Modelling and Numerical Simulation of Harvested Prey – Predator System Incorporating A Prey Refuge

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Abstract

prey-predator is defined as an interaction between the prey and predator in ecosystem. However, over-harvesting of wildlife resources is an important challenge facing protected area in Africa, a better understanding of the nature would improve the way in which it is managed.

This paper describes Modelling harvested prey–predator model incorporating a prey refuge in which a prey and predator species are affected by over-harvesting. The intention is to investigate the impacts of over-harvesting and make a possible suggestion on how to alleviate the problem. The results obtained from theoretical and numerical analysis of the prey-predator with harvesting showed that, overharvesting affect the prey-predator species negatively. However, the results obtained from numerical analysis of the prey-predator model with control strategies showed that catchibility coefficient and prey refuge has a great impact on both prey and predator species on their population densities.

Keywords: prey-predator system, harvesting, incorporating a prey refuge

1.Introduction

The prey-predator models has become one of the great interest to researchers in mathematics and ecology because they deal with number of factors in environmental problem, such as community morbidity and how to control it and optimal harvest policy to sustain a community (Sagamiko, 2015; A. B. Ashine, 2017.). Therefore, the developed mathematical model of prey-predator interaction of Lotka-Volterra model has motivated extensive study in the area of ecological modelling.

In dynamical system a definite activity done by individual area causes severe destruction to the ecosystem of that area. If such activity is unavoidable then the prevailing authority of the area should plan a regular policy which would keep the destruction of the ecosystems minimal (Kar, 2006). One of such activity is harvesting, which has a strong impact on the dynamic evolution of a population subjected to it however, it has been observed that over exploitation and over-harvesting of population species are commonly practiced in fishing, forestry and wildlife management which is done for the purpose of economic progress (Katsukawa, 2002). It is also agreed that biological species of prey–predator system is harvested unscientifically and exported with the aim of positive economic profit which regularly decreases the resources and eventually the ecosystems collapse.

(Ghosh, 2010; Kar, 2006) argued that using optimal harvesting efforts as controls can help discontinuities cyclic behaviour of the system of the prey-predator which may results to a required state of the ecosystem.

The study of the consequences of hiding behaviour of prey on the dynamics of predator-prey interactions can be recognized as a major issue in applied mathematics and theoretical ecology. However, prey refuge in Game reserve and National parks is mostly practiced by Wildebeest, Cape buffaloes that help them to protect from predator attack, hence reduces their predation rate. Therefore, under such situation it is expected that the addition of a small prey refuge stabilizes prey-predator interactions, the addition of a large refuge leads to almost changeability (i.e. random like prey population outbreak) (Li, 2013). Hence this study employed Holling Type II functional response on its model in which the rate of consumption of predator was assumed to depends on the availability of prey density as the only source of food.

2. Model and its Properties

In this section, we consider two different populations, the prey (x(t)) and predator (y(t)) interaction incorporating a prey refuge in which the model is formulated using deterministic differential equation and its stability analysis is done using Jacobian Matrix while simulation is done using MATLAB software

2.1 Model Assumptions

The ecological setup considers the following assumptions as follows;

(i) Both prey and predator are continuously harvested
(ii) Predator depend on the prey as its favourite food. Thus, in absence of f prey the predator goes to extinction
(iii) We also assumed that there is a refuge habitat where prey species are secured from predation and non-refuge habitat in which the prey are visible to predation
(iv) In absence of harvesting on both species, prey is assumed to grow logistically to the carrying capacity
(v) The rate of increase of the predator depends on the amount of biomass predator converts as food

Then from the above assumptions, we assume $x(t)$ and $y(t)$ represent the population density of prey and predator respectively at time t. with assumption we use Holling type (II) function response to formulate the pre-predator model as follows

$$\frac{dx}{dt} = r\left(1-\frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x \qquad (1)$$

$$\frac{dy}{dt} = -\mu y + \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_2h_2y$$

Where $x(t) > 0$ and $y(t) > 0$, also α, K, μ, a, b are all positive constants and r is the intrinsic growth rate of the prey. K is the carrying capacity of the prey in the absence of the predator and harvesting, the term $\frac{\alpha(1-p)xy}{1+\alpha(1-p)x}$ is the functional response of the predator which is a Holling type (II) response functional of the predator, μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator per unit time, b is the predators for each captured prey, q_1 and q_2 are catchibility coefficient of the prey and predator respectively. P is the proportion of prey population not exposed to predation, that it protects px and leaves $(1-p)x$ of the prey available to predation. Note that $p \in [0,1]$

3. Model analysis

3.1 Boundedness of the system

The solution of the prey-predator model developed in (1) represents the populations of living individuals and they have their ecological meaning that is to say they must be positive and bounded.

Lemma: All solutions of the system (1) which starts with $\mathcal{R}^{2+}$ are uniformly bounded.

Proof: To prove the theorem, we define a function

$$W(t) = x(t) + \frac{\alpha}{\alpha b} y(t) \tag{2}$$

which simplifies to

$$W(t) = x(t) + \frac{1}{b} y(t) \tag{3}$$

Where $W(t)$ represents total population of the prey and predator species, we differentiate equation (3) with respect to t above as;

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{b}\frac{dy}{dt} \tag{4}$$

Then substitute equation (1) into equation (4)

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1 h_1 x + \frac{1}{b}\left(-\mu y + \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_2 h_2 y\right) \tag{5}$$

Then equation (5) will be simplified as follows;

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1 h_1 x + \frac{1}{b}(-\mu - q_2 h_2)y + \frac{\alpha(1-p)xy}{1+a(1-p)x}$$

Then all terms of interspecific competition are cancelled out

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - q_1 h_1 x + \frac{1}{b}(-\mu - q_2 h_2)y$$

Also, on simplification we have

$$\frac{dW}{dt} = rx - \frac{rx^2}{K} - q_1 h_1 x + \frac{1}{b}(-\mu - q_2 h_2)y$$

We let $E_1 = q_1 h_1$ and $E_2 = q_2 h_2$

Then we have the simplified equation as follows $\frac{dW}{dt} = (r - E_1)\, x - \frac{r\,x^2}{K} - \frac{1}{b}(\mu + E_2)\, y$

Let the arbitrary constant to be Ω then the equation above will be written as follows

$$\frac{dW}{dt} = (r - E_1)\, x - \frac{r\,x^2}{K} - \frac{1}{b}(\mu + E_2)\, y + \Omega W(t) - \Omega W(t)$$

Thus;

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1)\, x - \frac{r\, x^2}{K} - \frac{1}{b}(\mu + E_2)\, y + \Omega\left(x(t) + \frac{1}{b} y(t)\right) \tag{6}$$

Using the concept of perfect square

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1 + \Omega)\, x - \frac{r\, x^2}{K} - \frac{1}{b}(\mu + E_2 - \Omega)\, y$$

Then it follows

$$\frac{dW}{dt} + \Omega W(t) \leq \frac{K}{4\, r}(r - E_1 + \Omega)^2 \; - \frac{r}{K}\left(x^2 - (r - E_2 + \Omega)\frac{K}{r}\right)^2 - \frac{1}{b}(\mu + E_2 + \Omega)\, y$$

$$\text{But } \frac{K}{4\,r}(r - E_1 + \Omega)^2 = \max\left[\frac{r}{K}\left(x^2 - (r - E_2 + \Omega)\frac{K}{r}\right)^2\right]$$

$$\text{Also letting the } \frac{K}{4\,r}(r - E_1 + \Omega)^2 = m_1$$

Thus

$$\frac{dW}{dt} + \Omega W(t) \leq m_1 \tag{7}$$

Solving equation (7) differential inequality using integrating factor $I = e^{\Omega t}$ yields

$$W(t) e^{\Omega t} \leq \frac{m_1}{\Omega} + C e^{-\Omega t} \tag{8}$$

At $t = 0$ equation in (8) becomes

$$W(0) = \frac{m_1}{\Omega} + \left(W(0) - \frac{m_1}{\Omega}\right) e^{-\Omega(0)} \tag{9}$$

As $t \to \infty$ (8)

$$0 \leq W(t) \leq \frac{m_1}{\Omega}$$

Therefore $W(t)$ is bounded and from positivity of x and y it follows

$$0 \leq x(t) \leq \frac{m_1}{\Omega}$$

and

$$0 \leq y(t) \leq \frac{m_1}{\Omega}$$

3.2 Analysis of the stability of the equilibrium points

In this section, we establish condition for the existence of equilibrium pons of the model equation (1) the system has at least four equilibrium points obtained by setting $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ by so doing we get the possible equilibrium points of the system as;

(i) $E_0(0,0)$ is the extinction of both species, prey and predator
(ii) $E_1(x, 0)$ is the predator extinction
(iii) $E_2(0, y)$ is the prey extinction
(iv) $E_3(x, y)$ the coexistence or equalibrium point of the system

But $E_0(0,0)$ point is trial. The existence of the rest of the fixed equilibrium points are described below

(i) The existence of $E_1(x^*, 0)$ with $x^* > 0$

Let $y = 0$ the system of equation reduces to

$$0 = r\left(1 - \frac{x^*}{K}\right)x^* - q_1h_1x^*$$

On simplifying we have

$$x^*\left(r - \frac{rx^*}{K} - q_1h_1\right) = 0$$

Thus $x^* = \frac{K(r-q_1h_1)}{r}$

Therefore $E_1(x^*, 0) = \left(\frac{K(r-q_1h_1)}{r}, 0\right)$

From the expression of x^* we observe that harvesting has negative impact on the prey growth hence affect the prey population density. However, for the predator free equilibrium point $E_1(x^*, 0)$ to exist $r - q_1h_1 > 0$ which implies $r > q_1h_1$. Therefore, in absence of predator the intrinsic growth rate of prey population should be greater than harvesting rate. Hence increasing harvesting of prey species results into decreasing of predator which affects survival of predator species. This is the fact prove that predator depends on the prey as their only source of food.

(ii) The existence of $E_2(0, y^*)$ with $y^* > 0$

Let $x = 0$ the system of equation (1) reduces to $y^*(-\mu - q_1h_1) = 0$ from which we obtain $y^* = 0$ which implies

$$E_2(0, y^*) = E_0(0,0) \tag{10}$$

The results above imply that the predator depend on prey as their only source of food. Thus, in absence of prey, predator populations become exist.

(iii) Co-existence of equilibrium point $E_3(x^*, y^*)$

We equate the equation (1) equals to zero that is to say $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ then the system reduces the following equations;

$$r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1 + a(1-p)x} - q_1h_1x = 0$$

$$-\mu y + \frac{\alpha(1-p)xy}{1 + a(1-p)x} - q_2h_2y = 0$$

Using MAPLE software, the co-existence point will be as;

$$x^* = \frac{\mu + H_2}{\big((\mu + H_2)a - \alpha b\big)(p-1)}$$

$$y^* = -\frac{b}{\big((\mu + H_2)a - \alpha b\big)(p-1)}\left(-r\left(\frac{\big((\mu + H_2)a - \alpha b\big)(p-1)K - \mu - H_2}{\big((\mu + H_2)a - \alpha b\big)(p-1)K}\right) + H_2\right)$$

For $H_1 = q_1h_1$ and $H_2 = q_2h_2$

Thus, the existence of the point

$$E_3(x^*, y^*) = \left(\frac{\mu+H_2}{\big((\mu+H_2)a-\alpha b\big)(p-1)}, -\frac{b}{\big((\mu+H_2)a-\alpha b\big)(p-1)}\left(-r\left(\frac{\big((\mu+H_2)a-\alpha b\big)(p-1)K-\mu-H_2}{\big((\mu+H_2)a-\alpha b\big)(p-1)K}\right) + H_2\right)\right) \qquad (11)$$

From the expression of $E_3(x^*, y^*)$ we observe that predators death rate and harvesting affect the convention factor b (predator biomass to the prey) of newly born predator negatively which in turn results into negative effects on predator population density. However, the co-existence equilibrium point (non -trivial) exist if $\big((\mu + H_2)a - \alpha b\big) > 0$ implying that $\frac{\alpha b}{a} < \mu + H_2$. Therefore, in the absence of both populations birth rate of predator should be greater than the sum of death rate and harvesting of predator. Increasing harvesting to predator population causes rapid decrease of predator which results in increasing of prey population density.

3.3 Stability analysis of the equilibrium points

The stability of the equilibrium points is analyzed by computing the Jacobian matrix and determining the eigenvalues of the Jacobian matrix of each fixed point $E_0(0,0)$, $E_1(x^*, 0)$, $E_2(0, y^*)$ and $E_3(x^*, y^*)$.The equilibrium points are asymptotically stable if the real parts of the eigenvalues of each jacobian matrix are negative. From the system equation (1) the general Jacobian matrix of the equations is given by;

$$J(E_i) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

This will be described as follows;

$$J(E_i) = \begin{pmatrix} r\left(1 - \frac{x^*}{K}\right) - \frac{r\,x^*}{K} - \frac{\alpha(1-p)^2x^*y^*a}{\big(1+a(1-p)\,x^*\big)^2} & -\frac{\alpha(1-p)\,x^*}{1+a(1-p)\,x^*} \\ \frac{b\alpha(1-p)y^*}{1+a(1-p)\,x^*} - \frac{\alpha(1-p)^2x^*y^*a}{\big(1+a(1-p)\,x^*\big)^2} & -\mu + \frac{b\alpha(1-p)\,x^*}{1+a(1-p)\,x^*} \end{pmatrix} \qquad (12)$$

Hence from the Jacobian matrix $J(E_i)$ above the equilibrium point;

(i) $\mathbf{E_0(0,0)}$ **is given by**

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -\mu \end{pmatrix}$$

Thus, using Maple software, the eigenvalues of the Jacobian matrix $J(E_0)$ are r and $-\mu$ However, $E_0(0,\ 0)$ is saddle point under condition that $r > 0$ and all saddles are unstable.

(ii) **For predator free equilibrium point** $\boldsymbol{E_1(x^*, 0) = \left(\frac{K(1-q_1h_1)}{r}, 0\right)}$

The corresponding matrix is written as

$$J(E_1) = \begin{pmatrix} 2q_1h_1 & -\frac{\alpha K(1-q_1K)(1-p)}{r+\alpha K(1-q_1K)(1-p)} \\ 0 & -\mu + \frac{\alpha b(r-q_1h_1)(1-p)}{r+\alpha K(r-q_1h_1)(1-p)} \end{pmatrix} \qquad (13)$$

Eigenvalues of $E_1(x^*,0)$ are $2q_1h_1$ and
$-\mu+\frac{\alpha b(r-q_1h_1)(1-p)}{r+\alpha K(r-q_1h_1)(1-p)}$ hence J is locally asymptotically stable if

$$\frac{\alpha b(r-q_1h_1)(1-p)}{r+\alpha K(r-q_1h_1)(1-p)}<\mu \quad (14)$$

(iii) The corresponding Jacobian matrix of the equilibrium point $\mathbf{E_2(0,y^*)}$

$$J(E_2) = \begin{pmatrix} r & 0 \\ 0 & -\mu \end{pmatrix} \quad (15)$$

Hence, we find that $E_0(0,0) = E_2(0,y^*)$ hence the eigen values for Jacobian matrix $J(E_2)$ are r and $-\mu$ where $r>0$ therefore the point at equilibrium $E_2(0,y^*)$ is unstable saddle.

(iv) For co-existence equilibrium point $\mathbf{E_3(x^*,y^*)}$

The jacobian matrix $J(\mathbf{E_3})$ is given by

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \quad (16)$$

Where

$$E_{11} = r\left(1-\frac{r}{K}\right)-\frac{r(H_2+\mu)}{G_2K}-\frac{G_1\alpha\big(1-p(G_4-G_3)\big)}{G_1+a(H_2+\mu)}+M \quad (17)$$

$$M=\frac{{G_2}^2\alpha(1-p)(H_2+\mu)(G_4-G_3)a}{\Big(G_1\big(G_2+a(1-p)(H_2+\mu)\big)\Big)^2}$$

Therefore, on simplification of equation (17)

$$r\left(1-\frac{r}{K}\right)-\frac{r(H_2+\mu)}{G_2K}-\frac{G_1Q}{G_1+a(H_2+\mu)}+\frac{{G_2}^2(H_2+\mu)Qa}{\Big(G_1\big(G_2+a(1-p)(H_2+\mu)\big)\Big)^2}$$

Where

$$Q=(G_4-G_3)\alpha(1-p)$$

$$G_1=a(H_2+\mu)-\alpha b$$

$$G_2=(1-p)[a(H_2+\mu)-\alpha b]$$

$$G_3=\frac{bH_1}{\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)}$$

$$G_4=\frac{br\Big(\Big(\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)\Big)K-H_2-\mu\Big)}{\big[\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)\big]^2K}$$

Again for

$$E_{12}=-\frac{\alpha(H_2+\mu)}{a\mu+aH_2+[a(H_2+\mu)-\alpha b]} \quad (18)$$

$$E_{21}=\frac{b\alpha(1-p)M}{1+G_5}-\frac{b\big(\alpha(1-p)\big)^2(\mu+H_2)Ma}{(1-p)[a(\mu+H_2)-\alpha b](aG_5+1)^2} \quad (19)$$

where

$$M = \frac{br\left(\left(\left(H_2(-1+p)+\mu(-1+p)\right)a(-1+p)-\alpha(-1+p)b(-1+p)\right)K - H_2 - \mu\right)}{\left[\left(H_2(-1+p)+\mu(-1+p)\right)a(-1+p)-\alpha(-1+p)b(-1+p)\right]^2 K} - D$$

$$G_5 = \frac{\mu + H_2}{[a(\mu + H_2) - \alpha b]}$$

And

$$D = \frac{bH_1}{\left((\mu + H_2) - \alpha b\right)}$$

$$E_{22} = -\mu + \frac{b\alpha(H_2 + \mu)}{2a(H_2 + \mu) - \alpha b} \qquad (20)$$

The stability of the $J(E_3)$ is stated using the characteristic of polynomial equation techniques using trace and determinant techniques proposition as follows

Preposition 3.1: suppose the jacobian matrix is evaluated at the co-existence equilibrium has characteristic polynomial equation

(21) $$\lambda^2 - \left(\text{trace}\left(J(E_3)\right)\right)\lambda + \text{determinant}\left(J(E_3)\right)=0$$

Such that trace $(J(\mathbf{E_3}))$ =E_{11}+E_{22} and determinant $\left(J(E_3)\right)$= $E_{11}E_{22}$-$E_{12}E_{21}$
The co-existence equilibrium point is locally stable or stable spiral if
$\text{trace}\left(J(E_3)\right) < 0$ and $\text{determinant}\left(J(E_3)\right) > 0$. Also, the interior equilibrium point is Centre (neutral stable) if $\text{trace}\left(J(E_3)\right) = 0$ and $\text{determinant}\left(J(E_3)\right) > 0$

4. Global stability of equilibrium point

Points $E_1$and E_2 is shown by linearizing the system of equation (1) and defining appropriate Lyapunov function to separately described each equilibrium point. The linearizing process is done using jacobian technique such that;

$$\frac{dX_i}{dt} = J(E_i)X_i \qquad (22)$$

Where $J(E_i)$ is the Jacobian Matrix and X_i is the small perturbation on x_i. Therefore, the system (1) reduces to the following linear system;

$$\frac{dX}{dt} = \left[r\left(1 - \frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1| + a(1-p)x^*)^2}\right]X - \left[\frac{\alpha(1-p)x^*}{(1 + a(1-p)x)}\right]Y \qquad (23)$$

$$\frac{dY}{dt} = \left[\frac{b\alpha(1-p)y^*}{1 + a(1-p)x^*} - \frac{\alpha b(1-p)^2 y^* x^* a}{(1| + a(1-p)x^*)^2}\right]X + \left[-\mu + \frac{\alpha(1-p)x^*}{(1 + a(1-p)x)}\right]Y$$

The Lyapunov function is chosen as

$$V(X, Y) = \frac{X^2}{2} + \frac{Y^2}{2} \qquad (24)$$

The function $V(X, Y)$ is positive definite function since $V(X, Y) \geq 0$ for any values of (X, Y) and it is minimum at the origin that is $V(0, 0) = 0$ the time derivative of $V(X, Y)$ is given by

$$\frac{dV(X,Y)}{dt} = \frac{\partial V}{\partial X}.\frac{dX}{dt} + \frac{\partial V}{\partial Y}.\frac{dY}{dt} \tag{25}$$

By substituting equation (23) and the partial V into (25) we obtain the relation below;

$$\frac{dV(X,Y)}{dt} = X\left[\left(r\left(1-\frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1| + a(1-p)x^*)^2}\right)X - \left(\frac{\alpha(1-p)x^*}{(1+a(1-p)x)}\right)Y\right] +$$

$$Y\left[\left(\frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{\alpha b(1-p)^2y^*x^*a}{(1|+a(1-p)x^*)^2}\right)X + \left(-\mu + \frac{\alpha(1-p)x^*}{(1+a(1-p)x)}\right)Y\right] \tag{26}$$

(i) For fixed $E_1(x^*, 0)$

We substitute the equation $E_1(x^*, 0) = \left(\frac{K(r-q_1h_1)}{r}, 0\right)$ into equation (26) above as follows

$$\frac{dV(X,Y)}{dt} = X^2(q_1h_1 - r) - \left(\frac{\alpha(1-p)(r-q_1h_1)}{1+\alpha K(1-p)(1-q_1h_1)}\right) \tag{27}$$

Therefore, from the equation (27) the equilibrium point $E_1(x^*, 0)$ is asymptotically stable if it satisfies the condition that

$$q_1h_1 - r < 0 \tag{28}$$

Thus, using simple algebraic mathematical manipulation results into $r > q_1h_1$
Hence in absence of the equilibrium point $E_1(x^*, 0)$ is globally stable if the intrinsic growth rate of the prey population is greater than the harvesting rate.

(ii) For steady state $E_3(x^*, y^*)$

Here, we substitute equation (11) into equation (26) to obtain

$$\frac{dV(X,Y)}{dt} = E_{11}X^2 + (E_{12} + E_{21})XY \tag{29}$$

With usual notation for E_{11}, E_{12} and E_{21} . Therefor the point is globally stable if the condition below holds

$$\frac{dV(X,Y)}{dt} = (E_{11}X^2 + (E_{12} + E_{21})XY) < 0 \tag{30}$$

5. Numerical Results and Simulation

Numerical simulation in this paper is done in two cases using MATLAB software. The two cases are phase diagram and variation of catchibility coefficient of prey and predator on harvesting rate. The corresponding parameter used in the developed model in equation (1) is described in table (1) below;

Table 1: The table of the corresponding parameters for developed model in equation (1) with their sources;

Parameter	Parameter Names	Parameter values
K	Carrying capacity of the prey	600 (Assumed)
R	Intrinsic growth rate of the prey	1
α		0.00000674
μ	Predator's death rate	0.01
p	Prey refuge	0.6 Chosen from $p \in [0, 1]$
q_1	Catchibility coefficient of prey	0.06
q_2	Catchibility Coefficient of predator	0.0375
h_1	Harvesting rate	2
h_2	Harvesting rate	4
B_1	Cost weight	100
B_2		200
A		1000
a		0.02

Case 1 phase diagram of the model in equation (1) after numerical simulation was

(i) Phase diagram for equilibrium point $E_1(x^*, c)$

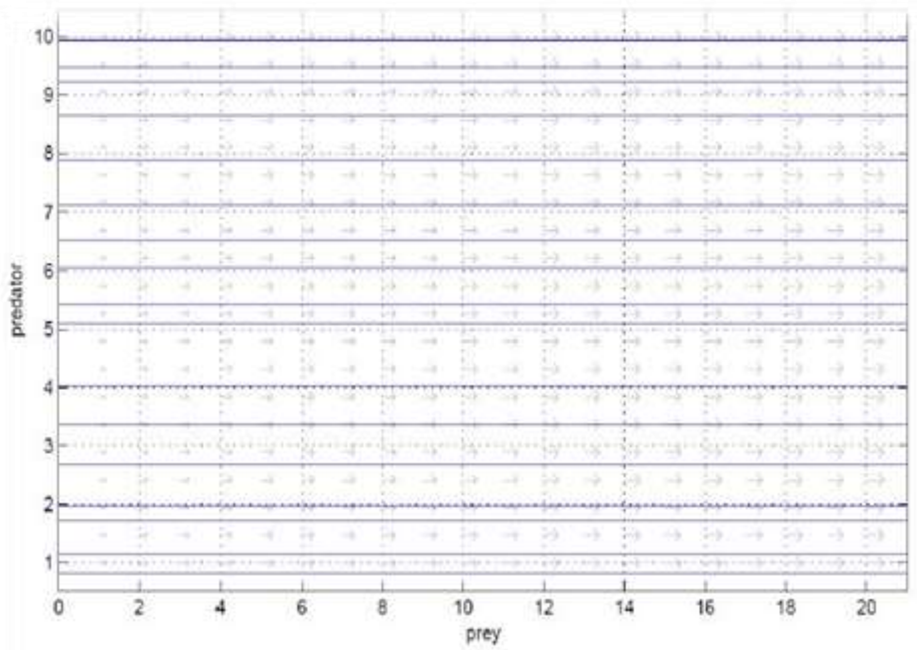


Figure 1: Phase diagram showing dynamic beaviour of $E_1(x^*, c)$

(ii) Phase diagram for equilibrium point $E_3(x^*, y^*)$

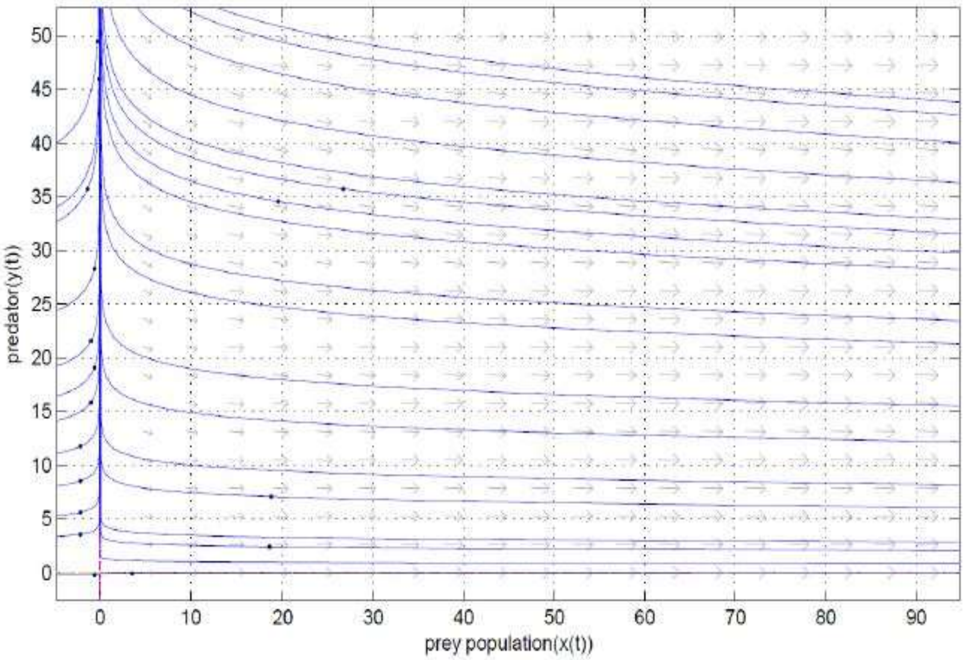


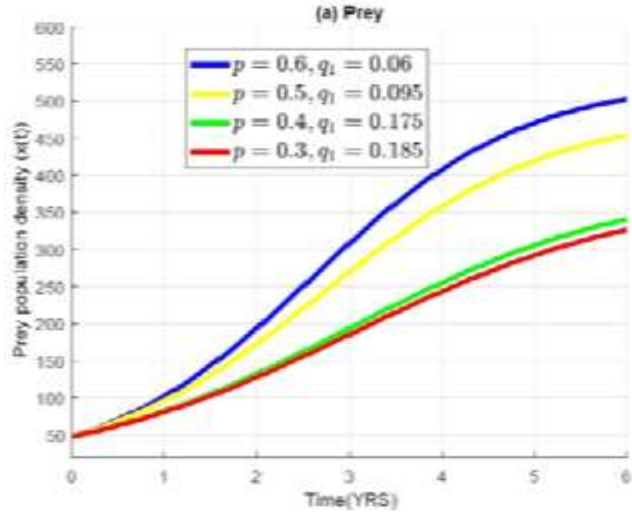
Figure 2: Phase diagram showing dynamical behaviour of $E_3(x^*, y^*)$

Figure (1) above indicate that in the absence of predator while presence of over-harvesting the dynamic equilibrium point of $E_1(x^*, c)$ is unstabe while the dynamic behaviour of co- existence equilibrium point $E_3(x^*, y^*)$ is spiral unstable surrounded by a stable convergence lines at point as shown in figure (2).

Case II: Effects of harvesting without any control strategy

In this section we present figures of harvesting prey and predator species without control using the parameter described in Table 1.

(i) The effect of varying catchibility coefficient on harvesting of prey with effect of prey refuge on prey population density;



igure 3: Showing Simulation of the harvested prey with variation of catchibility co-fiecient and prey refuge

Figure 3 illustrate that at a minimum prey refuge p and high catchibility coefficient q_1 the population density of prey decreases as we see in the figure 3 above. However the Red line shows the catchibility q_1 =0.185 and prey refuge p = 0.3 with only approximately 320 number of prey species , the Green line has catchibility coefficient q_1 = 0.175 and prey refuge 0.4 with approximately 350 number of prey species , the yellow line shows the catchibility coefficient q_1 =0.095 and p = 0.5 with approximately 450 number of prey species and blue line shows catchibility coefficient q_1 = 0.06 and prey refuge p= 0.6 with approximately 500 number prey species, while at maximum prey refuge and low catchibility coefficient q_1 the population density of prey increases. Therefore, from figure 3 we observed that the high the prey refuge and the lower the catchibility coefficient the greater the number of the prey species are saved as shown in the figure above thus we conclude that prey refuge and harvesting have a great impact on prey population density.

(ii) The effect of varying catchibility coefficient on harvesting of predator with effect on predator population density

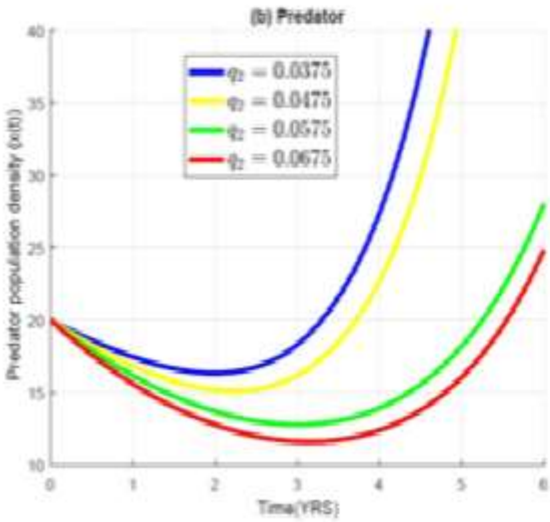


Figure 4: Showing Simulation of the harvested predator with variation of catchibility coeffiecient

Figure 4 illustrate that at a high catchibility coefficient q_2 the population density of predator decreases, while at low catchibility coefficient q_2 the population density of predator increases. Therefore, from figure 4 we observed that harvesting have a great impact on predator population density as we discussed in theoretically.

6. Discussion, Conclusion and Recommendation

In this paper, we presented Modelling and Numerical simulation of harvested prey predator model incorporating a prey refuge using a deterministic differential equation. The aim was to analyze the effect of harvested prey-predator species we observed that overharvesting, prey refuge and variation of catchibility coefficient of both prey and predator species has great impact on both species on their population growth.

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