Extension of Banach Cotraction Principle for Cone Metric space

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Abstract

In this paper an extension of Banach contraction principle is introdues for the rational contraction in cone metric space.

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2. Introduction and Preliminaries: The famous result in the field of fixed point theory is given by Banach which is known as Banach contraction principle. It says that if X is a complete metric space then every contraction has a fixed point. Many authors worked to extend this principle.*The cone metricspace introduced for common fixed point theorem in weakly compatible maps with implicit relations by A. Aliouche V. Popa* [8] *and further M.S. Khan and Imdad M*[14] *proves the Fixed and coincidence points in Banach and 2-Banach spaces.*

Let Q be a subset of X and X is a real Banach space, then Q is called a cone If Q satisfies the following axioms:

(i)Q is closed, nonempty and Q$\neq 0$

(ii) $ax + by \in Q$ $for\ all\ x, y \in Q\ and\ non\ negative\ real\ number\ a, b$

(iii) $Q \cap (-Q) = \{0\}$

Here we define a partial ordering $\leq$*on X with respect to Q by* $y - x \in Q, given\ a\ cone\ Q \subset X$

$If\ y - x \in\ intQ, i.e. x << y, denoted\ by\ \|.\| the\ norm\ on\ X, the\ cone\ Q\ is\ called\ normal$

$If there\ is\ a\ number\ k > 0\ such\ that\ for\ all\ x, y \in X$

$0 \leq x \leq y\ implies\ that \|x\|\ \leq r\|y\|$ *[1]*

Therefore the least number r satisfying the particular equation [1] is called the normal constant of Q.

Hence in this the author proves that there is no normal cone with normal constant M< 1 *and for each r*> 1*, there are cone with normal constant M>r.*

The cone Q is called regular if every increasing sequence which is bounded above is convergent, that is if $\{x_n\}_{n\geq 1}$*is a sequence such that* $x_1 \leq x_2 \leq \cdots \ldots \ldots \ldots \ldots \leq y$ *for some* $y \in X$,

Then there is $x \in X$ $\lim_{n\to\infty} \|x_n - x\| = 0$. *The cone Q is regular Iff every decreasing sequence which is bounded from below is convergent.*

Definition:2. 1 *Let X be anon empty set and X is a real Banach space, T is a mapping from X into itself such that, T satisfying following conditions,*

(i) $T(x,y) \geq 0, for\ all\ x, y \in X$
(ii) *T(x, y) = 0 if and only if x = y*
(iii) *T(x, y) = T(y, x)*
(iv) $T(x, y) \leq T(x,z) + T(z,y)$

Then T is called a cone metric on X and (X, T) is called cone metric space.

Definition: 2.2 *Let E and M be two mapping of a cone metric space (X, T) then it is said to be compatible if,* $\lim_{n\to\infty} T(EMx_n, MEx_n) = 0$, *whenever* $\{x_n\}$*is a sequence in X such that*

$$\lim_{n\to\infty} Ex_n = v\ and\ \lim_{n\to\infty} Mx_n = v\ for\ some\ v \in X.$$

Let E and M be two self mapping of a cone metric space (X, T) then it is said to be weakly compatible, If they commute at coincidence point , that is E x = M x implies that

EMx = MEx for x ∈ *X.*

Altering distance function for self-mapping on a metric space established by M.S. Khan in 1984 and it can be expanded by M. Swalesh, S. Sessa that they introduced a control function which they called as altering distance function in the research of fixed point theory. The author Mier- Keeler type (ε, δ)- contractive condition to study of fixed point by using a control function with extended contractive conditions.

Definition 2.3 A function $\psi: \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ is called an altering distance function if the following properties are satisfied.

(φ_1) $\psi(t) = 0 \Leftrightarrow t = 0$.

(φ_2) ψ is monotonically non decreasing.

(φ_3) ψ is continuous.

By ψ wedenotes the set of all altering distance function.

Using those control functions the author extend the Banach contraction principle by taking $\psi = \mathrm{Id}$, (the identity mapping), in the inequality contraction [2.4.1]of the following theorem.

Theorem2.4 *Let* (M, d) *be a complete metric space, let* $\psi \in \Psi$ *and let* $\mathrm{Q} : \mathrm{M} \to \mathrm{M}$

be a mapping which satisfies the following inequality

$\psi[d(Q_x, Q_y)] \leq a\psi[d(x, y)]$ [2.4.1]

for all $x, y \in M$ *and for some* $0 < a < 1$. *Then , shas a unique fixed point* $v_0 \in M$

and moreover for each $x \in M$, $\lim_{n\to\infty} Q^n x = v_0$.

Fixed point theorems involving the notion of altering distance functions has been widely studied, On the other hand, in 1975, B.K. Das and S. Gupta [1] proves the following result.

Theorem 2.5*Let* (M, d) *be a metric space and let* $Q: M \to M$ *be a given mapping*

such that,

(i) $d(Qx, Qy) \leq \alpha\, d(x, y) + \beta\, m(x, y)$ [2.5.1]

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < 1$ *where*

$m(x, y) = \left[\frac{d^2(x,Qx)+d(x,Qy)\, d(y,Qx)+d^2(y,Qy)}{1+d(x,Qx)d(y,Qy)}\right]$ [2.5.2]

for all $x, y \in M$.

(ii) *for some* $x_0 \in M$, *the sequence of iterates* $(Q^n x_0)$ *has a subsequence* $(Q^{nk} x_0)$

Withlim$_{k\to\infty} Q^{nk} x_0 = v_0$. *Then* v_0 *is the unique fixed point of* Q.

Definition2.7 Let (M, d) be a metric space for a self-mapping Q with a nonempty fixed point set E(Q). Then Q is said to satisfy the property P If E(Q) = E(Q^n) for each n∈ N.

Lemma 2.8. Let (M, d) be a metric space. Let $\{y_n\}$ be a sequence in M such that

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ 2.8.1

If $\{y_n\}$ is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequence of integers positive (m(k)) and (n(k)) with

(m(k))> (n(k)) $> k$, such that,

$d\left(y_{(\mathrm{m(k)})}, y_{(\mathrm{n(k)})}\right) \geq \varepsilon_0,\ d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})}\right) < \varepsilon_0$, and

i. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})+1}\right) = \varepsilon_0$

ii. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})}, y_{(\mathrm{n(k)})}\right) = \varepsilon_0$

iii. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})}\right) = \varepsilon_0$

Remark 2.9. From Lemma 2.8 is easy to get

$$\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})+1}, y_{(\mathrm{n(k)})+1}\right) = \varepsilon_0$$

3.Main Result

Theorem:3.1 Let (X, d)be a complete cone metric space and Q a normal cone with normal constant r. $\psi \in \Psi$. Suppose that the mapping S from X into itself satisfies the condition

$$\psi d(Sx,Sy) \leq a\,\psi d(x,y) + b\psi\,[d(x,Sx)+d(y,Sy)] + c\psi[d(x,Sy)+d(y,Sx)] + e\psi\left[\frac{[d(x,Sx)+d(y,Sy)]}{1+d(x,Sy)d(y,Sx)}\right] + f\psi\left[\frac{d^2(x,Sx)+d(x,Sy)d(y,Sx)+d^2(y,Sy)}{1+d(x,Sx)+d(y,Sy)}\right]$$

For all $x,\ y \in X$ *and* $a,b,c,f \geq 0$ *such that* $0 \leq a+b+e+c+f < 1$.Then S has unique fixed point in X.

Proof: for any arbitrary x_0 *in X, we have to choose* $x_1, x_2 \in$ X such that

$Sx_0 = x_1$ and $Sx_1 = x_2$

Also, in general we can define a sequence of elements in X such that

$$x_{2n+1} = Sx_{2n} \ \text{and}\ x_{2n+2} = Sx_{2n+1}$$

Now, ψ d (x_{2n+1}, x_{2n+2}) = ψ d$(Sx_{2n},\ Sx_{2n+1})$

From(1)

$$\psi\, d(Sx_{2n},Sx_{2n+1}) \leq\ a\,\psi d(x_{2n},x_{2n+1}) + b\psi\,[d(x_{2n},Sx_{2n}) + d(x_{2n+1},Sx_{2n+1})] + c\psi[d(x_{2n},Sx_{2n+1}) + d(x_{2n+1},Sx_{2n})] + e\,\psi\left[\frac{[d(x_{2n},Sx_{2n}) + d(x_{2n+1},Sx_{2n+1})]}{1+d(x_{2n},Sx_{2n})d(x_{2n+1},Sx_{2n})}\right] + f\,\psi\left[\frac{d^2(x_{2n},Sx_{2n}) + d(x_{2n},Sx_{2n})d(x_{2n+1},Sx_{2n}) + d^2(x_{2n+1},Sx_{2n+1})}{1+d(x_{2n},Sx_{2n}) + d(x_{2n+1},Sx_{2n+1})}\right]$$

$$\psi\, \mathrm{d}(x_{2n+1}, x_{2n+1}) \leq a\, \psi d(x_{2n}, x_{2n+1}) + b\, \psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$$
$$+ e\psi\left[\frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+1})}\right]$$
$$+ f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\psi\, \mathrm{d}(x_{2n+1}, x_{2n+1}) \leq a\, \psi d(x_{2n}, x_{2n+1}) + (b+e)\psi\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2})] + f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\leq a\, \psi d(x_{2n}, x_{2n+1}) + (b+e)\psi\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+\psi c[d(x_{2n}, x_{2n+2})] + f\psi\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$\leq a\, \psi d(x_{2n}, x_{2n+1}) + (b+e)\, \psi[d(x_{2n}, x_{2n+1})] + b[d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2})] + f\psi\, [d(x_{2n}, x_{2n+1})] + f\psi[d(x_{2n+1}, x_{2n+2})]$$

$$\psi \mathrm{d}(x_{2n+1}, x_{2n+1}) < (a + (b+e) + f)\psi d(x_{2n}, x_{2n+1}) + ((b+e) + c + f)\psi d(x_{2n+1}, x_{2n+2})$$

$$(1 - (b+e) - c - f)\psi d(x_{2n+1}, x_{2n+2}) \leq (a + (b+e) + f)\psi d(x_{2n}, x_{2n+1})$$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \frac{(a + (b+e) + f)}{(1 - (b+e) - c - f)}\psi d(x_{2n}, x_{2n+1})$$

Similarly we can show that

$$\psi d(x_{2n}, x_{2n+1}) \leq \frac{(a + (b+e) + f)}{(1 - (b+e) - c - f)}\psi d(x_{2n-1}, x_{2n})$$

In general we can write,

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \left[\frac{(a + (b+e) + f)}{(1 - (b+e) - c - f)}\right]^{2n+1} \psi d(x_0, x_1)$$

On taking $\left[\frac{(a+(b+e)+f)}{(1-(b+e)-c-f)}\right] = K$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq K^{2n+1}\, \psi d(x_0, x_1)$$

For n$\leq m$, we have

$$\psi d(x_{2n}, x_{2m}) \leq \psi d(x_{2n}, x_{2n+1}) + \psi d(x_{2n+1}, x_{2n+2}) + \cdots \ldots \ldots \ldots + \psi d(x_{2m-1}, x_{2m})$$

$$\psi d(x_{2n}, x_{2m}) \leq (K^n + K^{n+1} + K^{n+2} + \cdots \ldots \ldots \ldots K^m)\psi\, d(x_0, x_1)$$

$$\psi d(x_{2n}, x_{2m}) \leq \frac{K^n}{1-K}\, \psi\, d(x_0, x_1)$$

$$\psi\|d(x_{2n}, x_{2m})\| \leq \frac{K^n}{1-K}\quad r\,\psi\,\|d(x_0, x_1)\|\ \ as\, n \to \infty$$

$$\lim_{n\to\infty} \psi\|d(x_{2n}, x_{2m})\| \to 0$$

Hence $\{x_n\}$is a Cauchy sequence which converges to v in X.

Hence (X, d) is a complete cone metric space. Then $x_n \to v\ as\ n \to \infty,\ Sx_{2n} \to v\ as\ n \to \infty$.

Therefore v is a fixed point of S in X.

Uniqueness:- Let us suppose that there is another fixed point of S, i.e. w in X which is distinct from v, then

$$Sw = w\ and\ Sv = v$$

$$\psi d(v, w) = \psi d(Sv, Sw)$$

From (1)

$$\psi d(Sv, Sw) \leq a\, \psi d(v, w) + b\psi\, [d(v, Sv) + d(w, Sw)] + c\psi[d(v, Sw) + d(w, Sv)]$$

$$+ e\psi \left[\frac{[d(v, Sv) + d(w, Sw)]}{1 + d(v, Sw)\, d(w, Sv)}\right] + f\psi\, \left[\frac{d^2(v, Sv) + d(v, Sw)\, d(w, Sv) + d^2(w, Sw)}{1 + d(v, Sv) + d(w, Sw)}\right]$$

$$\psi d(Sv, Sw) \leq (a + 2c + f)\psi d(v, w)$$

This is a contradiction. Thus v is a unique fixed point of S in X.

Theorem: 3.2 Let (X, d) be a complete cone metric space and Q a normal cone with normal constant r. $\psi \in \Psi$. Suppose that the mapping S and P be the mapping from X into itself satisfies the condition

$$\psi d(Sx, Py) \leq a\, \psi d(x, y) + b\psi\, [d(x, Sx) + d(y, Py)] + c\psi[d(x, Py) + d(y, Sx)]$$
$$+ e\psi \left[\frac{d(x, Sx) + d(y, Py)}{1 + dd(x, Py)d(y, Sx))}\right]$$
$$+ f\psi \left[\frac{d^2(x, Sx) + d(x, Py)d(y, Sx) + d^2(y, Py)}{1 + d(x, Sx) + d(y, Py)}\right]$$

For all $x,\ y \in X$ *and* $a, b, c, f \geq 0$ *such that* $0 \leq a + b + c + e + f < 1$.Then S and P has unique fixed point in X.

Proof: for any $x_0 \in X$ we have

$$Sx_0 = x_1 \;\; and \;\; Px_1 = x_2$$

In general we can define a sequence of elements of X, such that

$$x_{2n+1} = Sx_{2n} \; and \; x_{2n+2} = Px_{2n+1}$$

Now, $\psi d(x_{2n+1}, x_{2n+2}) = \psi d(Sx_{2n}, Px_{2n+1})$

From (1)

$$\psi d(Sx_{2n}, Px_{2n+1}) \leq a\,\psi d(x_{2n}, x_{2n+1}) + b\psi[d(x_{2n}, Sx_{2n}) + \psi d(x_{2n+1}, Px_{2n+1})]$$

$$+c\psi[d(x_{2n}, Px_{2n+1}) + d(x_{2n+1}, Sx_{2n})] + e\psi\left[\frac{d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Px_{2n+1})}{1 + d(x_{2n}, Px_{2n+1}), d(x_{2n+1}, Sx_{2n})}\right]$$

$$+f\psi\left[\frac{d^2(x_{2n}, Sx_{2n}) + d(x_{2n}, Px_{2n+1}), d(x_{2n+1}, Sx_{2n}) + d^2(x_{2n+1}, Px_{2n+1})}{1 + d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Px_{2n+1})}\right]$$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq a\,\psi d(x_{2n}, x_{2n+1}) + (b+e)\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$$

$$+f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq a\psi\, d(x_{2n}, x_{2n+1}) + (b+e)\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2})] + f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\leq a\, d(x_{2n}, x_{2n+1}) + (b+e)b[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+2})] + \psi f[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$\leq a\psi\, d(x_{2n}, x_{2n+1}) + ((b+e)+f)\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+c\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + f\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$\leq (a + (b+e) + c + f)\psi\, d(x_{2n}, x_{2n+1}) + ((b+e) + c + f)\psi d(x_{2n+1}, x_{2n+2})$$

$$(1 - (b+e) - c - f)\psi\, d(x_{2n+1}, x_{2n+2}) \leq (a + (b+e) + c + f)\psi\, d(x_{2n}, x_{2n+1})$$

Therefore by using triangle inequality, we get

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \left|\frac{(a+(b+e)+c+f)}{(1-(b+e)-c-f)}\right| \psi d(x_{2n}, x_{2n+1})$$

Similarly we can show that

$$\psi d(x_{2n}, x_{2n+1}) \leq \left|\frac{(a+(b+e)+c+f)}{(1-(b+e)-c-f)}\psi\right| d(x_{2n-1}, x_{2n})$$

In general we can write

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \left|\frac{(a+(b+e)+c+f)}{(1-(b+e)-c-f)}\right|^{2n+1} \psi d(x_0, x_1)$$

On taking $\left|\frac{(a+(b+e)+c+f)}{(1-(b+e)-c-f)}\right| = \phi$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq K^{2n+1} \ \psi d(x_0, x_1)$$

For $n \leq m$, we have

$$\psi d(x_{2n}, x_{2m}) \leq \psi d(x_{2n}, x_{2n+1}) + \psi d(x_{2n+1}, x_{2n+2}) + \_\_\_\_ + \psi d(x_{2m-1}, x_{2m})$$

$$\psi d(x_{2n}, x_{2m}) \leq (K^n + K^{n+1} + K^{n+2} + \_\_\_\_ + K^m)\psi \ d(x_0, x_1)$$

$$\psi d(x_{2n}, x_{2m}) \leq \frac{K^n}{1-K} \ \psi d(x_0, x_1)$$

$$\psi \| d(x_{2n}, x_{2m}) \| \leq \frac{K^n}{1-K} \quad r\,\psi \, \| d(x_0, x_1) \|$$

As $\lim_{n\to\infty} \psi \| d(x_{2n}, x_{2m}) \| \to 0$

In this way $\lim_{n\to\infty} \psi d(x_{2n+1}, x_{2n+2}) \to 0$

Hence $\{x_n\}$is a cauchy sequence which converges to $v \in X$.

Hence (X, d) is complete cone metric space

Thus $x_n \to v \ \ as \, n \to \infty$

$Sx_{2n} \to v \ \ and \ Px_{2n+1} \to v \ as \ n \to \infty$ then v is fixed point of S and P in X, since SP = PS this gives

$v = Pv = PSv = SPv = Sv = v$

Uniqueness: Now Let w be another fixed point of S and P in X which is distinct from w, then

$Pv = v \text{ and } Pw = w \text{ also } Sv = v \text{ and } Sw = w$

$w\ \psi d(v,w) = \psi d(Sv,Pw)$

From (2)

$$\psi d(Sv,Pw) \le a\,\psi\, d(v,w) + b\psi\,[d(v,Sv) + d(w,Pw)] + c\psi[d(v,Pw) + d(w,Sv)]$$
$$+e\psi\left[\frac{d(v,Sv) + d(w,Pw)}{1 + d(v,Pw),\ d(w,Sv)}\right]$$
$$+f\psi\left[\frac{d^2(v,Sv) + d(v,Pw),\ d(w,Sv) + d^2(w,Pw)}{1 + d(v,Sv) + d(w,Pw)}\right]$$

$\psi d(Sv,Pw) \le (a+2c)\,\psi d(v,w) + f\,d(v,w)$

$$\psi d(Sv,Pw) \le (a+2c+f)\psi\, d(v,w)$$

This gives contradiction

Hence v is unique fixed point of S and P in X.

Theorem: 3.3 Let (X, d) be a complete cone metric space and Q a normal cone with normal constant r. $\psi \in \Psi$. Suppose that the mapping S, P and T be the mapping from X into itself satisfies the condition

$$\psi d(SPx,TPy) \le a\,\psi d(x,y) + b\psi\,[d(x,SPx) + d(y,TPy)] + c\psi[d(x,TPy) + d(y,SPx)]$$
$$+ e\psi\left[\frac{d(x,SPx) + d(y,TPy)}{1 + d(x,TPy), d(y,SPx)}\right]$$
$$+ f\psi\left[\frac{d^2(x,SPx) + d(x,TPy), d(y,SPx) + d^2(y,TPy)}{1 + d(x,SPx) + d(y,TPy)}\right]$$

For all $x, y \in X$ *and* $a, b, c, f \ge 0$ *such that* $0 \le a + b + c + e + f < 1$.Then S, P and T has unique fixed point in X.furthermore either SP = PS or TP = PT then it have unique common fixed point in X.

Proof: Here we choose$x_1, x_2 \in X$, *for any arbitrary element* x_0 *in* X *such that*

$SPx_0 = x_1 \text{ and } TPx_1 = x_2$

In general we can define a sequence of elements of X, such that

$$x_{2n+1} = SPx_{2n} \text{ and } x_{2n+2} = TPx_{2n+1}$$

Now, $\psi d(x_{2n+1}, x_{2n+2}) = \psi d(SPx_{2n}, TPx_{2n+1})$

From (3) $\psi d(SPx_{2n}, TPx_{2n+1}) \le a\,\psi[d(x_{2n}, x_{2n+1})]$

$$+b\psi\,[d(x_{2n}, SPx_{2n}) + d(x_{2n+1}, TPx_{2n+1})]$$

$$+ c\psi\,[d(x_{2n}, TPx_{2n+1}) + d(x_{2n+1}, SPx_{2n})]$$

$$+e\psi[\frac{(x_{2n}, SPx_{2n}) + d(x_{2n+1}, TPx_{2n+1})}{1 + d(x_{2n}, TPx_{2n+1}), d(x_{2n+1}, SPx_{2n})}]$$

$$+ f\psi\left[\frac{d^2(x_{2n}, SPx_{2n}) + d(x_{2n}, TPx_{2n+1}), d(x_{2n+1}, SPx_{2n}) + d^2(x_{2n+1}, TPx_{2n+1})}{1 + d(x_{2n}, SPx_{2n}) + d(x_{2n+1}, TPx_{2n+1})}\right]$$

$$\psi d(x_{2n+1}, x_{2n+2}) \leq a\psi\,[d(x_{2n}, x_{2n+1})] + b\psi\,[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+ c\psi\,[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$$

$$+e\,\psi\frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})}$$

$$+ f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\leq a\psi\,[d(x_{2n}, x_{2n+1})] + (b + e)\,\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+ c\psi\,[d(x_{2n}, x_{2n+2})] + f\psi\left[\frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}\right]$$

$$\leq a\psi\,[d(x_{2n}, x_{2n+1})] + (b + e)\psi\,[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$+ c\psi\,[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + f\,\psi[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$\leq (a + (b + e) + c + f)\,\psi d(x_{2n}, x_{2n+1}) + ((b + e) + c + f)\psi\,d(x_{2n+1}, x_{2n+2})$$

$$(1 - (b + e) - c - f)\psi\,d(x_{2n+1}, x_{2n+2}) \leq (a + (b + e) + c + f)\psi d(x_{2n}, x_{2n+1})$$

By using triangle inequality we get,

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \frac{(a + (b + e) + c + f)}{(1 - (b + e) - c - f)}\,\psi d(x_{2n}, x_{2n+1})$$

As similarly we can show that,

$$\psi d(x_{2n}, x_{2n+1}) \leq \frac{(a + b + c + f)}{(1 - b - c - f)}\psi\,d(x_{2n-1}, x_{2n})$$

In general we can write,

$$\psi d(x_{2n+1}, x_{2n+2}) \leq \left[\frac{(a + (b + e)b + c + f)}{(1 - (b + e) - c - f)}\right]^{2n+1}\psi\,d(x_0, x_1)$$

On taking $\left[\frac{(a+(b+e)+c+f)}{(1-(b+e)-c-f)}\right] = K$,

$$\psi d(x_{2n+1}, x_{2n+2}) \leq K^{2n+1}\,\psi d(x_0, x_1)$$

For $n \leq m$, we have

$$\psi d(x_{2n}, x_{2m}) \leq \psi\, d(x_{2n}, x_{2n+1}) + \psi d(x_{2n+1}, x_{2n+2}) + \_\_\_\_ + \psi d(x_{2m-1}, x_{2m})$$

$$\leq (K^n + K^{n+1} + K^{n+2} + \_\_\_\_\_\_ + K^m)\, \psi d(x_0, x_1)$$

$$\leq \frac{K^n}{1-K}\, \psi d(x_0, x_1)$$

$$\psi \| d(x_{2n}, x_{2m}) \| \leq \frac{K^n}{1-K} \quad r\psi\ \| d(x_0, x_1) \|$$

As $\lim_{n\to\infty} \psi\ \| d(x_{2n}, x_{2m}) \| \to 0$

In this way $\lim_{n\to\infty}\ \psi d(x_{2n+1}, x_{2n+2}) \to 0$

Hence $\{x_n\}$is a Cauchy sequence which converges to $v \in X$.

Therefore (X, d) is complete cone metric space Thus $x_n \to v \quad as\ n \to \infty$

SP$x_{2n} \to v \quad and \quad TPx_{2n+1} \to v \quad as\ n \to \infty$

$$\therefore\ v\ is\ fixed\ point\ of\ S\ and\ T\ in\ X, Since\ ST = TS\ this\ gives,$$

$v = Tv = TSv = STv = Sv = v$

$v\ is\ \ common\ fixed\ point\ of\ S\ and\ T.$

Uniqueness : Let w be another fixed point of S and T in X distinct from v, Then we have,

T $v = v \quad and \quad Tw = w\ also\ Sv = v\ and\ \ Sw = w$

$\psi d(v, w) = \psi d(Sv, Tw)$

From (3)

$$\psi d(Sv, Tw) \leq a\, \psi d(v, w) + b\psi\, [d(v, Sv) + d(w, Sw)] + c\psi\, [d(v, Tw) + d(w, Sv)]$$

$$+e\psi \left[\frac{d(v,Sv)+d(w,Sw)}{1+d(v,Tw),d(w,Sv)}\right]$$

$$+f\psi \left[\frac{d^2(v, Sv) + d(v, Tw),\ \ d(w, Sv) + d^2(w, Sw)}{1 + d(v, Sv) + d(w, Sw)}\right]$$

$$\psi d(Sv, Tw) \leq a\, \psi d(v, w) + (b + e)\psi\, [d(v, Sv) + d(w, Sw)] + c\psi\, [d(v, Tw) + d(w, Sv)]$$

$$+f\ \psi \left[\frac{d^2(v, Sv) + d(v, Tw),\ \ d(w, Sv) + d^2(w, Sw)}{1 + d(v, Sv) + d(w, Sw)}\right]$$

$\psi d(Sv, Tw) \leq (a + 2c + f)\ \psi d(v, w)$

This is a contradiction. So v is unique common fixed point of S and T in X.

Theorem: 3.4 Let (X, d) be a complete cone metric space and Q a normal cone with normal constant r. $\psi \in \Psi$. Suppose that the mapping E, F, S and P be the mapping from X into itself satisfies the condition

(i) $E(X) \subseteq P(X), \ F(X) \subseteq S(X)$
(ii) [E, S] and [F, P] are weakly compatible.
(iii) S or P is continuous
(iv) $$\psi d(Ex, Fy) \leq a\,\psi d(Sx, Py) + b\psi\,[d(Sx, Ex) + d(Py, Fy)] + c\psi\ [d(Sx, Fy) + d(Py, Ex)] + +e\,\psi[\frac{d(Sx, Ex) + d(Py, Fy)}{1 + d(Sx, Fy)\,d(Py, Ex)}] +f\psi\,[\frac{d^2(Sx, Ex) + d(Sx, Fy)\,d(Py, Ex) + d^2(Py, Fy)}{1 + d(Sx, Ex) + d(Py, Fy)}]$$

For all $x,\ y \in X$ *and* $a, b, c, f \geq 0$ *such that* $0 \leq a + b + c + e + f < 1$.Then E, F, S and P have unique fixed point in X.

Proof: Let us define a sequence$\{x_n\}$ *and* $\{y_n\}$ *in* X, *such that*
$Ex_{2n} = Px_{2n+1} = y_{2n}$ and $Fx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ $\forall\, n = 0,1,2, \dots \dots$

Now, $\psi d(y_n, y_{2n+1}) = \psi d(Ex_{2n},\ Fx_{2n+1})$

From (iv) $$\psi d(Ex_{2n}, Fx_{2n+1}) \leq a\psi\, d(Sx_{2n}, Px_{2n+1}) + b\psi\,[d(Sx_{2n}, Ex_{2n}) + d(Px_{2n+1}, Fx_{2n+1})]$$

$$+ c\psi\,[d(Sx_{2n}, Fx_{2n+1}) + d(Px_{2n+1}, Ex_{2n})] + e\psi\left[\frac{d(Sx_{2n}, Ex_{2n}) + d(Px_{2n+1}, Fx_{2n+1})}{1 + d(Sx_{2n}, Fx_{2n+1}), d(Px_{2n+1}, Ex_{2n})}\right]$$

$$+f\psi\ \left[\frac{d^2(Sx_{2n}, Ex_{2n}) + d(Sx_{2n}, Fx_{2n+1}), d(Px_{2n+1}, Ex_{2n}) + d^2(Px_{2n+1}, Fx_{2n+1})}{1 + d(Sx_{2n}, Ex_{2n}) + d(Px_{2n+1}, Fx_{2n+1})}\right]$$

$$\psi d(y_n, y_{2n+1}) \leq a\,\psi d(y_{2n-1}, y_{2n}) + b\,\psi[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

$$+ c\psi\,[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] + e\psi\left[\frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n+1})\,d(y_{2n}, y_{2n})}\right]$$
$$+ f\,\psi\,\left[\frac{d^2(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\,d(y_{2n}, y_{2n}) + d^2(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}\right]$$

$$\psi d(y_n, y_{2n+1}) \leq (a + b + e)\,\psi\,d(y_{2n-1}, y_{2n}) + (b + e)\,\psi[d(y_{2n}, y_{2n+1})] + \ c\psi\,[d(y_{2n-1}, y_{2n+1})]$$

$$+f\psi\ [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

$$\leq (a + (b + e) + f)\,\psi\,d(y_{2n-1}, y_{2n}) + ((b + e) + c + f)\psi\,[d(y_{2n}, y_{2n+1})]$$

$$(1 - (b + e) - c - f)\psi\,[d(y_{2n}, y_{2n+1})]\ \leq (a + (b + e) + f)\,\psi\,d(y_{2n-1}, y_{2n})$$

$$\psi[d(y_{2n}, y_{2n+1})] \le \left[\frac{(a+(b+e)+f)}{(1-(b+e)-c-f)}\right] \psi\, d(y_{2n-1}, y_{2n})$$

Similarly, in general we have

$$\psi[d(y_{2n}, y_{2n+1})] \le \left[\frac{(a+(b+e)+f)}{(1-(b+e)-c-f)}\right]^{2n+1} \psi\; d(y_0, y_1)$$

On taking $\left[\frac{(a+(b+e)+f)}{(1-(b+e)-c-f)}\right] = K \;\; and\; n \le m, we\; have$

$$\psi[d(y_{2n}, y_{2n+1})] \le (K^n + K^{n+1} + \phi^{n+2} + \_\_\_\_\_\_ + K^m)\psi\;\; d(y_0, y_1)$$

$$\le \frac{K^n}{1-K}\;\psi\, d(y_0, y_1)$$

$$\psi\|d(y_{2n}, y_{2m})\| \le \frac{K^n}{1-K}\quad r\,\psi\,\|d(y_0, y_1)\|$$

As $\lim_{n\to\infty}\;\psi\,\|d(y_{2n}, y_{2m})\| \to 0$

Hence $\{y_n\}$ is a Cauchy sequence which converges to $v \in X$, by the continuity of S and P. Also the sequence $\{x_n\}$ is also convergent sequence which converges to $v \in X$, Hence (X, d) is complete cone metric space and v is a fixed point of E, F, S and P.

Since {E, S} and {F, P} are weakly compatible implies that v is common fixed point of E, F, S and P.

Uniqueness: Let us assume that,w is another fixed point of E, F, S and P in X distinct from v, then

$$Ev = v\;\; and\; Ew = w\;\; also\; Fv = v\;\; and\; Fw = w$$

$$\psi d(v, w) = \psi d(Ev, Fw)$$

From (4) $\psi d(Ev, Fw) \le a\psi\, d(Sv, Pw) + b\psi\,[d(Sv, Ev) + d(Pw, Fw)]$

$$+c\psi\,[d(Sv, Fw) + d(Pw, Ev)]$$

$$+e\psi\left[\frac{d(Sv, Ev) + d(Pw, Fw)}{1 + d(Sv, Fw)\;\; d(Pw, Ev)}\right]$$

$$+f\psi\;\left[\frac{d^2(Sv, Ev) + d(Sv, Fw)\;\; d(Pw, Ev) + d^2(Pw, Fw)}{1 + d(Sv, Ev) + d(Pw, Fw)}\right]$$

$\psi d(Ev, Fw) \le (a + 2c)\psi\, d(v, w)$, this is a contraction.

Hence v is unique point of E, F, S and P in X.

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