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Exponentiated Exponential Lomax Distribution and its Properties

Maradesa Adeleke[1*], Akomolafe Abayomi Ayodele[2] and Adebola Femi Barnabas[3]

[1,2,3]Department of Statistics, Federal University of Technology Akure, Nigeria

Abstract

This research study the generalization of exponentiated version of Exponential lomax Distribution (ELD) called Exponentiated exponential lomax distribution (EEPD) through its distribution function and mathematical derivation of their moment, reliability, cumulative distribution function, Renyi Entropy and hazard rate function, Median, Quartile and Quantile Function. The distribution was found to generalize some known distributions thereby providing a great flexibility in modeling heavy tailed, skewed and bimodal distributions.

Keywords: Exponentiated-exponential Lomax Distribution (EELD), Moment generating function, Hazard Function, Entropy, Median, Quartile, Quantile Function**.**

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1 Overview of the research

Many researchers have worked on aspect of compounding two or more probability distributions to obtain family of hybrid distributions which are more efficient than their parent distributions due to addition of more parameters which increase the flexibility of the mixture of distributions in tracking many random phenomena which cannot be easily modeled by their parent distributions. Many authors have also worked on compounding beta distribution with other distributions. The beta family of distribution became popular some years back, which include beta-normal (Eugene &Famoye, 2002); beta-Gumbel (Nadarajah & Kotz, 2004), beta-Weibull (Famoye, Lee& Olugbenga, 2005), Beta-exponential (Nadarajah &Kotz, 2006); bata-Rayleigh (Akinsete &Lowe, 2009); beta-Laplace (Kozubowski Nadarajah, 2008) beta-pareto (Akinete, Faye & Lee, 200), Barreto-Souza, Santos, and Cordeiro (2009) constructed Beta generalized exponential, Beta-half-Cauchy was presented by Cordeiro, and Lemonte (2011), Gastellares, Montenegero, and Gauss derived Beta Log–normal **,** while Morais ,Cordeiro ,and Audrey (2011) introduced Beta Generalized Logistic, the Beta Burr III Model for Lifetime Data, Beta–hyperbolic Secant(BHS) by Mattheas, David(2007), Beta Fre´chet by Nadarajah, and Gupta(2004), beta-halfnormal, Akomolafe AA and Maradesa A (2017), beta-Gamma, beta-f, beta-t, beta-beta, beta-modified weibull, beta-nakagami among others. Some articles also evolved regarding exponential-pareto by Kareema Abdul Al-Kadim and Mohammed Abdulhussain (2013).Transmuted pareto, meronvci f. and Puka L. (2014), transmuted lomax, Ashour S.K and Eltehiwy M.A (2013), Exponential lomax, El-bassiouny et al (2015) and transmuted frechet , Mahmoud M.R and Mandouh R.M (2013). It is in this view that this research is structured to propose new hybrid distributions with a view to studying its properties and application to real life data to reflect the flexibility, stability and consistency of this hybrid model as compared to its parent distributions.

2 Derivation of Exponentiated Exponential Lomax Distribution (EEPD)

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The cdf of the exponentiated family of distribution according to Nadarajah and Kotz is defined as (1) below:

$$=F(x;\tau,\theta) = (1-(1-G(x;\tau))^{\theta} \ , \ \theta > 0 \quad (1)$$

To obtain the pdf $f(x;\tau,\alpha)$, obtain the derivative of (1)

$$f(x;\tau,\alpha) = \frac{d}{dx}\left[(1-(1-G(x;\tau))^{\theta})\right] = \frac{d}{dx}(1) - \frac{d}{dx}\left(1-G(x;\tau))^{\theta}\right) = -g\theta(x;\tau) \ .-(1-G(x;\tau))^{\theta-1}$$

$$=f(x;\tau,\alpha) = \theta g(x;\tau)(1-G(x;\tau))^{\theta-1} \quad (2)$$

Where

$g(x;\tau)$, $G(x;\tau)$ *and* τ *are pdf, cdf and parameter vector of parent distribution*

The cdf of Exponential lomax is defined according to El-bassiouny et al (2015) is given by (3).

$$=F(x;\beta,\lambda,\theta) = 1 - e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}} \quad (3)$$

$$=f(x;\beta,\lambda,\theta) = \frac{d}{dx}(1) - \frac{d}{dx}\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right) = \frac{\lambda\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}} \quad (4)$$

Where (3) and (4) are the cdf and pdf of exponential El-bassiouny et al (2015).

Now

let $g(x;\tau) = \frac{\lambda\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}$ *and* $G(x;\tau) = 1 - e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}$ *where* $\tau = (\beta,\lambda,\alpha)$

Substitute for $G(x;\tau) = 1 - e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}$ in (1) to obtain the cdf of Exponentiated exponential pareto distribution (EELD).

$$= F_{EELD}(x;\beta,\lambda,\theta,\alpha) = (1-(1-G(x;\tau))^{\theta} \ , \ \theta > 0$$

$$F_{EELD}(x;\beta,\lambda,\theta,\alpha) = (1-\left[(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta}\right] \ ,\alpha > 0,\theta > 0,\beta > 0,\lambda > 0 \ and \ x > 0 \quad (5)$$

The cumulative density function is represented by (5), the Exponentiated exponential lomax (EELD) has an increased number of parameter which can increase its flexibility.

By using equation (2) the pdf of EEPD can be obtained as (6) when substituting for $g(x;\tau) = \frac{\lambda\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}$ *and* $G(x;\tau) = 1 - e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}$ *where* $\tau = (\beta,\lambda,\alpha)$

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$$f_{(EELD)}(x;\beta,\lambda,\theta,\alpha) = \theta g(x;\tau)(1-G(x;\tau))^{\theta-1}$$

$$= f_{(EEPD)}(x;\beta,\lambda,\theta,\alpha) = \frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta-1} \quad \alpha > 0, \theta > 0, \beta > 0, \lambda > 0 \; and \; x > 0 \quad (6)$$

The equation (6) is the pdf of EEPD. The pdf of EELD has the unique property of reducing to the pdf of ELD when $\theta = 1$

2.1 Reliability

$$=R(x) = 1 - F(x;\beta,\lambda,\theta,\alpha) = 1 - \left[1-(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta}\right]$$

$$= (1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta} = (e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}$$

2.2 Hazard Rate Function

$$H(x) = \frac{f(x)}{R(x)} = \frac{\frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta-1}}{(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}}$$

$$= \frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta-1} \cdot \left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{-\theta}$$

$$\mathrm{H(x)} = \frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{1-\theta}(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta-1}$$

$$\mathrm{H(x)} = \frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{1-\theta} \cdot (e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta-1}$$

$$\mathrm{H(x)} = \frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}$$

2.3 Odd Function

$$O(x) = \frac{F(x;\alpha,\beta,\theta,\lambda)}{R(x)} = \frac{(\mathbf{1}-(\mathbf{1}-\left(\mathbf{1}-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{\theta})}{\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{\theta}} = \frac{(\mathbf{1}-(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta})}{\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{\theta}}$$

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$$= (1 - (e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}).\left(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}\right)^{-\theta}$$

2.4 Moment Generating Function

$$= M_x(t) = Ee^{tx} = \int_0^\infty e^{tx} f(x;\tau)\,dx = \sum_r^\infty \frac{t^r}{r!}\int_0^\infty x^r f(x;\tau)dx = \sum_r^\infty \frac{t^r}{r!} Ex^r \quad (7)$$

$$= Ex^r = \int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}))^{\theta-1}\,dx \quad (8)$$

$$= \int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}.(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta-1}dx$$

$$=\int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}}.e^{-\lambda(\theta-1)(\frac{\beta}{x+\beta})^{-\alpha}}dx \quad (9)$$

$$= \int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{(-\lambda-\theta\lambda+\lambda)(\frac{\beta}{x+\beta})^{-\alpha}}dx = \int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-\theta\lambda(\frac{\beta}{x+\beta})^{-\alpha}}dx \quad (10)$$

We can make use of the relations below and make necessary substitution in (10)

$$u = \theta\lambda(\frac{\beta}{x+\beta})^{-\alpha};\ \frac{u}{\theta\lambda} = (\frac{\beta}{x+\beta})^{-\alpha};\ (\frac{\beta}{x+\beta})^{\alpha} = \frac{\theta\lambda}{u};\ \frac{\beta}{x+\beta} = \frac{(\theta\lambda)^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}};\ \beta u^{\frac{1}{\alpha}} = (\theta\lambda)^{\frac{1}{\alpha}}(x+\beta)$$

$x = \frac{\beta u^{\frac{1}{\alpha}}}{(\theta\lambda)^{\frac{1}{\alpha}}} - \beta$; $\frac{dx}{du} = \frac{\beta u^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda)^{\frac{1}{\alpha}}}$. substitute for dx ,x and u in (10).

$$= \int_0^\infty x^r \frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x+\beta})^{-\alpha+1} e^{-u}.\frac{\beta u^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda)^{\frac{1}{\alpha}}}du = \frac{\lambda\theta}{(\lambda\theta)^{\frac{1}{\alpha}}}\int_0^\infty x^r \Big(\frac{\beta}{\frac{\beta u^{\frac{1}{\alpha}}}{(\theta\lambda)^{\frac{1}{\alpha}}}-\beta+\beta}\Big)^{-\alpha+1} e^{-u}.u^{\frac{1}{\alpha}-1}du$$

$$=\frac{\lambda\theta}{(\lambda\theta)^{\frac{1}{\alpha}}}\int_0^\infty x^r\Big(\frac{\beta(\theta\lambda)^{\frac{1}{\alpha}}}{\beta(u)^{\frac{1}{\alpha}}}\Big)^{-\alpha+1}.e^{-u}.u^{\frac{1}{\alpha}-1}du = \frac{\lambda\theta}{(\lambda\theta)^{\frac{1}{\alpha}}}\int_0^\infty x^r.\frac{(\theta\lambda)^{\frac{1}{\alpha}(-\alpha+1)}}{u^{\frac{1}{\alpha}(-\alpha+1)}}e^{-u}.u^{\frac{1}{\alpha}-1}du \quad (11)$$

$$= \frac{\lambda\theta}{(\lambda\theta)^{\frac{1}{\alpha}}}\int_0^\infty x^r.\frac{(\theta\lambda)^{-1+\frac{1}{\alpha}}}{u^{-1+\frac{1}{\alpha}}}e^{-u}.u^{\frac{1}{\alpha}-1}du = \frac{(\lambda\theta).(\theta\lambda)^{-1+\frac{1}{\alpha}}}{(\lambda\theta)^{\frac{1}{\alpha}}}\int_0^\infty x^r.\frac{1}{u^{-1+\frac{1}{\alpha}}}e^{-u}.u^{\frac{1}{\alpha}-1}du \quad (12)$$

$$= (\lambda\theta)^{1-\frac{1}{\alpha}}.(\theta\lambda)^{-1+\frac{1}{\alpha}}\int_0^\infty x^r.u^{-(-1+\frac{1}{\alpha})}e^{-u}.u^{\frac{1}{\alpha}-1}du \quad (13)$$

$$=\int_0^\infty x^r.u^{-(-1+\frac{1}{\alpha})}.u^{\frac{1}{\alpha}-1}e^{-u}du = \int_0^\infty x^r e^{-u}du \quad (14)$$

Now put for $x = \frac{\beta u^{\frac{1}{\alpha}}}{(\theta\lambda)^{\frac{1}{\alpha}}} - \beta$ in (14)

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$$=\int_0^{\infty}\left(\frac{\beta u^{\frac{1}{\alpha}}}{(\theta\lambda)^{\frac{1}{\alpha}}}-\beta\right)^r e^{-u}du = \sum_{j=0}^{r}\binom{r}{j}\left(\frac{\beta}{\sqrt[\alpha]{\theta\lambda}}\right)^j(-\beta)^{r-j}\int_0^{\infty}u^{\frac{j}{\alpha}}e^{-u}du \qquad (15)$$

$$=Ex^r = \sum_{j=0}^{r}\binom{r}{j}\left(\frac{\beta}{\sqrt[\alpha]{\theta\lambda}}\right)^j(-\beta)^{r-j}\,\Gamma\!\left(\frac{j}{\alpha}+1\right),\ r = 1,2,3\ldots \qquad (16)$$

Substitute for (16) in (7) to obtain the moment generating function of Exponentiated Exponential Lomax (EELD).

$$=\sum_{r}^{\infty}\frac{t^r}{r!}\sum_{j=0}^{r}\binom{r}{j}\left(\frac{\beta}{\sqrt[\alpha]{\theta\lambda}}\right)^j(-\beta)^{r-j}\,\Gamma\!\left(\frac{j}{\alpha}+1\right)\ \ r = 1,2,3 \qquad (17)$$

Let $W_r = \sum_{r}^{\infty}\frac{t^r}{r!}$, then (17) can be re-written as (18)

$$=M_x(t) = W_r\sum_{j=0}^{r}\binom{r}{j}\left(\frac{\beta}{\sqrt[\alpha]{\theta\lambda}}\right)^j(-\beta)^{r-j}\,\Gamma\!\left(\frac{j}{\alpha}+1\right) \qquad (18)$$

The equation (18) is the moment generating function of Exponential exponentiated lomax distribution (EELD) and it has many relations to the moment generating function of Exponential lomax distribution (ELD). This is because the mgf of EELD reduces to mgf of ELD when the parameter $\theta = 1$. It means the addition of another parameter θ made EELD four parameters-hybrid-distribution and the additional parameter gives it flexibility over ELD in modeling data with skewed distribution.

2.5 Renyi Entropy

The measure of the uncertain Situation for the random variable X with X~EELD($\lambda,\alpha,\beta,\theta$) can be calculated by the Renyi entropy (19) which can be expressed by the following relation.

$$=\delta_r = \frac{1}{1-r}\log\left(\int_0^{\infty}f^r(x)\,dx\right) \qquad (19)$$

2.5.1 Proposition 1

If X is a random variable that is distributed as EELD (X~EELD($\lambda,\alpha,\beta,\theta$)), then the measure of uncertain situation for the random variable X is given as: $\delta_r = \frac{1}{1-r}\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^r\frac{\beta}{\alpha}(\lambda\theta r)^{-r+\frac{r}{\alpha}-\frac{1}{\alpha}}\,\Gamma\!\left(\frac{(\alpha-1)(r-1)}{\alpha}+1\right)$

Proof

$$=\frac{1}{1-r}\log\int_0^{\infty}\left(\frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right)^r dx =$$

$$\frac{1}{1-r}\cdot\left(\frac{\alpha\lambda\theta}{\beta}\right)^r\log\int_0^{\infty}\left(\frac{\beta}{x+\beta}\right)^{r(-\alpha+1)}e^{-\theta\lambda r\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}dx \qquad (20)$$

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y = $\theta\lambda r(\frac{\beta}{x+\beta})^{-\alpha}$; $\frac{y}{\theta\lambda r} = (\frac{\beta}{x+\beta})^{-\alpha}$; $(\frac{\beta}{x+\beta})^{\alpha} = \frac{\theta\lambda r}{y}$; $\frac{\beta}{x+\beta} = \frac{(\theta\lambda r)^{\frac{1}{\alpha}}}{y^{\frac{1}{\alpha}}}$; $\beta y^{\frac{1}{\alpha}} = (\theta\lambda r)^{\frac{1}{\alpha}}(x+\beta)$

x = $\frac{\beta y^{\frac{1}{\alpha}}}{(\theta\lambda r)^{\frac{1}{\alpha}}} - \beta$; $\frac{dx}{dy} = \frac{\beta y^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}$, substitute for y, dx, and x in (20)

$$= \frac{1}{1-r}\cdot\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}\log\int_0^\infty \left(\frac{\beta}{\frac{\beta y^{\frac{1}{\alpha}}}{(\theta\lambda r)^{\frac{1}{\alpha}}}-\beta+\beta}\right)^{r(-\alpha+1)} e^{-y}\,\frac{\beta y^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}\,dy \qquad (21)$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}\int_0^\infty \left(\frac{(\theta\lambda r)^{\frac{1}{\alpha}}}{y^{\frac{1}{\alpha}}}\right)^{r(-\alpha+1)}.e^{-y}\,y^{\frac{1}{\alpha}-1}dy \qquad (22)$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}\int_0^\infty \frac{(\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)}}{y^{\frac{r}{\alpha}(-\alpha+1)}}e^{-y}\,y^{\frac{1}{\alpha}-1}dy \qquad (23)$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}.(\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)}\int_0^\infty \frac{1}{y^{\frac{r}{\alpha}(-\alpha+1)}}e^{-y}\,y^{\frac{1}{\alpha}-1}dy \qquad (24)$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha(\theta\lambda r)^{\frac{1}{\alpha}}}.(\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)}\int_0^\infty y^{-\frac{r}{\alpha}(-\alpha+1)}\,e^{-y}\,y^{\frac{1}{\alpha}-1}dy \qquad (25)$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha}.(\theta\lambda r)^{-\frac{1}{\alpha}}(\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)}\int_0^\infty y^{r-\frac{r}{\alpha}+\frac{1}{\alpha}-1}\,e^{-y}dy$$

$$= \frac{1}{1-r}.\log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r}.\ \frac{\beta}{\alpha}.(\theta\lambda r)^{-\frac{1}{\alpha}}(\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)}.M \qquad (26)$$

M = $\int_0^\infty y^{r-\frac{r}{\alpha}+\frac{1}{\alpha}-1}\,e^{-y}dy$

u = $y^{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}$; dy = $\frac{y^{\frac{r}{\alpha}-r-\frac{1}{\alpha}+1}}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}\ du$ and we can used y = $u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}}$

$$= -\frac{1}{\frac{r}{\alpha}-r-\frac{1}{\alpha}}\int_0^\infty e^{-u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}}}\,du. \qquad (27)$$

$$T = \int_0^\infty e^{-u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}}}\,du \qquad (28)$$

Equation (28)is a special integral of Incomplete gamma function

T= $-\left(-\frac{r}{\alpha}+r+\frac{1}{\alpha}\right)\Gamma(-\frac{r}{\alpha}+r+\frac{1}{\alpha}, u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}})$

Substitute for T in (27) to obtain (29)

$$= -\frac{1}{\frac{r}{\alpha}-r-\frac{1}{\alpha}} \cdot \left(-\frac{r}{\alpha}+r+\frac{1}{\alpha}\right) \Gamma\left(-\frac{r}{\alpha}+r+\frac{1}{\alpha}, u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}}\right) \quad (29)$$

Recall that y = $u^{\frac{1}{-\frac{r}{\alpha}+r+\frac{1}{\alpha}}}$, then

$$= \frac{1}{\frac{r}{\alpha}-r-\frac{1}{\alpha}} \cdot \left(-\frac{r}{\alpha}+r+\frac{1}{\alpha}\right) \Gamma\left(-\frac{r}{\alpha}+r+\frac{1}{\alpha}, y\right) = -\Gamma\left(\frac{(\alpha-1)r+1}{\alpha}, y\right) \quad (30)$$

Equation (30) can be rewritten as $-\Gamma\left(\frac{(\alpha-1)(r-1)}{\alpha}+1, y\right)$

Therefore M= $\Gamma\left(\frac{(\alpha-1)(r-1)}{\alpha}+1\right)$. We can substitute for M in (26) to obtain the proposed renyi entropy for EEPD.

$$= \frac{1}{1-r} . \log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r} . \frac{\beta}{\alpha} . (\theta\lambda r)^{-\frac{1}{\alpha}} (\theta\lambda r)^{\frac{r}{\alpha}(-\alpha+1)} \Gamma\left(\frac{(\alpha-1)(r-1)}{\alpha}+1\right)$$

$$\delta_r = \frac{1}{1-r} . \log\left(\frac{\alpha\lambda\theta}{\beta}\right)^{r} . \frac{\beta}{\alpha} . (\theta\lambda r)^{-r+\frac{r}{\alpha}-\frac{1}{\alpha}} \Gamma\left(\frac{(\alpha-1)(r-1)}{\alpha}+1\right) \quad (31) .$$

The (31) above is the proposed renyi entropy for EELD.

2.6 β-Entropy

β is defined as one parameter generalization of the Shannon entropy. β-entropy can be defined as:

$$= H_{\bar{\beta}} = \frac{1}{\bar{\beta}-1}\left[1 - \int_0^{\infty} f^{\bar{\beta}}(x)dx\right] \quad \text{for } \bar{\beta} \neq 1 \quad (32)$$

2.6.1 Proposition 2

Let X be a random variable that is distributed as X~EELD ($\alpha, \beta, \theta, \lambda$), then its one parameter generalization of the Shannon entropy (β-entropy) is defined as:

$$H_{\bar{\beta}} = \frac{\beta}{(\bar{\beta}-1)\alpha}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} (\theta\lambda\bar{\beta})^{-\bar{\beta}+\frac{\bar{\beta}}{\alpha}-\frac{1}{\alpha}} \Gamma\left(\frac{(\alpha-1)(\bar{\beta}-1)}{\alpha}+1\right)$$

Proof

$$= H_{\bar{\beta}} = \frac{1}{\bar{\beta}-1}\int_0^{\infty}\left(\frac{\lambda\theta\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\theta\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right)^{\bar{\beta}} dx =$$

$$\frac{1}{\bar{\beta}-1}.\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} \int_0^{\infty}\left(\frac{\beta}{x+\beta}\right)^{\bar{\beta}(-\alpha+1)} e^{-\theta\lambda\bar{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} dx \quad (33)$$

$$y = \theta\lambda\bar{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha};\ \frac{y}{\theta\lambda\bar{\beta}} = \left(\frac{\beta}{x+\beta}\right)^{-\alpha};\ \left(\frac{\beta}{x+\beta}\right)^{\alpha} = \frac{\theta\lambda\bar{\beta}}{y};\ \frac{\beta}{x+\beta} = \frac{(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}{y^{\frac{1}{\alpha}}};\ \beta y^{\frac{1}{\alpha}} = (\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}(x+\beta)$$

$x = \frac{\beta y^{\frac{1}{\alpha}}}{(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}} - \beta\ ;\ \frac{dx}{dy} = \frac{\beta y^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}$, substitute for y, dx, and x in (33)

$$= \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} \int_0^{\infty}\left(\frac{\beta}{\frac{\beta y^{\frac{1}{\alpha}}}{(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}-\beta+\beta}\right)^{\bar{\beta}(-\alpha+1)} e^{-y}\ \frac{\beta y^{\frac{1}{\alpha}-1}}{\alpha(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}\ dy \quad (34)$$

$$= \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} \frac{\beta}{\alpha(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}\int_0^{\infty}\left(\frac{(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}{y^{\frac{1}{\alpha}}}\right)^{\bar{\beta}(-\alpha+1)} .\, e^{-y}\ y^{\frac{1}{\alpha}-1}dy \quad (35)$$

$$= \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} (\theta\lambda\bar{\beta})^{\frac{\bar{\beta}}{\alpha}(-\alpha+1)} \frac{\beta}{\alpha(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}\int_0^{\infty}\frac{1}{y^{\frac{\bar{\beta}}{\alpha}(-\alpha+1)}} e^{-y}\ y^{\frac{1}{\alpha}-1}dy \quad (36)$$

$$= \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} (\theta\lambda\bar{\beta})^{\frac{\bar{\beta}}{\alpha}(-\alpha+1)} \frac{\beta}{\alpha(\theta\lambda\bar{\beta})^{\frac{1}{\alpha}}}\int_0^{\infty} y^{-\frac{r}{\alpha}(-\alpha+1)} e^{-y}\ y^{\frac{1}{\alpha}-1}dy \quad (37)$$

$$= \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} \frac{\beta}{\alpha}(\theta\lambda\bar{\beta})^{\frac{\bar{\beta}}{\alpha}(-\alpha+1)}.(\theta\lambda\bar{\beta})^{-\frac{1}{\alpha}}\int_0^{\infty} y^{-\frac{r}{\alpha}(-\alpha+1)} e^{-y}\ y^{\frac{1}{\alpha}-1}dy \quad (38)$$

$$H_{\bar{\beta}} = \frac{1}{\bar{\beta}-1}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} \frac{\beta}{\alpha}(\theta\lambda\bar{\beta})^{-\bar{\beta}+\frac{\bar{\beta}}{\alpha}-\frac{1}{\alpha}}\,\Gamma\left(\frac{(\alpha-1)(\bar{\beta}-1)}{\alpha}+1\right) \quad (39)$$

$$H_{\bar{\beta}} = \frac{\beta}{(\bar{\beta}-1)\alpha}\left(\frac{\lambda\theta\alpha}{\beta}\right)^{\bar{\beta}} (\theta\lambda\bar{\beta})^{-\bar{\beta}+\frac{\bar{\beta}}{\alpha}-\frac{1}{\alpha}}\,\Gamma\left(\frac{(\alpha-1)(\bar{\beta}-1)}{\alpha}+1\right) \quad (40)$$

Therefore the β −entropy is given in equation (40)

2.7 Median

$$0.5 = \int_0^{\infty} f(x;\tau)\,dx = 1-\left(1-\left(1-e^{-\lambda\left(\frac{\beta}{m+\beta}\right)^{-\alpha}}\right)^{\theta}\right.$$

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-0.5 = $-(1-(1-e^{-\lambda(\frac{\beta}{m+\beta})^{-\alpha}})^{\theta}$ = 0.5 = $(e^{-\lambda(\frac{\beta}{m+\beta})^{-\alpha}})^{\theta}$

0.5 = $e^{-\lambda\theta(\frac{\beta}{m+\beta})^{-\alpha}}$, therefore

ln 0.5 = $-\lambda\theta(\frac{\beta}{m+\beta})^{-\alpha}$ = -0.693 = $-\lambda\theta(\frac{\beta}{m+\beta})^{-\alpha}$

= 0.693 = $\lambda\theta(\frac{\beta}{m+\beta})^{-\alpha}$, then $\frac{0.693}{\lambda\theta} = \frac{(m+\beta)^{\alpha}}{\beta^{\alpha}}$

$=\beta^{\alpha}.\frac{0.693}{\lambda\theta}$ = $(m+\beta)^{\alpha}$, which can be simplify as : $m+\beta = \beta.\left(\frac{0.693}{\lambda\theta}\right)^{\frac{1}{\alpha}}$

= $m = \beta\sqrt[\alpha]{\frac{0.693}{\lambda\theta}} - \beta$

Therefore the median of EEPD is $m = \beta\left(\sqrt[\alpha]{\frac{0.693}{\lambda\theta}} - 1\right)$

2.8 Quartiles

2.8.1 First Quartile

0.25 = $\int_0^{q1} f(x;\tau)\,dx = 1-(1-(1-e^{-\lambda(\frac{\beta}{q1+\beta})^{-\alpha}})^{\theta}$

-0.75 = $-(1-(1-e^{-\lambda(\frac{\beta}{q1+\beta})^{-\alpha}})^{\theta}$ = 0.75 = $(e^{-\lambda(\frac{\beta}{q1+\beta})^{-\alpha}})^{\theta}$

0.75 = $e^{-\lambda\theta(\frac{\beta}{q1+\beta})^{-\alpha}}$, therefore

ln 0.75 = $-\lambda\theta(\frac{\beta}{q1+\beta})^{-\alpha}$ = -0.2877 = $-\lambda\theta(\frac{\beta}{q1+\beta})^{-\alpha}$

= 0.2877 = $\lambda\theta(\frac{\beta}{q1+\beta})^{-\alpha}$, then $\frac{0.693}{\lambda\theta} = \frac{(q1+\beta)^{\alpha}}{\beta^{\alpha}}$

$=\beta^{\alpha}.\frac{0.2877}{\lambda\theta}$ = $(q1+\beta)^{\alpha}$, which can be simplify as : $q1+\beta = \beta.\left(\frac{0.693}{\lambda\theta}\right)^{\frac{1}{\alpha}}$

= $q1 = \beta\sqrt[\alpha]{\frac{0.693}{\lambda\theta}} - \beta$

Therefore the first quartile of EEPD is $q1 = \beta\left(\sqrt[\alpha]{\frac{0.2877}{\lambda\theta}} - 1\right)$

2.8.2 Third Quartile

0.75 = $\int_0^{q3} f(x;\tau)\,dx \;= 1-(1-(1-e^{-\lambda(\frac{\beta}{q3+\beta})^{-\alpha}})^{\theta}$

-0.25 = $-(1-(1-e^{-\lambda(\frac{\beta}{q3+\beta})^{-\alpha}})^{\theta}$ = 0.25 = $(e^{-\lambda(\frac{\beta}{q3+\beta})^{-\alpha}})^{\theta}$

0.25 = $e^{-\lambda\theta(\frac{\beta}{q3+\beta})^{-\alpha}}$, therefore

ln 0.25 = $-\lambda\theta(\frac{\beta}{q3+\beta})^{-\alpha}$ = -0.1.3863 = $-\lambda\theta(\frac{\beta}{q3+\beta})^{-\alpha}$

= 1.3863 = $\lambda\theta(\frac{\beta}{q3+\beta})^{-\alpha}$, then $\frac{1.3863}{\lambda\theta} = \frac{(q3+\beta)^{\alpha}}{\beta^{\alpha}}$

=$\beta^{\alpha}.\frac{1.3863}{\lambda\theta}$ = $(q3+\beta)^{\alpha}$, which can be simplify as : $q3+\beta = \beta\,.\left(\frac{1.3863}{\lambda\theta}\right)^{\frac{1}{\alpha}}$

=$q3 \;=\; \beta\sqrt[\alpha]{\frac{1.3863}{\lambda\theta}} \;-\; \beta$

Therefore the first quartile of EEPD is $q3 \;=\; \beta\left(\sqrt[\alpha]{\frac{1.3863}{\lambda\theta}} \;-\; 1\right)$

2.9 Quantile Function

$$F(x) = 1-(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}$$

If we let u = F(x), then $u \;=\; 1-(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}$

u-1 = $-(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta} = 1-u = \;(1-(1-e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta}$

1-u = $(e^{-\lambda(\frac{\beta}{x+\beta})^{-\alpha}})^{\theta} = 1-u = \; e^{-\lambda\theta(\frac{\beta}{x+\beta})^{-\alpha}}$

ln (1-u) = $-\lambda\theta(\frac{\beta}{x+\beta})^{-\alpha}$

= -ln(1-u) = $\lambda\theta(\frac{\beta}{x+\beta})^{-\alpha} = \ln(1-u)^{-1} \;=\; \lambda\theta(\frac{\beta}{x+\beta})^{-\alpha} = \ln\frac{1}{(1-u)} \;=\; \lambda\theta(\frac{\beta}{x+\beta})^{-\alpha}$

=$\ln\frac{1}{(1-u)} \;=\; \frac{\lambda\theta(x+\beta)^{\alpha}}{\beta^{\alpha}} \;=\; \frac{\beta^{\alpha}}{\lambda\theta}\,.\ln\frac{1}{(1-u)} \;=\; (x+\beta)^{\alpha} = \frac{\beta}{(\lambda\theta)^{\frac{1}{\alpha}}}\left(\ln\frac{1}{(1-u)}\right)^{\frac{1}{\alpha}} \;=\; x+\beta$

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x = $\sqrt[\alpha]{\ln\frac{1}{(1-u)}\cdot\frac{1}{\lambda\theta}}\ \beta - \beta = \beta\left(\sqrt[\alpha]{\ln\frac{1}{(1-u)}\cdot\frac{1}{\lambda\theta}} - 1\right)$

Therefore the quantile function of EELD is represented by x = $\beta\left(\sqrt[\alpha]{\ln\frac{1}{(1-u)}\cdot\frac{1}{\lambda\theta}} - 1\right)$

3 Maximum Likelihood Estimation (MLE)

Lf(x;$\alpha,\beta,\theta,\lambda$) = $\prod_{i=1}^{n}\frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x_i+\beta})^{-\alpha+1}\, e^{-\lambda(\frac{\beta}{x_i+\beta})^{-\alpha}}(1-(1-e^{-\lambda(\frac{\beta}{x_i+\beta})^{-\alpha}}))^{\theta-1}$ (41)

Lf(x;$\alpha,\beta,\theta,\lambda$) = $\prod_{i=1}^{n}\frac{\lambda\theta\alpha}{\beta}(\frac{\beta}{x_i+\beta})^{-\alpha+1}\, e^{-\theta\lambda(\frac{\beta}{x_i+\beta})^{-\alpha}}$ (42)

ln Lf(x;$\alpha,\beta,\theta,\lambda$) = nln $\alpha + nln\lambda + \text{nln}\theta - nln\beta + (-\alpha+1)\sum_{i=1}^{n}\ln\left(\frac{\beta}{x_i+\beta}\right) - \lambda\theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}$

(43)

Now, obtain the partial derivative of (43) with respect to each parameter

= $\frac{\partial lnLf(x;\alpha,\beta,\theta,\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}$ = 0 (44)

= $\frac{\partial lnLf(x;\alpha,\beta,\theta,\lambda)}{\partial\theta} = \frac{n}{\theta} - \lambda\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}$ = 0 (45)

= $\frac{\partial lnLf(x;\alpha,\beta,\theta,\lambda)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\ln\left(\frac{\beta}{x_i+\beta}\right) + \lambda\theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}\ln\left(\frac{\beta}{x_i+\beta}\right)$ = 0 (46)

= $\frac{\partial lnLf(x;\alpha,\beta,\theta,\lambda)}{\partial\beta} = \frac{-n}{\beta} + (-\alpha+1)\frac{\partial}{\partial\beta}\sum_{i=1}^{n}\ln\left(\frac{\beta}{x_i+\beta}\right) - \lambda\theta\frac{\partial}{\partial\beta}\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} = 0$ (47)

We can obtain the estimate of the parameter through numerical method, specifically Newton Raphson Algorithm.

From (44), $\frac{n}{\lambda} - \theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} = 0$; $\frac{n}{\lambda} = \theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}$

= $\hat{\lambda} = \frac{n}{\theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}$ and $\hat{\theta}$ can be obtained from (45) as $\hat{\theta} = \frac{n}{\lambda\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}$

We can obtain $\hat{\lambda}, \hat{\theta}, \hat{\beta}$ *and* $\hat{\alpha}$ through numerical method and the fisher information matrix of Newton Raphson can be used to test for goodness of fit EELD over ELD. The numerical implementation of this hybrid distribution will be done in our subsequent research.

Conclusions

By compounding two or more probability distributions, we get the corresponding hybrid distribution with increased number of parameters which is believed to give the newly compounded distribution more flexibility, consistency, stability, sufficiency, uniqueness and wider applicability as compare to its parent distribution. Therefore, Exponentiated-Exponential-Lomax distribution is said to have vast applicability in modeling statistical behavior of stochastic processes such as the growth of tumors' cells in an oncology's study of benign transmogrifying to malignant tumor, studying the consumers' buying behavior and in complex epidemiological studies because of its increased number of parameters which give the hybrid distribution more flexibility to model many stochastic phenomena. The hybrid distribution normally has different patterns or shapes when fitted to simulated data that has well established distribution. It can be used to model data with highly skewed distribution, bimodal or multimodal density. For this reason, it tends to captures non normal data reasonably well; it can be used to model heavily skewed distribution.

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