

A Note on Derivation of Minimal set of Compatibility Classes for Covering

[1]T.O William-west
Department of Mathematics, Ahmadu Bello University
PO box 770, Zaria, Nigeria
Tel: +234-7031122411, E-mail: westtamuno@gmail.com

[2]Yahaya Shagaiya Daniel. (Corresponding author)
Department of Mathematics, Kaduna State University
Kaduna, Nigeria
Tel: +234-8080344792, E-mail: shagaiya3@yahoo.com

Abstract

This paper briefly describes the concept of compatibility relation defined on a finite set S and thereby, that of maximal compatibility classes. For a given compatibility relation defined on S, not all the maximal compatibles are needed to ensure covering for S. A technique to derive a minimal set of maximal compatibility classes which covers S is proposed and some results obtained.

Keywords: Compatibility relation, Maximal compatibility classes, Minimal covering

1. Introduction

A lot of literatures describe the concept of *covering* of a finite set S (see [4], [2], and the references there). Usually, this concept is concerned with selection of as *few* subsets of S as possible whose union covers S. On consequent to this, the notion of *minimal covering* of S has been investigated [3] and its application [1] is being considered. In [1] an application of this notion to *social choice* is described, where the elements of S are viewed as socially preferred candidates. Also, S may be viewed as set of alternatives. This paper considers S as a finite set endowed with a *compatibility relation* R, and the *covering* of S induced by R.

Further, a technique to derive *minimal* set of *maximal compatibles* for *covering* a finite set S endowed with a *compatibility relation* is proposed and some new results obtained.

2. Notation and definitions

Let S be a set with n elements, usually denoted n- set. A family $\{A_1, A_2, \dots, A_n\}$ of non- empty subsets of S is called a *covering* of S if $S = \bigcup_{k=1}^{n} A_k$. For example, the set $S_1 = \{1\}$ has only a single cover, namely,

$\{\{1\}\}$. However, there are five covers of the set $S_2 = \{1, 2\}$, namely, $\{\{1\}, \{2\}\}$, $\{\{1, 2\}\}$, $\{\{1\}, \{1, 2\}\}$, $\{\{2\}, \{1, 2\}\}$ and $\{\{1\}, \{2\}, \{1, 2\}\}$. Note that A_k's are not necessarily disjoint and hence, it may not define a partition.

A cover of a finite set S is called *minimal* if none of its proper subclasses covers S. A *minimum* covering for S is a covering of S with the least cardinality.

A relation on a set S which is reflexive and symmetric is called a *compatibility* relation, sometimes denoted $\approx$. Also, if R be a compatibility relation on a finite set S, then $x, y \in S$ are called compatible if xRy. Note that a compatibility relation not being necessarily transitive, may not define a partition. However, it does define a covering ([5], for details). Essentially, a compatibility relation defined on a finite set decomposes the set into its possibly pairwise non-disjoint subclasses, henceforth called *compatibility classes* (CC*)*. It follows that the elements of a compatibility class are pairwise compatible.

Let S be an n- set and R a compatibility relation on S. A subclass $M \subseteq S$ is called a *maximal compatibility class* (MCC) if any element of M is compatible to its every other element and no other element of $S - M$ is compatible to *all* the elements of M. Equivalently, a compatibility class of S is maximal if it is not a proper subclass of any other compatibility class of S. Graphically, maximal compatibility classes for a given compatibility relation R can also be viewed as the largest *complete* polygons in the graph of R. Thus a triangle is always a complete polygon and, for a quadrilateral to be a complete polygon, we need both the diagonals. Also, any element of the set that relates only to itself is a maximal compatibility block, and any two elements of S which are compatible to one another but to no other elements of S form a maximal compatibility class.

Example1

Consider the set $S = \{x_1, x_2, \dots, x_5\} = \{\{a, b\}, \{b, c, d\}, \{c, d, e\}, \{b, c, d\}, \{d, e\}\}$, where x_1= {a, b}, $x_2 = \{b, c, d\}, \dots etc.$ Let R be given by:

$R = \{(x_i, x_j) \mid x_i, x_j \in S \wedge x_i R x_j \; if \; x_i \; and \; x_j \; contain \; some \; common \; elements\}$. Then R is a compatibility relation defined on S, a simplified graph of R represented in figure 1.

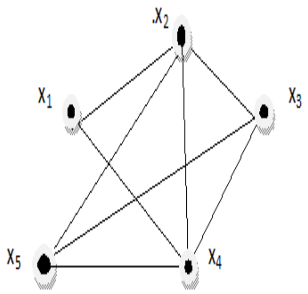

Figure 1: Simplified graph of R (Since R is reflexive and symmetric, loops at each node are not drawn and only one of x_iRx_j and x_jRx_i is drawn).

It may be observed that $x_1Rx_2 \wedge x_2Rx_3 \nRightarrow x_1Rx_3$. From the diagram, it follows that the elements in each of the classes $C_1 = \{x_1, x_2, x_4\}$, $C_2 = \{x_2, x_3, x_4\}$, $C_3 = \{x_3, x_5\}$ and $C_4 = \{x_2, x_3, x_4, x_5\}$ are mutually compatible and the sets are not mutually disjoint. These are the compatibility classes of S, the class $\{x_1, x_3\}$ does not have compatible elements. Clearly, $M_1 = \{x_1, x_2, x_4\}$ and $M_2 = \{x_2, x_3, x_4, x_5\}$ are the only maximal compatibility classes and these two classes define a minimal covering of S. Observe that x_2Rx_5 holds, x_2 belongs to both M_1 and M_2, but x_5 belongs to M_2 and does not belong to M_1. This prompts us to conclude that for any pair of compatible elements, the inclusion of one element in a maximal compatibility class does not necessarily implies the inclusion of the other in it.

Remark 1

For a given compatibility relation defined on an n- set S, not all the maximal compatibility classes are needed to cover S.

For an illustration, let $S = \{x_1, x_2, \dots, x_6\} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{3,4,5,6\}, \{4,5,6\}, \{1,6\}\}$, where $x_1 = \{1,2\}$, $x_2 = \{2,3\}$, ...etc. Let R be defined by:

$R = \{(x_i, x_j) \mid x_i, x_j \in S \wedge x_iRx_j \text{ if } x_i \text{ and } x_j \text{ contain some common elements}\}$. Then R is a compatibility relation and $\{x_1, x_2\}$, $\{x_2, x_3, x_4\}$, $\{x_3, x_4, x_5\}$, $\{x_4, x_5, x_6\}$, $\{x_1, x_6\}$ are compatibility classes of S. In fact, these classes are all maximal compatibility classes, but $\{x_1, x_2\}$, $\{x_3, x_4, x_5\}$ and $\{x_1, x_6\}$ is sufficient to cover S.

3. Main results

Definition 1

Let S be a finite set and x_k an element of S. We shall call $x_k \in C$ a generating element of a compatibility class C if x_k is compatible with every element of C and some (but not all) elements of S.

We note that $x_k \in C$ is a generating element of C if it is not compatible with any element of $S - C(\neq \emptyset)$.

Definition 2

Let S be a finite set endowed with a compatibility relation. The degree of a generating element $x_k \in S$, denoted deg(x_k), is the number of compatibility classes x_k appears.

Clearly, in the example1 above, x_1 and x_5 are generating elements of M_1 and M_2 respectively, but x_2 and $x_4$are not generating elements for any compatibility class.

Definition 3

A compatibility class will be called an essential compatibility class if it contains an element of the least degree.

Proposition 1

Every minimum covering of a finite set S endowed with a suitable compatibility relation contains at least one maximal compatibility class.

Proof: Let ξ be a minimum covering of S. Suppose for contradiction, ξ does not contain any maximal compatibility class. Then every element (class) in ξ must be a subclass of some maximal compatibility class. Therefore, there must exist a covering Γ of S consisting of maximal compatibility classes such that $|\xi| = |\Gamma|$. This contradicts the fact that ξ is a minimum covering of S. Hence, we obtain the result.

The gist of this proposition is that; selection of a minimum covering of S consisting of maximal compatibles is a preferable choice for application and that a covering generated by MCCs may yield a minimum covering of S.

Note 1: In an unpublished paper submitted to ICIST, Singh and William-west proved that every compatibility class is either itself a maximal compatibility class or a subclass of a maximal compatibility class.

Proposition 2

Let S be a n-set endowed with a suitable compatibility relation and $\{x_1, x_2, x_3, \ldots, x_k\}$ be a set of generating elements of all the compatibility classes of S. Let $\deg(x_1) = \deg(x_2) = \deg(x_3) = \cdots = \deg(x_k) \geq 2$, $(n - 2 \leq k \leq n)$, where n is even. Then there exists a minimal covering of S of cardinality 2 or 3.

Proof: Let $\deg(x_1) = \deg(x_2) = \deg(x_3) = \cdots = \deg(x_k) \geq 2, (n - 2 \leq k \leq n)$. (1)

T.hen $|S| \geq 4$ (by (1) and definition 1). Suppose $|S| = 4$, since a compatibility relation defined on S decomposes it into (possibly) non- disjoint compatibility classes. By (1), we must have at most four compatibility classes such that any three of these classes cover S. Such a covering is a minimal covering of cardinality 3.

Further, suppose $|S| = 6$, we can find at most eight compatibility classes satisfying (1), which generate a minimal covering of S of cardinality 3. Also, it is easy to observe that as n gets lager, we could obtain a minimal covering of cardinality 2. Hence, by induction hypothesis on the cardinality of S, the result follows.

Remark 2

i. If corresponding to each maximal compatibility class of S there exists a unique generating element, then a minimal covering for S is the union of the MCCs.

ii. By proposition 2, it can be argued that any compatibility relation on S which satisfies (1) is a preferable choice for application.

iii. It is observed that every subclass of S may not be a compatibility class.

In order to present a technique to compute the minimal set of maximal compatibility classes that covers S, it is important to note that an MCC may not have a unique generating element. However, our approach in describing the technique for deriving the minimal set of maximal compatibles which covers S, eliminates those none generating elements and any generating element of maximum degree from the compatibility matrix. Essentially, it aims at reducing the compatibility matrix to a level which constitutes all the essential subclasses for covering. Also, it is significant to note that every compatibility relation R can be represented by a matrix consisting of 0 – 1 entries. Usually, for each pair of nodes (x_i, x_j) in the compatibility matrix, a one (1) is assigned to it if x_iRx_j and, a zero (0) if it is not the case that x_iRx_j.

We develop the technique by working out an example. Accordingly, for further discussion, consider the compatibility matrix below, which represents the example presented in remark 1 above.

Thus, the rules of the technique could be stated in the following manner:

Rule1. – If a generating element is of maximum degree, it could be eliminated from the compatibility table. Also, any none generating element in S needs not be included in the compatibility table.

Rule2. – If all the generating elements are of the same degree, then no elimination is required. Then from proposition 2, by inspection we may obtain the minimal covering for S.

Rule3. – Add a generating element with the maximum degree to a subclass C (of S) derived from the reduced compatibility table if C is not a compatibility class.

Table 1: Compatibility table (CT_1).

	x_1	x_2	x_3	x_4	x_5
x_2	1				
x_3	0	1			

x_4	0	1	1		
x_5	0	0	1	1	
x_6	1	0	0	1	1

Clearly, x_4 is a generating state with the maximum degree (deg (x_4) =3). Hence we apply Rule 1, and obtain Table 2.

Table 2: Compatibility table (CT_2).

	x_1	x_2	x_3	x_5
x_2	1			
x_3	0	1		
x_5	0	0	1	
x_6	1	0	0	1

From table 2, we obtain $\{x_1,x_2\}, \{x_1,x_6\}, \{x_2,x_3\}, \{x_3,x_5\}$ *and* $\{x_5,x_6\}$.

Observe that $\{x_1,x_2\}, \{x_1,x_6\}$ and $\{x_3,x_5\}$ are essential compatibility classes in forming a minimal covering of S. By Rule 3, we add x_4 to $\{x_3,x_5\}$ and obtain $\{x_3,x_4,x_5\}$. Hence a minimal covering for S is $\{\{x_1,x_2\}, \{x_1,x_6\}, \{x_3,x_4,x_5\}\}$.

4. Concluding remarks

In view of the fact that applications of compatibility relation and thereby, that of maximal compatibility classes, have been found useful in different fields (see [5], and the references there for details), it has increasingly become pertinent to closely investigate issues pertaining to derivation of minimal set of maximal compatibility classes for covering of a finite set. In fact, the observation that not all maximal compatibility classes are needed to ensure a minimal covering for a finite set endowed with suitable compatible relation reveals a new technique for minimal covering selection.

Hopefully, in a foreseeable future this concept will be applied to information retrieval, air crew scheduling and assembly line balancing etc., which are meant to achieve optimal solutions where there are large number of incomplete specifications.

The direction of our future research is to construct an algorithm that computes minimal sets of maximal compatibles for covering of a finite set endowed with a compatibility relation.

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