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On the Exponential Power Distribution

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Abstract

The paper examines the nature of the exponential power distribution (EPD) in terms of its location, μ, scale, β and shape, σ, parameters. It establishes conditions under which the distribution is legitimate and reliable. It derives among others the moment and kurtosis of the distribution as well as the maximum likelihood estimators of the parameters. It then uses data on health to assess the departure of the distribution from normality. Three main softwares are used, namely; EasyFit, MATLAB and Minitab.

In the application, we find that the EPD, for some values of β, significantly fits Weight, Height and Body Mass Index out of seven variables covered. We deduce that the EPD would be inappropriate for fitting asymmetrical datasets, since the variables which are not significant are found to be highly skewed.

Keywords: Exponential power distribution, Kurtosis, Legitimacy, Statistical Distributions

1. Introduction

The defining characteristics of statistical distributions are their dependence on parameters and the incorporation of stochastic terms. The properties of the distributions and the properties of quantities derived from them are studied in a long-run, average sense through expectations, variances, skewness and kurtosis. The fact that the parameters of the distribution are estimated from the data introduces a stochastic element in applying a statistical distribution. This is because the distribution is not deterministic but includes randomness. Parameters and related quantities derived from the distribution are likewise random.

A statistical distribution of a variable is an approximate representation of its population distribution which may be parametric or non-parametric. A theoretical parametric distribution generally provides a simple parsimonious (and usually smooth) representation of the population distribution. It can be used for inference of the centiles (or quantiles) and moments of the population distribution and other population measures. Inference of the moments can be particularly sensitive to misspecification of the theoretical distribution and especially to misspecification of the heaviness of the tail(s) of the population distribution (Forbes, Evans, Hastings & Peacock, 2011).

Over the last two decades, many researchers have developed interest in the construction of flexible parametric classes of statistical distributions that are more flexible than the normal distribution. Many practical applications require models of data exhibiting a skewed or peaked distributions, and some researchers suggest the use of distributions which are more robust for such data. Some of these applications are in areas that include health, environmental and finance. There are several parametric classes of distributions to choose from. Rigby, Stasinopoulos, Heller and Bastiani (2017) have reviewed many of them. Subbotin (1923) introduced a class of distribution called the exponential power distribution (EPD) which is believed to be more flexible than the normal distribution in terms of kurtosis.

Subbotin in his study on the Law of Frequency of Error formulated an axiom which states that the probability of a random error ε depends only on the absolute value of the error itself and can be expressed by a function $f(\varepsilon)$ having continuous first derivative almost everywhere. Based on this axiom, Subbotin obtained a density function called Subbotin's family of distributions given by

$$f(\varepsilon) = \frac{mh}{2\Gamma\left(\frac{1}{m}\right)} exp\{-h^m|\varepsilon|^m\}, \qquad (1)$$

with $-\infty < \varepsilon < \infty$, $h > 0$ and $m \geq 1$. This class of distributions is said to be symmetric, but with variation in kurtosis. It was noted that this distribution has many structural properties close to the normal distribution. There is also a link in the axiom considered by Subbotin and that of Gauss, as Gauss used similar axiom to derive the usual normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}, \quad (2)$$

with two parameters; $\infty < \mu < \infty$, (mean or location parameter), and $\sigma > 0$ (standard deviation or scale parameter). Several researchers (Coin, 2017; Giller, 2005; Nadarajah, 2005; Pogány & Nadarajah, 2009; Tahir, Cordeiro, Alizadeh, Mansoor, Zubair & Hamedani, 2015) have introduced various classes of distribution relating to the Subbotin's family of distributions. Some studies have used the name the Generalized Gaussian Distribution, Generalized Normal Distribution or Generalized Error Distribution to refer to the Exponential Power Distribution.

Giller (2005) in his study expressed the EPD as

$$P(y|\mu,\sigma,q) = \frac{2^{q+1}\sigma^{-1}}{\Gamma(q+1)} exp\left\{-\frac{1}{2}\left|\frac{y-\mu}{\sigma}\right|^{\frac{1}{q}}\right\}, \quad (3)$$

Giller stated that if $q = 1/2$, then $P(y|\mu,\sigma,q) \sim N(\mu,\sigma^2)$ (Normal) and if $q = 1$, then $P(y|\mu,\sigma,q) \sim L(\mu, 4\sigma^2)$ (Double Exponential or Laplace). In the limit as $q \to 0$, $P(y|\mu,\sigma,q) \sim U(\mu-\sigma,\mu+\sigma)$ (Uniform).

Pogány and Nadarajah (2009) modified the EPD and expressed it as

$$P(y|\mu,\sigma,q) = \frac{q\sigma^{-1}}{2\Gamma\left(\frac{1}{q}\right)} exp\left\{-\left|\frac{y-\mu}{\sigma}\right|^{\frac{1}{q}}\right\}. \quad (4)$$

This distribution has three parameters given as μ, σ and q which represent the location, scale (or dispersion) and shape of the distribution, respectively. They further noted that $P(y|\mu,\sigma,1)$ is Laplace (or Double Exponential) and $P(y|\mu,\sigma,2) \sim N(\mu,\sigma^2/2)$. Also, the pointwise $lim_{q\to\infty} P(y|\mu,\sigma,q)$ coincides with the density function of uniform distribution, $U(\mu-\sigma,\mu+\sigma)$.

Mineo and Ruggieri (2005) expressed their EPD as

$$P(y|\mu,\sigma,q) = \frac{1}{2q^{\frac{1}{q}}\sigma_q\Gamma\left(1+\frac{1}{q}\right)} exp\left\{-\frac{1}{q}\left|\frac{y-\mu}{\sigma_q}\right|^q\right\}, \quad (5)$$

Mineo and Ruggieri explained that the parameter q determines the shape of the curve; in this way, it is linked to the thickness of the tails, and thus to the kurtosis, of the distribution. In fact, by changing the parameter q, the EPD describes both leptokurtic $(0 < q < 2)$ and platikurtic $(q > 2)$ distributions.

Purczyriski and Bednarz-Okrzyriska (2014) adopted a class of EPDs of the form

$$f(y) = \frac{\lambda q}{2\Gamma\left(\frac{1}{q}\right)} exp\{-|y-\mu|^q\}, \quad (6)$$

where $\Gamma(1/q)$ is an Euler's gamma function. For $q = 1$, the EPD turns into the Laplace distribution (bi-exponential), and for $q = 2$, a Normal distribution is obtained given $\lambda = 1/\sigma\sqrt{2}$.

More generally, the EPD can be deduced as

$$P(y|\mu,\sigma,q) = k\sigma^{-1} exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^q\right\}, \quad (7)$$

for $for\ -\infty < y < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ and $q > 0$. This distribution function is characterized by a location parameter μ, a scale parameter σ, and a shape parameter q, where k is the normalizing constant and c is a constant which may depend on q. The Normal distribution is obtained from this distribution when $q = 2$, whereas heavier (or lighter) tail distributions are produced for $q < 2$ $(or\ q > 2)$. In particular, we obtain the double exponential distribution for q = 1 and the uniform distribution for $q \to \infty$.

In Equation (7), if we consider

$$q = \frac{2}{1+\beta}, \tag{8}$$

(Elsalloukh, 2010), then the EPD is given as

$$P(y|\mu,\sigma,\beta) = k\sigma^{-1}exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^{\frac{2}{1+\beta}}\right\}. \tag{9}$$

This will be the basis for the study. The rationale for the expression for q is intended to enable us track the changes in the characteristics of the distribution as a result of the values of β. Also, the constant c will be chosen to ensure that it does not affect the variability or the scale parameter of the distribution. This would ensure the flexibility of the tails whether heavy or thinner tails. If $\beta = 0$, the distribution becomes a Normal distribution; if $\beta = 1$, the distribution becomes a Laplace distribution, but if $\beta \to -1$, then the distribution turns to a rectangular or uniform distribution.

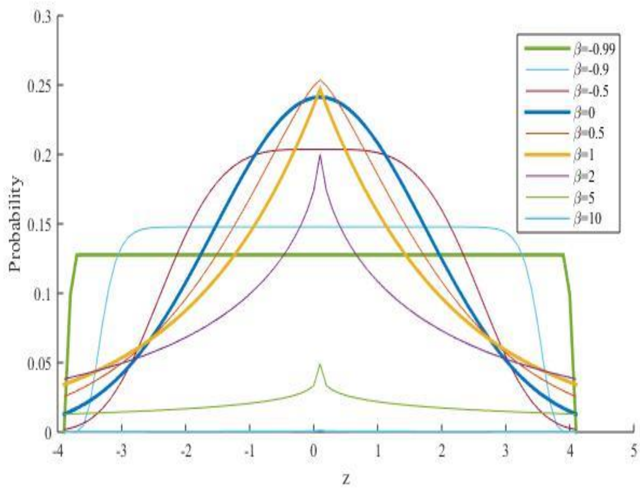


Figure 1: Exponential Distribution for some values of β

Figure 1 presents a typical example of EPD for various values of the parameter, β. In Figure 1, it can be observed that for a small value of β (e.g., $\beta = -0.99$), the EPD has a flat top, whiles for large values of β (e.g., $\beta = 1$ and $\beta = 2$) the EPD has a pencil-like top. Thus, for a decreasing values of β, the EPD approaches uniform or a rectangular distribution.

Many researchers have adopted different forms of EPD, by adopting different values of the constant c in Equation (9) which affect the scale parameter of the distribution as well as the normalizing constant k. For many of the research, the choice of the constant c depends on the shape parameter q. Vianelli (1963) developed EPDs with the constant c deduced as $c = 1/q$. Vianelli called the distribution "A Normal Distribution of Order q". Rahnamaei, Nematollahi and Farnoosh, (2012) adopted the distribution proposed by Vianelli in their data modelling. Giller (2005) adopted a case whereby $c = 1/2$ which does not depend on q. Olosunde (2013) argued that there are limitation on using such family of EPDs, explaining that EPD exhibits thinner tails and care needs to be taken to ensure that the tails are not affected by the choice of c. Due to Olosunde's interest in analysing data from heavy-tailed distribution, he adopted EPD with c as a constant function, $c(q)$, which he explained to regulate the tail region of the distribution. This study will adopt the approach of Olosunde to estimate the constant c, but will ensure that it is estimated to make the variance of the EPD the same as the scale parameter, σ.

Although, most research have addressed the characteristics of the EPDs of different kinds, information is rarely provided to address the reliability of the density function of their adopted distribution. Thus, the study will examine the flexibility and the characteristics of EPD of various kinds and address the issue of its legitimacy and reliability. In the process, we will derive and estimate the parameters of the distribution with respect to a dataset and examine its fitness.

2. Methodology

2.1 The Gamma Distribution

The gamma function is very important in mathematical statistics. It is a continuous extension to the factorial function, which is only defined for the non-negative integers. The gamma function (or gamma integral) is given by

$$\Gamma(s) = \int_0^\infty \mathrm{y}^{s-1}\mathrm{e}^{-\mathrm{y}}\,dy, \qquad s > 0, \qquad (10)$$

or sometimes

$$\Gamma(s) = 2\int_0^\infty y^{2s-1}e^{-y^2}\,dy, \qquad s > 0. \qquad (11)$$

Also, the function

$$\Gamma(\mathrm{s},\mathrm{t}) = \int_t^\infty y^{\mathrm{s}-1}\mathrm{e}^{-\mathrm{y}}\,dy, \qquad (12)$$

for all $\mathrm{s} > 0$and $\mathrm{y} \geq 0$ is the incomplete gamma function. Now, if $\mathrm{s} > 1$, then $\Gamma(s) = (s-1)\Gamma(s-1)$. For any non-negative integer, the logarithmic derivative of $\psi(s)$ is the psi or digamma function denoted $\psi(s)$ and given as

$$\psi(s) = \frac{d}{ds}\left(\ln\left(\Gamma(s)\right)\right) = \frac{\Gamma'(s)}{\Gamma(s)}, \qquad (13)$$

and expressed as

$$\psi(s) = \int_0^\infty \left(\frac{e^{-y}}{y} - \frac{e^{-sy}}{1-e^{-y}}\right)dy. \qquad (14)$$

While there are other continuous extensions to the factorial function, the gamma function is the only one that is convex for positive real numbers. Figure 2, presents a typical gamma function in the plane.

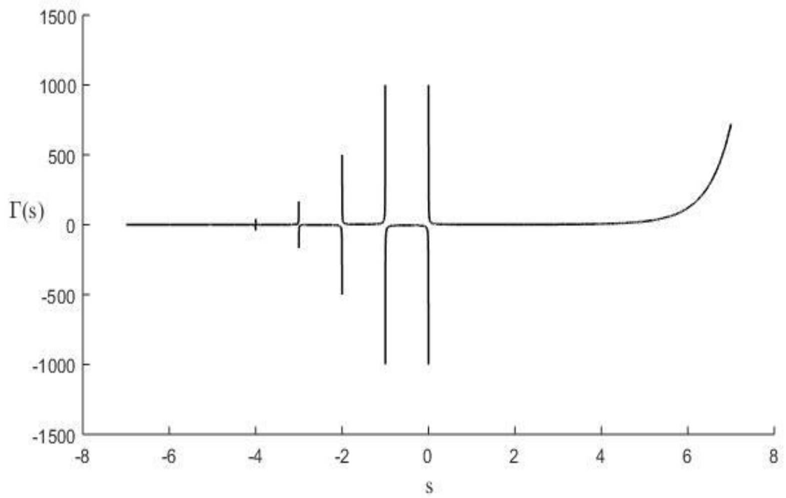


Figure 2: Gamma function $[\Gamma(s)]$ in the whole complex plane

From Figure 2, the function is defined for non-negative values of *s*, but undefined (discontinues) for some negative values of *s*. The gamma function however plays a major role in the characteristics and properties of EPD, as we will demonstrate in following Sections.

2.2 The Legitimacy of the Exponential Power Distribution

Let Y be a continuous random variable. The function $f(y)$ is said to be a proper or legitimate probability density function (pdf) of the continuous variable Y if $f(y)$ is positive for all values of y within $\mathbb{R}_y$, that is $f(y) \geq 0$ for $y \in \mathbb{R}$ (non-negativity) and if

$$\int_{\mathbb{R}_y} f(y)dy = 1. \qquad (15)$$

Thus, the EPD is a proper pdf of the continuous variable Y if $P(y|\mu,\sigma,\beta)$ is for all values of y within $\mathbb{R}_y$ and if

$$\int_{\infty}^{-\infty} P(y|\mu,\sigma,\beta)\,dy = \int_{\infty}^{-\infty} k\sigma^{-1}exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^{\frac{2}{1+\beta}}\right\}dy = 1. \tag{16}$$

Also, $P(y|\mu,\sigma,\beta) = 0$ for all y in the real line $\mathbb{R}$ not in $\mathbb{R}_y$. We note that probabilities are given by areas under $P(y|\mu,\sigma,\beta)$ as

$$P(a \le Y \le b) = \int_a^b P(y|\mu,\sigma,\beta)dy. \tag{17}$$

We also note the peculiarity that $P(Y = a) = \int_a^a P(y|\mu,\sigma,\beta)dy = 0,$ for any arbitrary value a. This can be circumvented by defining the probability on a small interval $(a-\Delta y, a+\Delta y)$ around a, where Δy has a small value. Then

$$P\big(Y \in (a-\Delta y, a+\Delta y)\big) = \int_{a-\Delta y}^{a+\Delta y} P(y|\mu,\sigma,\beta)dy, \tag{18}$$

is properly defined.

Given the two conditions for the legitimacy of the EPD, it can be deduce that if the integral

$$I = \int_{-\infty}^{\infty} P(y|\mu,\sigma,\beta)\,dy, \tag{19}$$

exists and is finite and strictly positive, then $1/_I$ is called the normalizing constant. In this paper, the coefficient k in Equation (9) is the normalizing constant so that the area under the graph of the EPD is 1. This ensures the legitimacy of the EPD.We will now derive an expression for the normalizing constant *k*.

2.2 Deducing the Normalizing Constant

The normalizing constant k, can be deduced by ensuring that

$$k\sigma^{-1}\int_{-\infty}^{\infty} exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^q\right\}dy = 1. \tag{20}$$

In relation to the absolute term $\left|\frac{y-\mu}{\sigma}\right|$, Equation (20) can be expressed as

$$k\sigma^{-1}\left\{\int_{-\infty}^{0} exp\left[-c\left(\frac{y-\mu}{\sigma}\right)^q\right]dy + \int_0^{\infty} exp\left[-c\left(\frac{y-\mu}{\sigma}\right)^q\right]dy\right\} = 1. \tag{21}$$

Let$x = c\left(\frac{y-\mu}{\sigma}\right)^q$ so that $c^{-1}x = \left(\frac{y-\mu}{\sigma}\right)^q$ and $c^{-1/_q}x^{1/_q} = \frac{y-\mu}{\sigma}$. Thus, we can deduce that $y = \sigma c^{-1/_q}x^{1/_q} + \mu$,

for which $\frac{dy}{dx} = \frac{1}{q}\sigma c^{-1/_q}x^{\frac{1}{q}-1}$. Thus, we will have Equation (21) as

$$\begin{aligned}
k\sigma^{-1}\left[\int_{-\infty}^{0} e^{-x}\frac{1}{q}\sigma c^{-1/_q}x^{\frac{1}{q}-1}dx + \int_0^{\infty} e^{-x}\frac{1}{q}\sigma c^{-1/_q}x^{\frac{1}{q}-1}dx\right] &= 1,\\
k\sigma^{-1}\frac{1}{q}\sigma c^{-1/_q}\left[\int_{-\infty}^{0} x^{\frac{1}{q}-1}e^{-x}dx + \int_0^{\infty} x^{\frac{1}{q}-1}e^{-x}dx\right] &= 1,\\
k\frac{1}{q}c^{-1/_q}\left[\int_{-\infty}^{0} x^{\frac{1}{q}-1}e^{-x}dx + \int_0^{\infty} x^{\frac{1}{q}-1}e^{-x}dx\right] &= 1.
\end{aligned} \tag{22}$$

Since the two integrals in Equation (22) are symmetrical, we should have

$$2k\frac{1}{q}c^{-1/_q}\left[\int_0^{\infty} x^{\frac{1}{q}-1}e^{-x}dx\right] = 1. \tag{23}$$

From Equation (10), the integral in Equation (23) are a family of gamma function given as

$$\int_0^{\infty} x^{\frac{1}{q}-1} e^{-x} dx = \Gamma\left(\frac{1}{q}\right).$$

Thus,

$$2kc^{-1/q}\left[\frac{1}{q}\Gamma\left(\frac{1}{q}\right)\right] = 1. \quad (24)$$

Thus, the normalizing constant k is given as

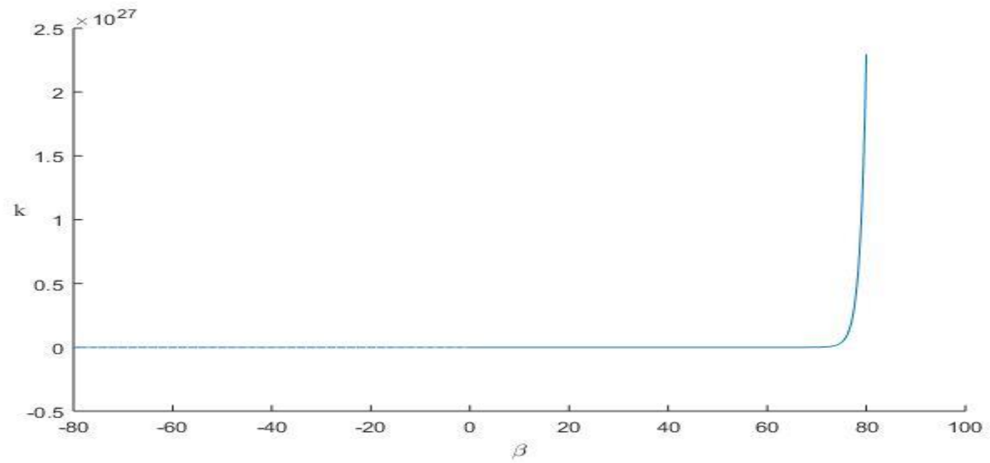
$$k = \frac{1}{2c^{-1/q}\left[\Gamma\left(\frac{1}{q}+1\right)\right]}. \quad (25)$$

Now, for $q = \frac{2}{1+\beta}$ so, we have

$$k = \frac{1}{2c^{-\frac{(1+\beta)}{2}}\left[\Gamma\left(\frac{3+\beta}{2}\right)\right]}. \quad (26)$$

From Equations (25) and (26), the normalizing constant, k, is undefined for some values q and β, respectively. This makes the EPD illegitimate. Figure 3 presents the graphical relationship between k and the shape parameter, β. It can be observed that k is undefined for some negative values of β, and for higher values of β ($\beta > 80$). This makes the integral over $\mathbb{R}$ not equal to 1, and thus, the EPD is illegitimate for such values of β.

Figure 3: Relationship between the normalized constant, k and the shape parameter, β



2.3 The Central Moment of the Exponential Power Distribution

The *i*th central moment of a random variable Y for EPD function, $P(y|\mu,\sigma,q)$ is given by

$$E[(y-\mu)^i] = \int_{-\infty}^{\infty} (y-\mu)^i \, P(y|\mu,\sigma,q)\, dy,$$

$$= k\sigma^{-1}\int_{-\infty}^{\infty} (y-\mu)^i exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^q\right\} dy.$$

Since the integral is symmetrical about the location parameter, μ, we have

$$E[(y-\mu)^i] = [1+(-1)^i]k\sigma^{-1}\int_{\mu}^{\infty} (y-\mu)^i exp\left\{-c\left(\frac{y-\mu}{\sigma}\right)^q\right\} dy, \quad (27)$$

so that $E[(y-\mu)^i] = 0$ for odd values of *i*.

By standardization, setting z= $\left(\frac{y-\mu}{\sigma}\right)$ so that $z^i = \frac{(y-\mu)^i}{\sigma^i}$ and $\sigma^i z^i = (y-\mu)^i$. Now, $\sigma dz = dy$ and $E(z) = 0$.

Substituting these deductions into Equation (27), we have

$$E[(y-\mu)^i] = k\sigma^{-1}[1+(-1)^i]\int_0^\infty z^i\sigma^i exp\{-cz^q\}\,\sigma dz,$$

$$= k\sigma^i[1+(-1)^i]\int_0^\infty z^i exp\{-cz^q\}\,dz, \quad (28)$$

Now, we let $x = cz^q$, so that $c^{-1}x = z^q$. Thus, $z = c^{-1/q}x^{1/q}$, $z^i = c^{-i/q}x^{i/q}$ and $\frac{dz}{dx} = \frac{1}{q}c^{-1/q}x^{\frac{1}{q}-1}$.

Equation (28) then gives

$$E[(y-\mu)^i] = k\sigma^i[1+(-1)^i]c^{-i/q}c^{-1/q}\frac{1}{q}\int_0^\infty x^{i/q}x^{\frac{1}{q}-1}e^{-x}\,dx,$$

$$= \frac{1}{q}k\sigma^i[1+(-1)^i]c^{-\frac{i+1}{q}}\int_0^\infty x^{\frac{i+1}{q}-1}e^{-x}\,dx. \quad (29)$$

Equation (29) simplifies as

$$E[(y-\mu)^i] = \frac{1}{q}k\sigma^i[1+(-1)^i]c^{-\frac{i+1}{q}}\left[\Gamma\left(\frac{i+1}{q}\right)\right].$$

Making substitution for *k*, we obtain

$$E[(y-\mu)^i] = \frac{1}{2c^{-1/q}\left[\frac{1}{q}\Gamma\left(\frac{1}{q}\right)\right]}\sigma^i[1+(-1)^i]c^{-\frac{i+1}{q}}\left[\frac{1}{q}\Gamma\left(\frac{i+1}{q}\right)\right],$$

which simplifies as

$$E[(y-\mu)^i] = \frac{\left[\Gamma\left(\frac{i+1}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^i\frac{[1+(-1)^i]}{2}. \quad (30)$$

For q= $\frac{2}{1+\beta}$ so, we have

$$E[(y-\mu)^i] = \frac{\left[\Gamma\left(\frac{(i+1)(1+\beta)}{2}\right)\right]}{\left[\Gamma\left(\frac{1+\beta}{2}\right)\right]}\left(\frac{\sigma}{c^{\frac{1+\beta}{2}}}\right)^i\frac{[1+(-1)^i]}{2}. \quad (31)$$

We will now use the *i*th central moment to derive the mean, variance, skewness and kurtosis of the EPD. The following sections present the results.

2.3.1 Mean and Variance of the Exponential Power Distribution

It can be deduced from Equation (30) that $E(y) = \mu$ when $i = 1$. In Equation (30) again, if $i = 2$, then

$$E[(y-\mu)^2] = \frac{\left[\Gamma\left(\frac{2+1}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^2\frac{[1+(-1)^2]}{2}.$$

Thus,

$$var(y) = \frac{\left[\Gamma\left(\frac{3}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^2, \quad (32)$$

$$= \frac{c^{-2/q}\left[\Gamma\left(\frac{3}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\sigma^2 . \quad (33)$$

Now, for $q = \frac{2}{1+\beta}$ so, we have

$$var(y) = \frac{c^{-\frac{1+\beta}{2}}\left[\Gamma\left(\frac{3(1+\beta)}{2}\right)\right]}{\left[\Gamma\left(\frac{1+\beta}{2}\right)\right]}\sigma^2. \quad (34)$$

From Equation (33), it can be observed that, the $var(y)$ is affected by the shape parameter, q, and the constant, c. We want to derive c such that $var(y) = \sigma^2$.

Let

$$h = \frac{c^{-2/q}\left[\Gamma\left(\frac{3}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]} . \quad (35)$$

Thus, if $h = 1$, then $var(y) = \sigma^2$. Therefore, we would find c such that

$$\frac{c^{-2/q}\left[\Gamma\left(\frac{3}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]} = 1. \quad (36)$$

This gives

$$c = \left[\Gamma\left(\frac{3}{q}\right)\right]^{q/2}\left[\Gamma\left(\frac{1}{q}\right)\right]^{-q/2}, \quad (37)$$

or

$$c = \left[\Gamma\left(\frac{3(1+\beta)}{2}\right)\right]^{\frac{1}{1+\beta}}\left[\Gamma\left(\frac{1+\beta}{2}\right)\right]^{-\frac{1}{1+\beta}}. \quad (38)$$

2.3.2 Skewness and Kurtosis of the Exponential Power Distribution

The third central moment given by

$$\mu_3 = E[(y-\mu)^3], \quad (39)$$

is used to determine the symmetry of the distribution. As we know, μ_3 alone is a poor measure of skewness since the size is influenced by the units used to measure the values of *X*. To make this measure dimensionless, we use

$$a_3 = \frac{E[(y-\mu)^3]}{\sqrt{Var(y)}^3}, \quad (40)$$

which is a measure of lack of symmetry. Since $E[(y-\mu)^3] = 0$, $a_3 = 0$.

Generally, the coefficient of kurtosis, also known as the fourth standardized comulant, is given by

$$a_4 = \frac{E[(y-\mu)^4]}{var(y)^2} - 3. \quad (41)$$

In terms of EPD, the coefficient of kurtosis is given by

$$a_4 = \frac{\int_{-\infty}^{\infty}(y-\mu)^4 \, P(y|\mu,\sigma,q)\,dy}{var(y)^2} - 3. \quad (42)$$

This measures the nature of the spread of the values around the mean. Thus, it is a measure of the peakedness of EPD or how heavy the tails of EPD are. If a random population has kurtosis above or below zero (0), it cannot be adequately represented by a normal distribution. From Equation (30),

$$E[(y-\mu)^4] = \frac{\left[\Gamma\left(\frac{5}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^4. \quad (43)$$

Thus, the coefficient of Kurtosis could then be deduced as

$$a_4 = \frac{\frac{\left[\Gamma\left(\frac{5}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^4}{var(y)^2} - 3. \quad (44)$$

Substituting the expression for var(y) into Equation (44), we have

$$a_4 = \frac{\frac{\left[\Gamma\left(\frac{5}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^4}{\left[\frac{\left[\Gamma\left(\frac{3}{q}\right)\right]}{\left[\Gamma\left(\frac{1}{q}\right)\right]}\left(\frac{\sigma}{c^{1/q}}\right)^2\right]^2} - 3,$$

which simplifies as

$$a_4 = \frac{\left[\Gamma\left(\frac{5}{q}\right)\right]\left[\Gamma\left(\frac{1}{q}\right)\right]}{\left[\left[\Gamma\left(\frac{3}{q}\right)\right]\right]^2} - 3. \quad (45)$$

In terms of β, we have

$$a_4 = \frac{\left[\Gamma\left(\frac{5(1+\beta)}{2}\right)\right]\left[\Gamma\left(\frac{1+\beta}{2}\right)\right]}{\left[\left[\Gamma\left(\frac{3(1+\beta)}{2}\right)\right]\right]^2} - 3. \quad (46)$$

Table 1: Estimation of the values of the constant, *c*, the normalized constant, *k* and the kurtosis for some values of β

β	q	c	k	Kurtosis (a_4)
-1.0	∞	∞	∞	∞
-0.8	10.000	0.003	0.295	-1.116
-0.6	5.000	0.060	0.310	-0.930
-0.4	3.333	0.180	0.333	-0.678
-0.2	2.500	0.332	0.363	-0.369
0.0	2.000	0.500	0.399	0.000
0.2	1.667	0.676	0.443	0.433
0.4	1.429	0.857	0.494	0.939
0.6	1.250	1.041	0.555	1.527
0.8	1.111	1.227	0.625	2.209
1.0	1.000	1.414	0.707	3.000
1.2	0.909	1.602	0.802	3.915
1.4	0.833	1.791	0.913	4.975
1.6	0.769	1.980	1.041	6.200
1.8	0.714	2.169	1.190	7.618
2.0	0.667	2.359	1.363	9.257

Table 1 presents some estimations of the normalizing constant, k, the constant c which ensures that $var(y) = \sigma^2$ and the coefficient of kurtosis of the EPD for some values of β. The estimations were based on the derivations of q, k, c and α_4 in Equations (8), (26), (38) and (46), respectively. The values were estimated based on the values of β of -1, -0.8, -0.6, …, 2. From the table, it can be observed that for $\beta = 0$, the coefficient of kurtosis, 4, is zero (0), indicating a mesokurtic distribution with identical distribution as that of the Normal. Also, for negative values of β, α_4 is also negative, with exception of $\beta = -1$ for which the distribution is rectangular and hence α_4 is undefined. For positive values of β, α_4 is also positive. For $\beta = 1$, α_4 is equal to 3, indicating a double exponential or Laplace distribution. This is similar to the twice of chi-square distribution with 2 degrees of freedom.

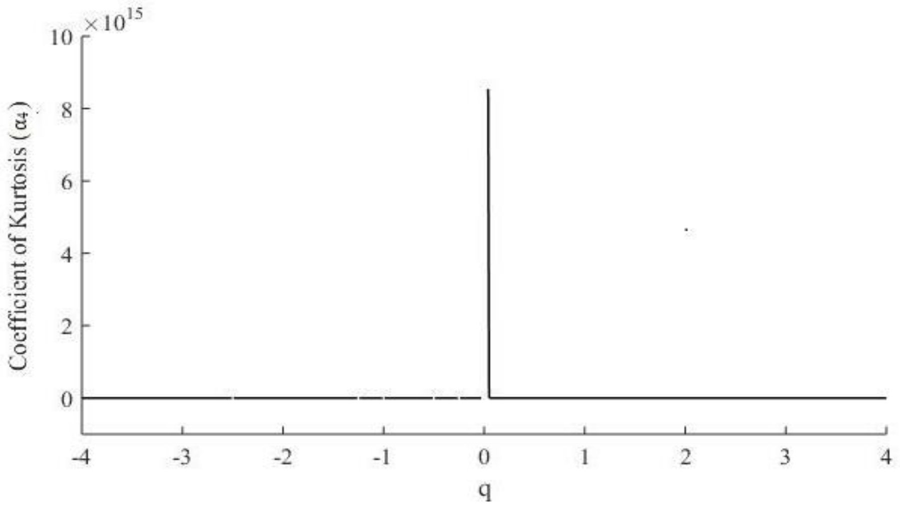


Figure 4: Kurtosis of the Exponential Power Distribution for various values of q

Figures 4 and 5 are graphs showing the relationship between α_4 and the shape parameters, q and β, respectively. As indicated earlier, α_4 is undefined for some negative values of q (and β). The two graphs show the effect of α_4 as a result of the reciprocal relationship between q and β. For large values of β (particularly for $\beta > 60$), we see that α_4 is ∞. Figure 6 shows the nature of the increase in α_4 for $\beta \in (-1,10)$. It is clear that the kurtosis is extremely large for even small values of β which are less than 10.

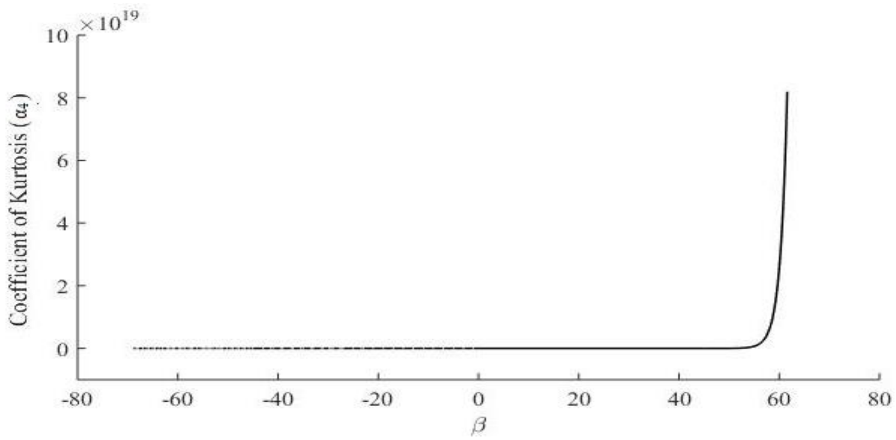


Figure 5: Kurtosis of the Exponential Power Distribution for various values of β

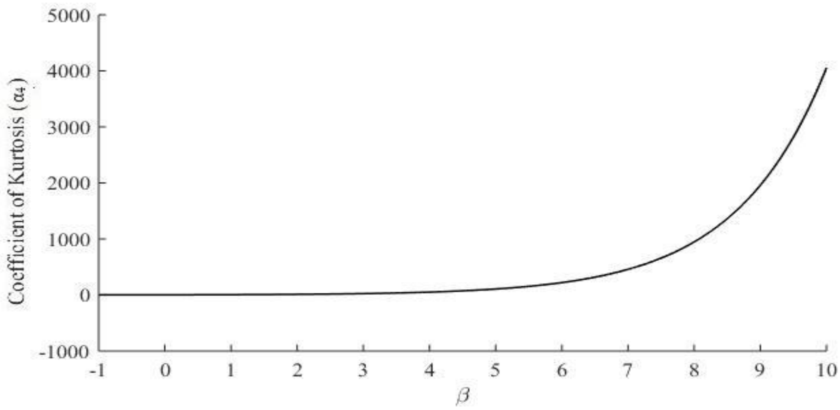


Figure 6: Kurtosis of the Exponential Power Distribution for various values of β between -1 and 10

2.4 Parameter Estimation

2.4.1 Maximum Likelihood Estimation of the Location and Scale Parameters

We assume to have a sample of n i.i.d. observations drawn from a population distributed as EPD, then the likelihood function is

$$L(y|\mu,\sigma,q) = \prod_{i=1}^{n} k\sigma^{-1} exp\left\{-c\left|\frac{y-\mu}{\sigma}\right|^{q}\right\},$$

$$= \left(\frac{k}{\sigma}\right)^{n} exp\left\{-c\sum_{i=1}^{n}\left|\frac{y-\mu}{\sigma}\right|^{q}\right\}.$$

The log-likelihood function is given as

$$l(y|\mu,\sigma,q) = \ln L(y|\mu,\sigma,q) = n\ln\left(\frac{k}{\sigma}\right) - c\sum_{i=1}^{n}\left|\frac{y-\mu}{\sigma}\right|^{q}.$$

By differentiating the log-likelihood function with respect to μ, equating the obtained expressions to zero and noting that $\sigma \geq 0$, we have

$$-\frac{c}{\sigma^q}\frac{d}{d\mu}\left[\sum_{i=1}^{n}|y-\mu|^{q}\right] = 0,$$

$$-\frac{qc}{\sigma^q}\left[\sum_{i=1}^{n}|y-\mu|^{q-1}\frac{|y-\mu|}{(y_i-\mu)}\right] = 0.$$

Now,

$$\frac{|y-\mu|}{(y_i-\mu)} = sign(y_i-\mu), \qquad (47)$$

so that

$$sign(y_i-\mu) = \begin{cases} -1 & \text{if} \quad y_i < \mu \\ 0 & \text{if} \quad y_i = \mu \\ 1 & \text{if} \quad y_i > \mu. \end{cases} \qquad (48)$$

Thus,

$$-\frac{qc}{\sigma^q}\left[\sum_{i=1}^{n}|y-\mu|^{q-1} sign(y_i-\mu)\right] = 0. \qquad (49)$$

We note that the above equation does not yield, in general, an explicit solution, by which the parameter, μ, may be derived by numerical methods. Thus, we will approximate the location parameter as the usual arithmetic mean given as

$$\mu = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

This is due to the nature of the EPD being a symmetrical distribution.

Again, by differentiating the log-likelihood function with respect to σ and equating to zero, we have

$$\frac{d}{d\sigma}[n\ln(k)] - \frac{d}{d\sigma}[n\ln(\sigma)] - \frac{d}{d\sigma}\left[c\sigma^{-q}\sum_{i=1}^{n}|y-\mu|^q\right] = 0,$$

$$-\frac{n}{\sigma} - (-q)[c\sigma^{-q-1}]\sum_{i=1}^{n}|y-\mu|^q = 0,$$

$$-\frac{n}{\sigma} + \frac{qc}{\sigma^{q+1}}\sum_{i=1}^{n}|y-\mu|^q = 0.$$

Solving for σ gives

$$\sigma = \left(\frac{qc}{n}\sum_{i=1}^{n}|y-\mu|^q\right)^{\frac{1}{q}}. \qquad (50)$$

This expression may explain why Vianelli (1963) referred to σ as the power deviation of order q and it can be seen as a variability index which generalises the standard deviation. In terms of β, it may be given as

$$\sigma = \left(\frac{2c}{n(1+\beta)}\sum_{i=1}^{n}|y-\mu|^{\frac{2}{1+\beta}}\right)^{\frac{1+\beta}{2}}. \qquad (51)$$

2.4.2 Estimation of the Shape Parameter

The estimate of the shape parameter β is an open problem. Several procedures have been proposed (Nadarajah, 2005; Vasudeva & Vasantha Kumari, 2013; Purczyriski & Bednarz-Okrzyriska, 2014). In literature, the main proposals are based on the maximum likelihood estimation method or on the computation of kurtosis indices. By differentiating the log-likelihood function with respect to q and equating to zero, noting that $\sigma \geq 0$, we have

$$\frac{d}{dq}\left[n\ln\left(\frac{k}{\sigma}\right) - c\sum_{i=1}^{n}\left|\frac{y-\mu}{\sigma}\right|^q\right] = 0,$$

$$\frac{d}{dq}[n\ln(k)] - \frac{d}{dq}[n\ln(\sigma)] - \frac{d}{dq}\left[c\sigma^{-q}\sum_{i=1}^{n}|y-\mu|^q\right] = 0,$$

$$\frac{d}{dq}[n\ln(k)] - \frac{d}{dq}\left[c\sigma^{-q}\sum_{i=1}^{n}|y-\mu|^q\right] = 0. \qquad (52)$$

Now,

$$\frac{d}{dq}[n\ln(k)] = \frac{d}{dq}\left[n\ln\left(\frac{1}{2c^{-1/q}\left[\frac{1}{q}\Gamma\left(\frac{1}{q}\right)\right]}\right)\right],$$

$$= n\frac{d}{dq}\left[-\ln(2) + \frac{1}{q}\ln(c) + \ln(q) - \ln\Gamma\left(\frac{1}{q}\right)\right],$$

$$= n\left[\frac{d}{dq}\frac{1}{q}\ln(c) + \frac{d}{dq}\ln(q) - \frac{d}{dq}\ln\Gamma\left(\frac{1}{q}\right)\right].$$

Also,

$$\frac{1}{q}\ln(c) = \frac{1}{q}\ln\left(\left[\Gamma\left(\frac{3}{q}\right)\right]^{q/2}\left[\Gamma\left(\frac{1}{q}\right)\right]^{-q/2}\right),$$

$$= \frac{1}{q}\left(\ln\left[\Gamma\left(\frac{3}{q}\right)\right]^{q/2} + \ln\left[\Gamma\left(\frac{1}{q}\right)\right]^{-q/2}\right),$$

$$= \frac{1}{2}\ln\left[\Gamma\left(\frac{3}{q}\right)\right] - \frac{1}{2}\ln\left[\Gamma\left(\frac{1}{q}\right)\right].$$

Thus, in relation to the definition of digamma or psi function in Equation (13), we have

$$\frac{d}{dq}[n\ln(k)] = n\left\{\frac{1}{2}\left[\psi\left(\frac{3}{q}\right)\right] - \frac{1}{2}\left[\psi\left(\frac{1}{q}\right)\right] + \frac{1}{q} - \psi\left(\frac{1}{q}\right)\right\}. \quad (53)$$

Now, from Equation (52)

$$\frac{d}{dq}\left[c\sigma^{-q}\sum_{i=1}^{n}|y-\mu|^q\right] = \sum_{i=1}^{n}|y-\mu|^q\frac{d}{dq}[c\sigma^{-q}] + c\sigma^{-q}\frac{d}{dq}\left[\sum_{i=1}^{n}|y-\mu|^q\right].$$

Thus, Equation (52) is simplified as

$$\frac{n}{2}\left[\psi\left(\frac{3}{q}\right)\right] - \frac{n}{2}\left[\psi\left(\frac{1}{q}\right)\right] + \frac{n}{q} - n\,\psi\left(\frac{1}{q}\right) - \sum_{i=1}^{n}|y-\mu|^q\frac{d}{dq}[c\sigma^{-q}] - c\sigma^{-q}\frac{d}{dq}\left[\sum_{i=1}^{n}|y-\mu|^q\right] = 0. \quad (54)$$

The maximum likelihood estimators have suitable properties, at least asymptotically, but in the case of the estimation of the shape parameter q, likewise β, we could meet with computational difficulties by using numerical methods to solve. In this paper, therefore, we examine the shape of the EPD for various chosen values of β and then compare the corresponding distribution to the Normal and that from Easyfit based on the dataset.

2.5 Accuracy of the Exponential Power Distribution and Goodness-of-fit

We will use model error measures to determine how well the EPD fits the dataset. Some of the accuracy measures include Mean Absolute Deviation (MAD), Mean Square Error and Root Mean Square Error. In this paper, the Kolmogorov-Simirnov test will be considered. It is argued that the problem with the KS statistic is that it cares only about the maximum level of discrepancies, without considering whether the distribution as a whole fits reasonably well (Cruz, 2002). Thus, it tends to over-fit the data, that is, it tends to be too lenient. This is especially true for small samples.

The selection of KS test is because it is among the best for small samples and it can also be used for large samples (Romeu, 2003). To use the test, we pre-specify the distribution which is the EPD for various values of β and estimate the parameters for the distribution. The KS statistic is

$$Dn = \max[|F_n(y) - F(y)|], \quad (55)$$

where D_n is known as the KS distance; n is the number of the data points; $F_n(y)$ is the step function CDF of the actual dataset.

Table 2: Significant and Critical Values for the Kolmogorov-Smirnov Statistics

Critical Value	Significance Level
$1.07/\sqrt{n}$	0.20
$1.22/\sqrt{n}$	0.10
$1.36/\sqrt{n}$	0.05
$1.63/\sqrt{n}$	0.01

Source: Cruz (2002)

By the KS, if the maximum departure between the assumed (CDF) and the CDF of the underlying dataset is small, then the assumed CDF will likely be correct. But if this discrepancy is "large" the assumed distribution is likely not to follow the underlying distribution. Thus, a rule for making a decision on the distribution using the KS test is by comparing the KS distance with the appropriate critical value (CV). KS tables of CVs can be found in Table 2. If KS distance is less than KS critical value, then the CDF of the actual data follows the assumed theoretical CDF, which in our case, is the EPD.

3 Analysis and Results

The data was a second data, collected in 2012 and it covered 453 indigenous people of some six towns in the Central region of Ghana. The portion of the data that is used for this study is on the following variables; Random Blood Sugar Level (RBS), Blood Pressure Systolic (BPS), Diastolic Blood Pressure (BPD), Height, Weight, Body Mass Index (BMI) and Waist Circumference (WC).

3.1 Descriptive Analysis of the Data

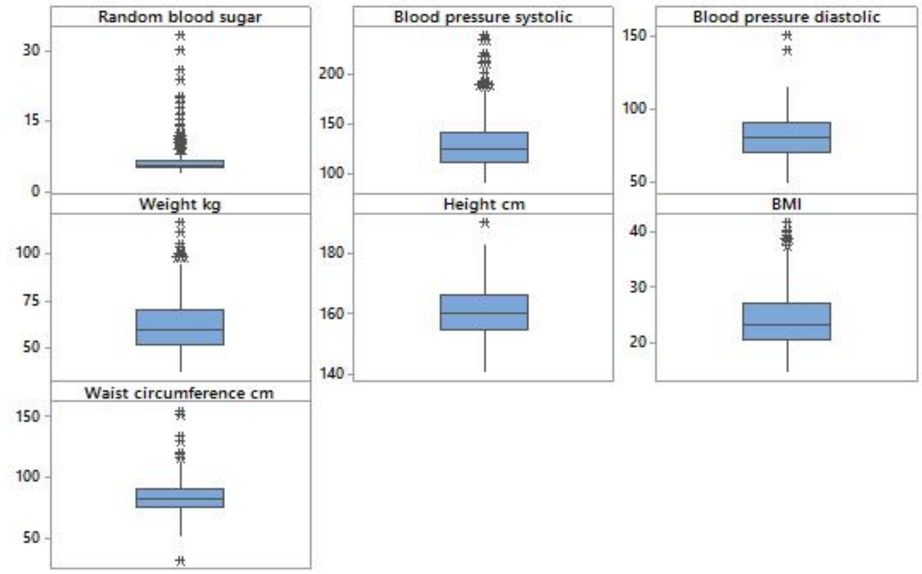


Figure 7: Box Plot of RBS, BPS, BPD, Height, Weight, BMI and WC

Figure 7 shows boxplot for each of the seven variables. From Figure 7, all the variables recorded extreme values. Particularly, random blood sugar level and blood pressure systolic recorded high level of skewness. Only the distribution of height seems to be symmetrically distributed. The remaining four variables appear generally positively skewed. We therefore expect the EPD to fit well for the distribution of height in particular.

Table 3: Descriptive Statistics for RBS, BPS, BPD, Height, Weight, BMI and WC

Var	Mean	SD	Min	Max	Range	Skewness	Kurtosis
RBS	6.10	2.90	3.40	33.30	29.90	5.37	37.09
BPS	130.17	25.16	88.00	238.00	150.00	1.29	2.26
BPD	79.06	13.70	48.00	150.00	102.00	0.69	1.82
Weight	62.15	13.54	37.00	116.00	79.00	0.78	0.74
Height	160.60	8.42	140.50	190.00	49.50	0.36	0.00
BMI	24.10	4.97	14.45	41.59	27.14	0.84	0.66
WC	84.86	13.06	32.00	154.70	122.70	1.04	3.63

Table 3 gives corresponding descriptive statistics of the seven variables. The table shows that the RBS level is highly positively skewed with a high level of kurtosis, which is a leptokurtic distribution, having very high peak level with most data clustering around the mean. Though variables such as the BPS also have high skewness and kurtosis, deviation from normality would be much less than that of RBS. The statistics of Height also buttress the fact that the distribution is quite close to normality.

3.2 Estimation of the Location and Scale Parameters

This part of the analysis presents the results on the estimation of the location, μ, and scale, σ, parameters for some values of β. Figure 8 is a plot of values of σ against β. From the graph, it can be observed that, for the variables, the values of σ decrease sharply, as β increases, indicating an inverse relationship between the two parameters. We see that σ approaches 0 for values of β beyond 2. This is the reason why all assignment made use of values of β between -1 and 2.

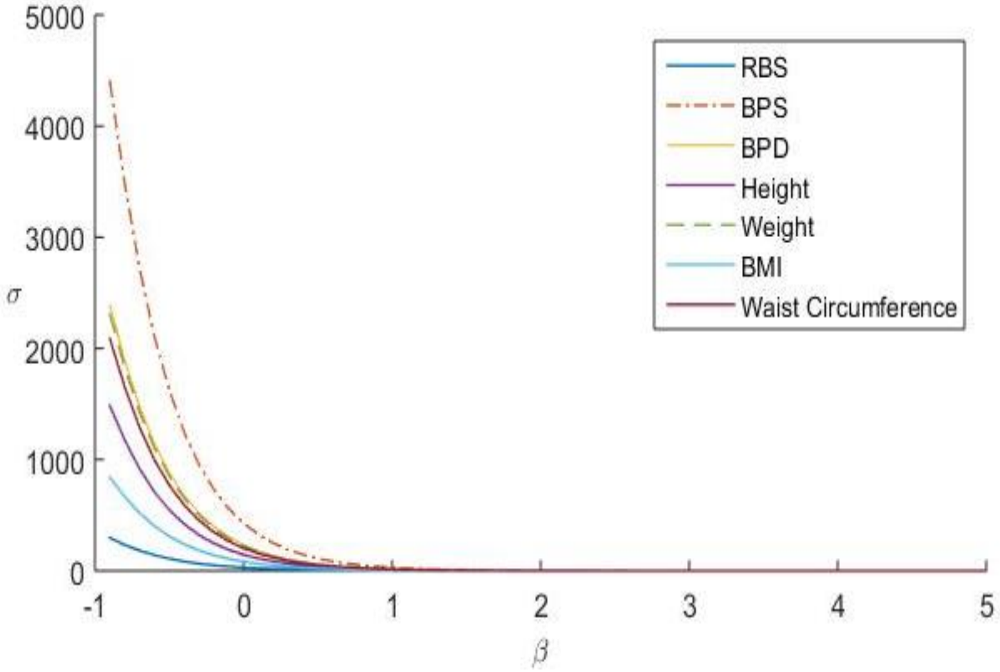


Figure 8: Estimation of the scale Parameter, σ, for some values of β

Table 4: Estimation of μ and σ for all the Variables

β	RBS	BPS	BPD	Weight	Height	BMI	WC
	σ	σ	σ	σ	σ	σ	σ
-1.0	∞	∞	∞	∞	∞	∞	∞
-0.8	233.52	3487.32	1893.20	1176.07	1826.04	666.00	1653.27
-0.6	141.01	2105.84	1143.22	710.18	1102.67	402.17	998.34
-0.4	83.49	1246.87	676.90	420.50	652.89	238.12	591.12
-0.2	49.02	732.02	397.40	246.87	383.30	139.80	347.04
0.0	28.65	427.83	232.26	144.28	224.02	81.71	202.83
0.2	16.70	249.38	135.38	84.10	130.58	47.63	118.23
0.4	9.72	145.11	78.78	48.94	75.98	27.71	68.80
0.6	5.65	84.34	45.79	28.44	44.16	16.11	39.98
0.8	3.28	48.98	26.59	16.52	25.65	9.35	23.22
1.0	1.90	28.43	15.43	9.59	14.89	5.43	13.48
1.2	1.10	16.49	8.95	5.56	8.64	3.15	7.82
1.4	0.64	9.56	5.19	3.23	5.01	1.83	4.53
1.6	0.37	5.54	3.01	1.87	2.90	1.06	2.63
1.8	0.22	3.21	1.74	1.08	1.68	0.61	1.52
2.0	0.12	1.86	1.01	0.63	0.98	0.36	0.88
μ	6.10	130.17	79.06	62.15	160.60	24.10	84.86

Table 4 shows the actual decreasing values of σ for values of β for all variables. If we assume normality (i.e., $\beta = 0$), the largest variation is observed in BPS, whilst the lowest is in RBS.

3.3 Estimation of the Shape Parameter

We will estimate the shape parameter β for determining the distribution of variables by assessing the error of the EPD for some values of β as well as the Kolmogorov-Smirnov (KS) test for the best fit of the EPD for different values of β. Tables 5 presents the results for RBS and Height.

Table 5: Measurement of Error and KS Statistic for RBS and Height for Some Values of β

β	RBS			Height		
	MAD	RMSE	KS	MAD	RMSE	KS
-0.8	1.5537	12.0513	0.4069	0.6164	3.9331	0.1759
-0.6	1.4331	10.6506	0.3722	0.4580	3.3058	0.1057
-0.4	1.3001	9.1036	0.3229	0.4240	3.0332	0.0725
-0.2	1.1218	7.7386	0.2808	0.4184	2.9210	0.0654
0.0	0.9308	6.5419	0.2450	0.4174	2.8642	0.0597
0.2	0.7696	5.5646	0.2103	0.4175	2.8275	0.0541
0.4	0.6490	4.8898	0.1810	0.4170	2.7997	0.0485
0.6	0.5644	4.4902	0.1581	0.4139	2.7777	0.0466
0.8	0.5072	4.2851	0.1407	0.4095	2.7608	0.0472
1.0	0.4761	4.1997	0.1318	0.4081	2.7494	0.0481
1.2	0.4717	4.1826	0.1339	0.4133	2.7440	0.0493
1.4	0.4701	4.2029	0.1360	0.4216	2.7449	0.0523
1.6	0.4705	4.2431	0.1381	0.4294	2.7522	0.0549
1.8	0.4720	4.2933	0.1402	0.4366	2.7659	0.0571
2.0	0.4744	4.3483	0.1422	0.4427	2.7856	0.0588
Critical Value for KS = 0.0639						

From Table 5, for $\beta = 1.2$, the standard error or the root mean square error (RMSE) is 4.1826 for random blood sugar level. Again, the mean absolute deviation is 0.4717 for the same $\beta = 1.2$. These values are the smallest for all the β values. Thus, it can be suggested that the EPD for $\beta = 1.2$ is adequate to fit the distribution of RBS. From Table 5 again, it can be observed that, for RBS, all the β values recorded KS statistics which are greater than the critical value of 0.1339. Thus, it can be concluded that the distribution of RBS does not significantly follow the EPD. However, the error measures are consistently smallest for $\beta = 1.2$. Again, for the Height, the β values between 0 and 2 recorded KS statistics smaller than the critical value. Thus, the distribution of Height of respondents could significantly be fitted with the EPD of values for β between 0 and 2. However, $\beta = 1.2$ appears to produce a more consistent lowest error measures. The results for the remaining variables are placed in the Appendix. We observe that the EPD may be suitable for only three variables: Weight, Height and BMI.

Table 6: Optimal Value of β for fitting EPD

Variable	Optimal Value of β	Remarks
RBS	1.2	Not Significant
BPS	0.6	Not Significant
BPD	0.2	Not Significant
Weight	0.2	Significant
Height	1.2	Significant
BMI	0.2	Significant
WC	0.2	Not Significant

Table 6 gives a summary of optimal values of β for fitting the data on each variable with the EPD. It is indicated that for such value, the EPD is either significantly suitable or not suitable.

Table 7: Parameter Estimation from EasyFit and the Computation Method

Variables	Approach	q	β	μ	σ	RMSE
RBS	Computed	0.909	1.200	6.100	1.100	4.183
	EasyFit	1.000	1.000	6.050	2.739	5.123
BPS	Computed	1.250	0.600	103.170	84.340	2.545
	EasyFit	1.135	0.763	131.410	26.446	2.624
BPD	Computed	1.667	0.200	79.060	135.380	1.573
	EasyFit	1.228	0.629	78.503	14.140	1.760
Weight	Computed	1.667	0.200	62.150	84.100	1.226
	EasyFit	1.596	0.254	61.880	13.290	1.244
Height	Computed	0.909	1.200	160.600	8.640	2.744
	EasyFit	2.148	-0.069	160.690	8.588	2.881
BMI	Computed	1.667	0.200	24.100	47.630	1.956
	EasyFit	1.525	0.311	23.970	4.855	2.066
WC	Computed	1.667	0.200	84.860	118.230	2.251
	EasyFit	1.047	0.909	84.820	12.450	2.771

Table 7 presents the parameters derived by using EasyFit and the one derived by computational method. It also compares their standard errors to determine how well the two approaches best fit the datasets. It can be observed from the table that the estimated parameters for RBS are almost the same for both approaches, except for the scale parameter, σ. Thus, though the distribution from the two approaches may have the same shape and location, there will be differences in their scale. However, the standard error shows that the parameters from the computed fit will be suitable for RBS.

It is striking that estimates of the scale, σ, is much greater for the computed value than the Easyfit for almost all variables. This is an indication that EasyFit may not be sensitive to the shape in the estimate of the variation. However, the error associated with the two methods is either the same or slightly lower for computed fit. Estimates for β are distinctly different for all variables.

3.5 Graphical Fit of the Exponential Power Distribution to Data

We now present the graphical fit the distribution of the EPD using the parameters from EasyFit and that of the computed fit. We will also fit the Normal distribution ($\beta = 0$) and compare for all variables. Figures 9 presents the graphs for all seven variables.

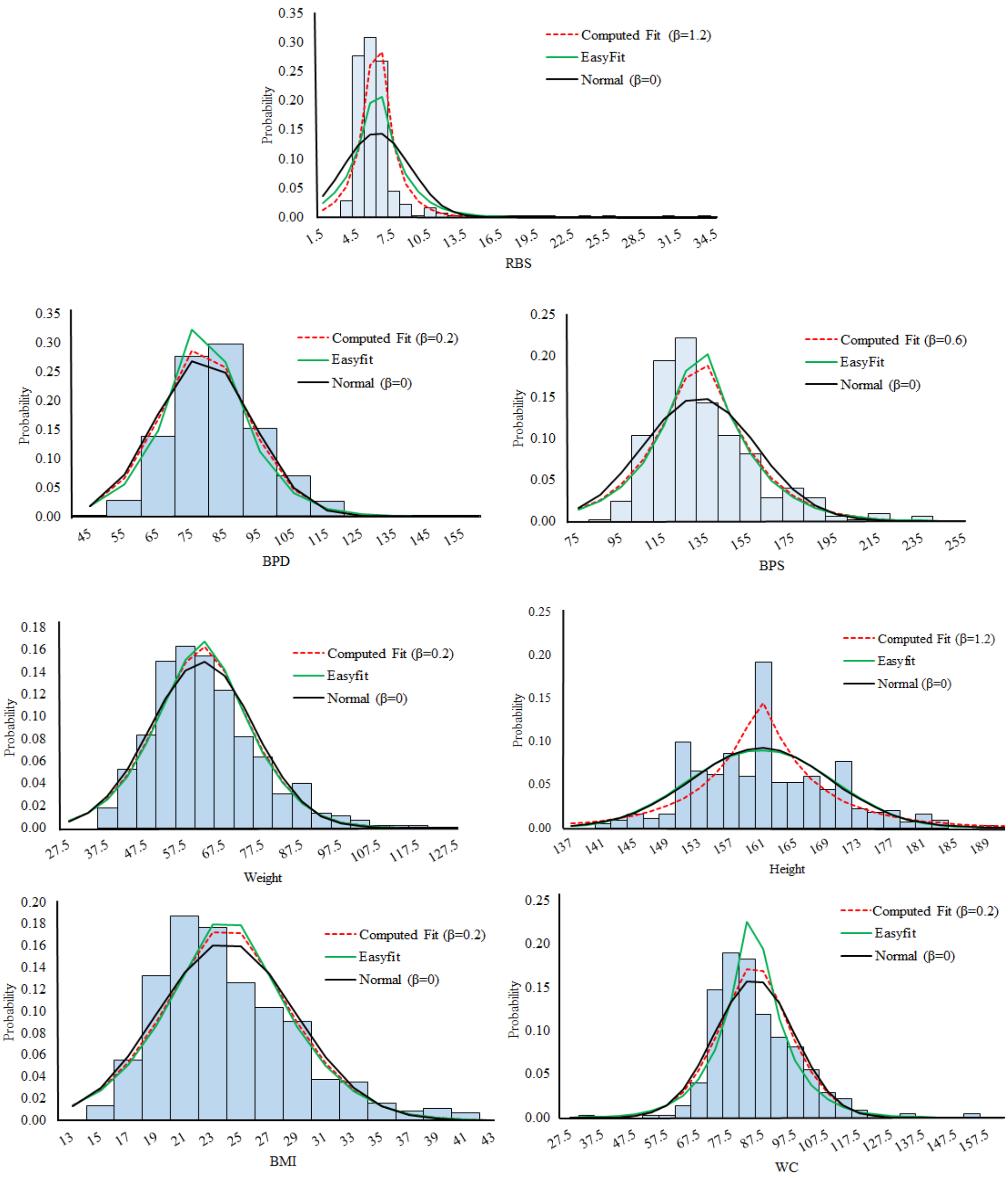


Figure 9: Distribution of the Variables and their Estimated Probability Density functions

In all the variables, the fits from the three approaches do not coincide. It is observed that the computed fit shows a better representation of the Kurtosis of the distribution and hence provides a more legitimate fit of the distribution. The use of the Normal fit could therefore introduce large errors in estimate of probabilities.

4 Conclusion

The study has examined extensively the EPD as given in Equation (9). In particular, the study has examined the characteristics of the EPD and its legitimacy, and assessed its deviation from normality using the kurtosis coefficient. The derivation and estimation of the parameters of the distribution using health data was carried out and the examination of the fitness of the EPD to the data is conducted.

The relationship of the normalizing constant to the gamma function makes the EPD illegitimate for some values of the shape parameter, β. Also, the infinitesimal nature of the scale parameter σ for large values of β made the EPD unreliable for such values. Thus, care needs to be taken in using the EPD as it does not exist for some point of β values. The study finds that the values of the shape parameter could hardly be greater than 2 for the data used.

The optimal value of β for fitting EPD to the data were found to be 0.2, 0.6, and 1.2 for the variables. The distribution is however found to fit significantly for only three variables: Weight, Height and BMI. It is deduced that these are variables with low coefficient of skewness, suggesting that the EPD would be suitable for non-skewed data.

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Appendix: Measurement of Error and KS Statistic for Variables Covered in Data

Measurement of Error and KS Statistic for BPD, BPS and WC for Some Values of β

β	BPD			BPS			WC		
	MAD	RMSE	KS	MAD	RMSE	KS	MAD	RMSE	KS
-0.8	0.9007	6.9318	0.2741	0.9516	6.0937	0.2743	0.9608	5.5515	0.2731
-0.6	0.5139	4.7399	0.1924	0.7056	4.7869	0.2031	0.6247	4.1059	0.1849
-0.4	0.3503	3.1641	0.1512	0.5128	3.8036	0.1430	0.4084	2.8819	0.1114
-0.2	0.2579	2.2626	0.1232	0.4179	3.1200	0.1082	0.3430	2.3105	0.0755
0.0	0.2107	1.7718	0.1037	0.3707	2.7241	0.0908	0.3102	2.1787	0.0748
0.2	0.2031	1.5733	0.0987	0.3683	2.5416	0.0824	0.2974	2.2510	0.0741
0.4	0.2151	1.5753	0.1031	0.3695	2.5002	0.0786	0.2978	2.4020	0.0734
0.6	0.2272	1.6885	0.1075	0.3729	2.5453	0.0774	0.3245	2.5765	0.0817
0.8	0.2393	1.8506	0.1118	0.3781	2.6399	0.0788	0.3508	2.7514	0.0894
1.0	0.2541	2.0289	0.1160	0.3917	2.7609	0.0812	0.3738	2.9177	0.0958
1.2	0.2771	2.2083	0.1200	0.4056	2.8946	0.0837	0.3946	3.0721	0.1014
1.4	0.2985	2.3825	0.1239	0.4191	3.0331	0.0862	0.4136	3.2142	0.1061
1.6	0.3189	2.5490	0.1276	0.4323	3.1718	0.0886	0.4324	3.3443	0.1103
1.8	0.3387	2.7072	0.1311	0.4451	3.3084	0.0910	0.4504	3.4635	0.1139
2.0	0.3574	2.8572	0.1346	0.4575	3.4415	0.0934	0.4671	3.5729	0.1171

Measurement of Error and KS Statistic for Weight and BMI for Some Values of β

β	Weight			BMI		
	MAD	RMSE	KS	MAD	RMSE	KS
-0.8	0.7076	4.3495	0.1991	0.7157	4.5239	0.1968
-0.6	0.4562	2.9275	0.1185	0.4584	3.2011	0.1238
-0.4	0.3201	2.0312	0.0739	0.3214	2.2892	0.0748
-0.2	0.4562	2.9275	0.1185	0.2809	1.8922	0.0629
0.0	0.1963	1.2673	0.0364	0.2637	1.8387	0.0626
0.2	0.1952	1.2257	0.0345	0.2582	1.9556	0.0623
0.4	0.2053	1.3213	0.0357	0.2822	2.1381	0.0681
0.6	0.2230	1.4879	0.0402	0.3102	2.3378	0.0763
0.8	0.2408	1.6841	0.0451	0.3347	2.5343	0.0833
1.0	0.2578	1.8891	0.0498	0.3564	2.7201	0.0892
1.2	0.2746	2.0931	0.0542	0.3770	2.8927	0.0942
1.4	0.2928	2.2917	0.0584	0.3956	3.0523	0.0985
1.6	0.3142	2.4829	0.0625	0.4125	3.1994	0.1023
1.8	0.3365	2.6658	0.0663	0.4287	3.3354	0.1056
2.0	0.3580	2.8405	0.0699	0.4458	3.4613	0.1084