Pedroza (2006) Hyndman and Ullah (2006), Girosi and King (2007), Li and Chan (2007), Koissi and Shapiro (2008), Zhao (2012). According to Reese (2015), despite the popularity of the Lee-Carter model, the theoretical properties underpinning the model are *obscure* and have not been thoroughly investigated.

A major challenge with the Lee and Carter (1992) model framework is the long-time span data requirement. Li *et al.* (2002) extended the Lee-Carter approach to situations in which mortality data are available at only a few points in time or at unevenly spaced intervals; situations often encountered in statistics for Third World Countries. They proposed that the improved Lee-Carter method can provide accurate mean mortality forecasts for Countries with historical data at only a few time points, if the earliest and latest points are sufficiently far apart in time**.** Their model was used on China's sex- combined mortality data for the years 1974, 1981 and 1990. It is important to state here that Li *et al.*'s major contribution is on the mortality time trend (parameter k_t) built in the Lee-Carter model. Their method was able to account for extra variability arising from fewer time points when forecasting the k_t parameter but does not account for the extra variability which will also be present in the age-specific parameters. Furthermore, their work does not address the distribution of the error term inherent in the original Lee-Carter model.

Chukwu and Oladipupo (2012) applied the Li *et al.* approach on Nigerian mortality data for both males and females. The data set used was the age-specific mortality rates of males and females aged 15-84 years for the time periods 1990, 2000 and 2009. They observed that the model did not give a good fit to the female data due to fluctuating increase and decrease pattern especially within the child-bearing age groups. Chukwu and Adegbilero-iwari (2015) also observed higher residuals within the middle age-groups after using the model on sex-combined mortality data.

Taruvinga *et al.* (2016) applied the Lee-Carter (1992) model to Zimbabwe mortality data for the age groups 0-84 years between the time periods 1983 to 2004. Results from their empirical analysis showed that the Lee- Carter model provided a better fit for thirteen age groups but failed to fit for four age groups namely 50-54, 55-59, 75-79 and 80-84 due to a fluctuating mortality pattern.

Manton *et al.* (2009) points out that the gamma distribution is usually used in mortality models for heterogeneous population especially in situations where the distribution of risk levels is flatter and has thicker tails.

Zografos and Balakrishnan (2009) proposed two generalized gamma-generated distributions with an extra positive parameter for any continuous baseline cdf G(x), x$\in R$. Their distribution emanates from the generalized gamma distribution as introduced by Stacy (1962). The Zografos–Balakrishnan-*G* family of distributions is particularly useful for heterogeneous data because it allows for greater flexibility of its tails. Nadarajah *et al.* (2014) presented some special cases of the Zografos–Balakrishnan-*G* family which includes the Gamma–Normal, the Gamma–Weibull, the Gamma–Gumbel, Gamma-log-normal and the Gamma-log-logistic distribution. Lima *et al.* (2015) took the works of Zografos-Balakrishnan (2009) further by considering a three-parameter distribution called the Gamma-Normal distribution.

1.3 Objectives of the Study

The main objective of this work is to develop a modelling framework with improved modelling properties with the aim of capturing a fluctuant mortality pattern. The specific objectives are to:

i. To derive the probability density function of the proposed distribution.
ii. To investigate some statistical features of the proposed model.
iii. To use robust methods such as the maximum likelihood in estimating the model's parameters

2.0 Methodology

2.1 The Lee-Carter model

The Lee-Carter methodology for forecasting mortality rates is a simple bi-linear model in the variables x (age) and t (calendar year). The model is defined as;

$$\ln m_{xt} = a_x + b_x k_t + \varepsilon_{xt} \qquad (1)$$

Where;

m_{xt} : is the matrix of the observed age-specific death rate at age *x* during year t. It is obtained from observed deaths divided by population exposed to risk.

a_x: is the average of $\ln m_{xt}$ over time t. It describes the (average shape of the age profile) general pattern of mortality by age.

b_x: indicates the relative pace of change in mortality by age as k_t varies. It describes the pattern of deviations from the age profile when the parameter k_t varies. It modifies the main time trend according to whether change at a particular age is faster or slower than the main trend.

k_t: is the time trend for the general mortality. It captures the main time trend on the logarithmic scale in mortality rates at all ages. It is also referred to as the mortality index.

ε_{xt} : is the residual term at age x and time t. It reflects the age specific influences not captured by the model. It is expected to be Gaussian i.e. $\varepsilon_{xt} \sim N(0, \sigma^2)$.

2.1.1 Forecasting the mortality index

One feature of the Lee-Carter (1992) model and its variant is that once the data are fitted to the model and the values of the vectors $\hat{a}_x$, $\hat{b}_x$ and $\hat{k}_t$ are found, only the mortality index $\hat{k}_t$ needs to be predicted. After obtaining the predicted values of the mortality index $\hat{k}_t$, the values are then plugged back into the original Lee-Carter model to obtain forecast of the mortality rates. The forecast of the mortality index is modeled as;

$$k_t = k_{t-1} + \hat{\theta} + \varepsilon_t, \quad (2)$$

$\varepsilon_t \sim N(0,1)$

The drift parameter $\hat{\theta}$ is expressed as;

$$\hat{\theta} = \frac{1}{T-1} \sum_{t=1}^{T-1} (\hat{k}_{t+1} - \hat{k}_t) \quad (3)$$

2.2 The Li *et al.* model

The model framework of the Li *et al.* (2002) bears essentially the same features as that of Lee and Carter (1992). The model is expressed as;

$$\ln m_{xu(t)} = a_x + b_x k_{u(t)} + \varepsilon_{xu(t)} \quad (4)$$

The notations in this model are essentially the same as in the Lee and Carter (1992) model.

Now let data be collected at times $u(0), u(1), \ldots, u(T)$, the forecast of the mortality index in equation (4) is expressed as:

$$\hat{k}_{u(\mathrm{t})} = \hat{k}_{u(t-1)} + \hat{\theta}[u(t) - u(t-1)] + [(\varepsilon_{u(t-1)} + 1) + \ldots + \varepsilon_{u(t)}] \tag{5}$$

Where the drift parameter $\hat{\theta}$ is:

$$\hat{\theta} = \frac{\sum_{\mathrm{t=1}}^{\mathrm{T}}[\hat{k}_{u(\mathrm{t})} - \hat{k}_{u(t-1)}]}{\sum_{\mathrm{t=1}}^{\mathrm{T}}[u(t) - u(t-1)]} = \frac{\hat{k}_{u(\mathrm{t})} - \hat{k}_{u(0)}}{u(\mathrm{T}) - u(0)} \tag{6}$$

2.3 The Gamma-normal generator

The distribution arising from record value densities using the gamma link as introduced by Zografos and Balakrishnan (2009) will be considered for this study.

Suppose $X_{U(1)}, X_{U(2)}, \ldots \ldots, X_{U(n)}$ are upper record values arising from a sequence of i.i.d. continuous random variables from a population with cdf $\mathrm{F}(X)$and pdf $\mathrm{f}(X)$. Then, the pdf of the nth upper record value, $X_{U(n)}$ is;

$$f[x_{U(n)}] = \frac{\{-ln[1-F(x)]\}^{n-1}}{n-1^{!}} f(x), \qquad -\infty < x < \infty \text{ for n = 1, 2, …} \tag{7}$$

Converting the positive integral parameter n to a positive real parameter say α, the family of densities with pdf which is precisely the class of gamma- generated densities is obtained;

It is given by;

$$g(x) = \frac{\{-ln[1-F(x)]\}^{\alpha-1}}{\Gamma\alpha} f(x) \qquad -\infty < x < \infty, \alpha > 0. \tag{8}$$

While its cumulative distribution function is given by;

$$\mathrm{G}(x) = \frac{1}{\Gamma\alpha}\int_0^{-\ln(1-F(x)} t^{\alpha-1}e^{-t}dt \tag{9}$$

Assumption of the Error term

The Lee-Carter approach assumes that its error term ε_{xt}, follows a normal distribution whose pdf must satisfy the condition below.

$$f(y) = \frac{1}{\sigma}\phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right] \tag{10}$$

Where $y = ln\, m_{xt}$

$$\phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\}}$$

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\}} \qquad ,\sigma > 0,\ b_x > 0, k_t > 0 \tag{11}$$

Its cumulative distribution function is given by;

$$\mathrm{F}(y) = \Phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right] \tag{12}$$

$$\mathrm{F}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}\{\frac{[t-(a_x+b_xk_t)]^2}{\sigma^2}\}}dt \tag{13}$$

3.0 Summary of Results

3.1 Probability density function

The probability density function of the proposed approach is derived by substituting equations (10) and (12) into (8)

to give;

$$g(y) = \frac{1}{\Gamma\alpha}\frac{1}{\sigma}\phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right] \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1}$$

$$= \frac{1}{\Gamma\alpha}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\}}\left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1}, \ \alpha > 1, \ \sigma > 0 \tag{14}$$

The cumulative distribution function is derived by substituting equation (12) into (9) to give;

$$\mathrm{p}(Y \leq y) = \frac{1}{\Gamma\alpha}\int_{0}^{-\ln(1-\Phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right]} t^{\alpha-1}e^{-t}dt \tag{15}$$

This reduces to;

$$G(y) = \frac{1}{\Gamma\alpha}\gamma[\alpha, -\ln(1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)] \tag{16}$$

3.2 Verification of proper probability density function

We need to show that the $\int_0^\infty g(y)\, dy = 1$ (i)

From (14), $g(y) = \frac{1}{\Gamma\alpha}\left\{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}[\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}]}\right\} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1}$

Substituting equation (14) into equation (i), we have;

$$\int_0^\infty \frac{1}{\Gamma\alpha}\left\{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}[\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}]}\right\} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1} dy \tag{ii}$$

We set $\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)] = z$

$$\frac{dz}{dy} = \frac{1}{\sigma}\phi\left[\frac{y-(a_x+b_xk_t)}{\sigma}\right] = \left[\frac{e^{-\frac{1}{2}\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\}}}{\sigma\sqrt{2\pi}}\right]$$

$$dy = \frac{\sigma\sqrt{2\pi}\, dz}{e^{-\frac{1}{2}\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\}}}$$

Equation (ii) becomes;

$$\int_0^\infty \frac{1}{\Gamma\alpha}\left\{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}[\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}]}\right\} [-\ln(1-z)]^{\alpha-1} \frac{\sigma\sqrt{2\pi}\, dz}{e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma^2}\right\}}}$$

$$=\frac{1}{\Gamma\alpha}\int_0^\infty \left[-\ln(1-z)\right]^{\alpha-1} dz \qquad \text{(iii)}$$

Applying the log function rule to (iii), we have

$$=\frac{1}{\Gamma\alpha}\int_0^\infty \left[\ln(1-z)^{-1}\right]^{\alpha-1} dz$$

$$=\frac{1}{\Gamma\alpha}\int_0^\infty \left[\ln(1-z)^{-1}\right]^{\alpha-1} dz$$

$$=\frac{1}{\Gamma\alpha}\int_0^\infty \left[\ln(\tfrac{1}{1-z})\right]^{\alpha-1} dz \qquad \text{(iv)}$$

$\int_0^\infty \left[\ln(\frac{1}{1-z})\right]^{\alpha-1} dz$ reduces to a one parameter gamma function: $\int_0^\infty z^{\alpha-1}e^{-z}dz$

Where $\int_0^\infty z^{\alpha-1}e^{-z}dz = \Gamma\alpha$

Equation (iv) becomes $\frac{\Gamma\alpha}{\Gamma\alpha} = 1.$

Hence the proposed distribution integrates to 1.

3.3 Asymptotic Behaviour

We seek to investigate the behaviour of the model in equation (14) as $y \to 0$

$$g(y) = \frac{1}{\Gamma\alpha}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1} \qquad (17)$$

- If $\alpha = 1$, we have

$$\lim_{y\to 0} g(y) = \lim_{y\to 0}\frac{1}{\Gamma\alpha_i}\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1} \qquad (18)$$

Equation (18) becomes $\lim_{y\to 0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}}$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{(a_x+b_xk_t)^2}{\sigma^2}\right]} = \frac{1}{\sigma}\Phi\left[\frac{(a_x+b_xk_t{}^2)}{\sigma}\right] \qquad (19)$$

Equation (19) is the pdf of the Normal distribution under the Lee-carter and Li *et al.* framework

- If $\alpha < 1$, we have

$$\lim_{y\to 0} g(y) = \lim_{y\to\infty}\frac{1}{\Gamma\alpha}\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1}$$

$$= \frac{1}{\Gamma\alpha}\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1}$$

$$= \frac{1}{\Gamma\alpha}\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}} \left\{-ln[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma}\right)]\right\}^{\alpha-1} \qquad (20)$$

3.4 Estimation of parameters

Let y_1, y_2 , …, y_n be a random sample of size n distributed according to the likelihood function of vector of parameter

s, $\theta = (a_x,\ b_x, k_t, \alpha\ and\ \sigma_i\ )$.Given $g(y)$ as;

$$g(y) = \frac{1}{\Gamma\alpha}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{[y-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}}\left\{-\ln\left[1-\Phi\left(\frac{y-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\}^{\alpha-1} \tag{21}$$

Its likelihood function is:

$$L(\theta) = \prod_{i=1}^{n}\frac{1}{\Gamma\alpha}\frac{1}{\sigma_i\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{[y_i-(a_x+b_xk_t)]^2}{\sigma_i^2}\right\}}\left\{-\ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\}^{\alpha-1}$$

$$= \prod_{i=1}^{n} e^{-\frac{1}{2\sigma_i^2}[y_i-(a_x+b_xk_t)]^2}\left\{-\ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\}^{\alpha-1}(\sigma_i\Gamma\alpha)^{-n}(2\pi)^{-\frac{n}{2}} \tag{22}$$

Then its log-likelihood function is expressed as:

$$lnL(\theta) = -n(ln\,\sigma_i + ln\Gamma\alpha) - \frac{n}{2}(ln2\pi) - \frac{1}{2\sigma_i^2}\sum_{i=1}^{n}[y_i-(a_x+b_xk_t)]^2 +$$

$$(\alpha-1)\sum_{i=1}^{n} ln\left\{-\ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\} \tag{23}$$

The following derivatives; $\boldsymbol{\alpha}$, $\boldsymbol{\sigma_i}$, $\boldsymbol{a_x}$, $\boldsymbol{b_x}$ ***and*** $\boldsymbol{k_t}$ are taken with respect to equation (23) respectively. Their final results are given as;

$$\frac{\partial \ln L(\theta)}{\partial\alpha} = \frac{-n\Gamma\alpha'}{\Gamma\alpha} + \sum_{i=1}^{n}\ln\left\{-\ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\}$$

Since $\frac{\Gamma\alpha'}{\Gamma\alpha}$ is a di − gamma function $= \psi(\alpha)$

$$\frac{\partial \ln L(\theta)}{\partial\alpha} = \sum_{i=1}^{n}\ln\left\{-\ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\right\} - n\psi(\alpha) \tag{24}$$

$$\frac{\partial \ln L(\theta)}{\partial\sigma_i} = \frac{n}{\sigma_i} + \frac{1}{\sigma_i}\sum_{i=1}^{n}\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right]^2 + \frac{(\alpha-1)}{\sigma_i}\sum_{i=1}^{n}\frac{\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right]\phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)}{\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]} \tag{25}$$

$$\frac{\partial \ln L(\theta)}{\partial a_x} = \frac{1}{\sigma_i}\sum_{i=1}^{n}\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right] + \frac{(\alpha-1)}{\sigma_i}\sum_{i=1}^{n}\frac{\phi\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right]}{\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]\ ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]} \tag{26}$$

$$\frac{\partial \ln L(\theta)}{\partial b_x} = \frac{k_t}{\sigma_i}\sum_{i=1}^{n}\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right] + \frac{k_t(\alpha-1)}{\sigma_i}\sum_{i=1}^{n}\frac{\phi\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right]}{1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]} \tag{27}$$

$$\frac{\partial \ln L(\theta)}{\partial k_t} = \frac{b_x}{\sigma_i}\sum_{i=1}^{n}\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right] + \frac{b_x(\alpha-1)}{\sigma_i}\sum_{i=1}^{n}\frac{\phi\left[\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right]}{1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)ln\left[1-\Phi\left(\frac{y_i-(a_x+b_xk_t)}{\sigma_i}\right)\right]} \tag{28}$$

Our proposed model is expressed mathematically as:

$$\ln m_{xt} = a_x + b_x k_t + \varepsilon_{xt}{}^{*} \tag{29}$$

Where $\varepsilon_{xt}{}^{*} \sim GN(\alpha, \mu, \sigma^2)$

The parameters $a_x,\ b_x\ and\ k_t$ bear essentially the same meaning as in the Lee-Carter approach except for the newly introduced shape parameter α. In our new contribution, parameter α denotes additional effects that might exist and could act constantly across age and time in human mortality experience especially in developing Countries.

Conclusion

Assumptions used by mortality models with respect to developed Countries should be modified when it comes to applying them to developing Countries because of our own complexities. Hence, in this paper, we present a modified Lee-Carter model using the Gamma-normal distribution. This work benefits from the contributions of Zografos and Balakrishnan (2009) and Lima *et al.* (2015). Various structural properties of the new model were investigated. We also discussed the estimation of its parameters using the method of maximum likelihood.

References

Booth, H., Maindonald, J. and Smith, L. (2002). Applying Lee-Cater under Conditions of Variable Mortality Decline. *Population Studies* 6:325-336.

Brouhns, N., Denuit, M. and Vermunt J. (2002). A Poisson log-bilinear regression approach to the construction of projected life-tables. *Insurance: Mathematics and Economics* 31:373–393

Chukwu, A.U. and Adegbilero-Iwari, O.E. (2015).The performance of the Lee-carter model on heterogeneous adult mortality data. *Mathematical Theory and Modeling*, Vol.5, No.4, 2015.

Chukwu, A.U. and Oladipupo, O.E. (2012). Modelling adult mortality in Nigeria: An analysis based on the Lee-carter model. *Studies in Mathematical Sciences* 5(2), pp. 1-11

Girosi, F. and King, G. (2007). Understanding the Lee-Carter mortality forecasting method. Retrieved from *http://gking.harvard.edu/files/lc.pdf.*

Hyndman, R. J. and Ullah, M. S. (2007). Robust forecasting of mortality and fertility rates: a functional data approach. *Computational Statistics and Data Analysis* 51(10): *4942-4956.*

Koissi, M., Shapiro, A. (2008). The lee-carter model under the condition of variables age-specific parameters. *Presented at the 43rd Actuarial Research Conference, Regina, Canada August 2008*

Lee, R. and Carter, L. (1992). Modeling and forecasting the time series of U.S. mortality. *Journal of the American Statistical Association* 87: 65-71.

Lee, R. and Miller, T. (2001). Evaluating the performance of the Lee-Carter method for forecasting mortality *Demography*, Vol. 38, No. 4 (Nov., 2001), pp. 537-549

Li, S. and Chan, W. (2007). The Lee-Carter model for forecasting mortality, revisited". *North American Actuarial Journal* 11(1), pp.68-87

Li, N., Lee, R. and Tuljapurkar, S. (2002). Using the Lee-Carter method to forecast mortality in populations with limited data. *International Statistical Review* 72:19-36

Lima, M., Cordeiro, G., and Ortega, E. (2015). A new extension of the normal distribution. *Journal of Data Science*. 13:2015, pp. 385-408

Manton, K. G., Akushevich, I., and Kravchenko, J. (2008). *Cancer mortality and morbidity patterns in the US population: an interdisciplinary approach*. Springer Science & Business Media.

Nadarajah, S., Cordeiro, G., and Ortega, E. (2015). The Zografos–Balakrishnan G family of distributions: mathematical properties and application. *Communication in Statistics- Theory and Methods,* 44: 186–215

Pedroza, C. (2006). A Bayesian forecasting model: predicting U.S. male mortality. *Biostatistics* 7(4), pp. 530–550.

Renshaw, A. E., and Haberman, S. (2006). A cohort-based extension to the Lee-Carter model for mortality reduction factors. *Insurance: Mathematics and Economics* 38: 556–570

Reese, S. (2015). Asymptotic inference in the Lee-Carter model for modelling mortality rates *Working Paper* 2015:16 *Department of Economics, School of Economics and Management*

Stacy, E.W. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, 33, 1187-1192.

Taruvinga, R., Gachira, W., Chiwanza, W. and Nkomo, D. (2017). Comparison of the Lee-Carter and arch in modelling and forecasting mortality in Zimbabwe. *Asian Journal of Economic Modelling* 5(1): 11-22

Wilmoth, J. (1993). Computational methods for fitting and extrapolating the Lee-Carter model of mortality change. *Technical Report, University of California, Berkeley U.S.A.*

Zhao, B. B. (2012). A modified Lee-Carter model for analysing short-base period data. *Population Studies*, 66(1), pp. 39-52

Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology* 6 (2009) 344- 362