

Principally Dual Stable Modules

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Abstract.

Another generalization of fully d-stable modules, in this paper was introduced. A module is principally d-stable if every cyclic submodule of it is d-stable. Quasi-projective principally d-stable module is fully d-stable. For finitely generated modules over Dedekind domains the two concepts (full and principal) d-stability of modules coincide. For regular modules over commutative rings, principal d-stability of modules is equivalent to commutativity and full d-stability of there endomorphism rings.

Keywords: fully(principally) d-stable module; quasi-projective, duo, regular module; Dedekind domain; endomorphism ring; hollow module; exchange property.

1. Introduction.

In two previous papers ([2] and [3]), we introduced the concept of fully d-stable modules and studied some generalizations of it. A submodule N of an R-module M is said to be d-stable if $N \subset Ker(\alpha)$ for every homomorphism $\alpha : M \to M/N$, the module M is said to be fully d-stable, if each of its submodules is d-stable [2]. Full d-stability is dual to the concept of full stability introduced by Abbas in [1], and both of these concepts are stronger than duo property of modules. A submodule N of an R-module M is said to be stable if $f(N) \subset N$, for any homomorphism $f : N \to M$, a module is fully stable of all of its submodules are stable [1]. In [1], it was proved that a module is fully stable if and only if each cyclic submodule is stable. Unfortunately it is not the case in full d-stability. This motivates introducing the concept of principally d-stable module which is a generalization of full d-stability. A module will be called principally d-stable if every cyclic submodule of it is d-stable. In this paper we studied this new concept and the conditions that make a principally d-stable module into a fully d-stable. In section 2 main properties of principal d-stable were investigated in addition, we see that quasi-projectivity is a sufficient condition for a principal d-stable module to be fully d-stable. Also we show that over Dedekind domain and integral domain with certain conditions, the two concepts, full (and principal) d-stability coincide. Links between the two dual concepts full stability and full d-stability, in certain conditions, also, was found . In section 3, under regular modules (in some sense), many characterizations to principally d-stable module, via endomorphism rings, were investigated.

Throughout, rings are associative having an identity(unless we state) and all modules are unital. R is a ring and M is a left R-module (simply we say module).

2. principally d-stable modules

Definition 2.1. A module is said to be principally d-stable if each of its cyclic submodule is d-stable.

Proposition 2.2. Any quasi-projective principally d-stable is fully d-stable.

Proof: By (Proposition 3.6. [2]). ◊

Proposition 2.3. Every principally d-stable module is duo.

Proof: Let M be an R-module, f an endomorphism of M, and N a submodule of M. Let $x \in N$, π_x be the natural epimorphism of M onto M/Rx and $\alpha = \pi_x \circ f$, then by assumption $\alpha(x) = 0$, hence $f(x) \in Rx \subset N$, that is $f(N) \subset N$. ◊

Definition 2.4. A ring R is right (left) principally d-stable if R_R ($_R R$) is principally d-stable.

The rings in this paper are assumed to have identity, this makes the concepts of duo, fully d-stability and principal d-stability coincide for rings. Note that a ring is right (left) duo if and only if every right (left) ideal is two sided ideal.

Proposition 2.5. A ring R is right (left) principally d-stable if and only if it is right (left) fully d-stable.

Proof: The (if part) is clear. We will prove the (only if part, the left case).
Assume that R is left principally d-stable, I a left ideal of R and $\alpha : R \to R/I$ is an R-homomorphism. By assumption and the note before the proposition, I is a two sided ideal too, if $x \in I$ then $\alpha(x) = x\alpha(1) = xx_0 + I$ =0, since $xx_0 \in I$. Therefore, R is left fully d-stable. ◊

In [3] we introduced minimal d-stable modules in which minimal submodules are d-stable. Since any minimal submodule is cyclic, so we conclude that any Principally d-stable module is minimal d-stable. The converse of this result is not true, as the Z-module Q is minimal d-stable (trivially) but not principally d-stable (see remark 2.14).

Another condition which versus principal d-stability into full d-stability is in the following. First we need to introduce the following concept.

Definition 2.6. An R-module, M is said to have the quotient embedding property (qe-property, for short) if M/N can be embedded into M/Rx for each submodule N of M and each $0 \neq x \in N$.

Remark 2.7. Let M be an R-module. If M/Rx is semisimple for each $0 \neq x \in M$, then M has the qe-property. In particular every semisimple module has the qe-property.

Proof: If $x \in N$, where N is a submodule of M, then there is a natural epimorphism $\delta : M/Rx \to M/N$ ($a + Rx \mapsto a + N$) with $ker(\delta) = N/Rx$. Since M/Rx is semisimple , N/Rx is a direct summand of M/Rx, that is, δ is split epimorphism, hence δ has a right inverse which is a monomorphism from M/N into M/Rx. ◊

Proposition 2.8. Let M be a principally d-stable R-module**.** If M has the qe- property, then M is fully d-stable.

Proof: Assume that M is a principally d-stable module , $\alpha : M \to M/N$ is an R-homomorphism, where N is a submodule of M. Let $x \in N$, then by hypothesis there is a monomorphism $\beta : M/N \to M/Rx$. Now $\beta\alpha$ is an R-homomorphism from M into M/Rx, so $Rx \subset ker(\beta\alpha) = ker(\alpha)$, since β is a monomorphism. Hence $N \subset ker(\alpha)$, since x is an arbitrary element of N, and then M is fully d-stable. ◊

From Proposition 2.8 and Remark 2.7 we conclude that, if M is principally d-stable and M/Rx is semisimple for each $x \in M$ (or M itself is semisimple), then M is fully d-stable.

Note that the Z-module Z has the qe- property, but $Z/4Z$ (for example) is not simisimple . On the other hand $Z_{(p^\infty)}$ has the qe- property, which is not principally d-stable (see Remark 2.14). So we restate
Proposition 2.8 in this way.

Corollary 2.9. Let M be a module, with the qe- property. Then the following two statements are equivalent:

(i) M is principally d-stable.

(ii) M is fully d-stable. ◊

Note that the Z-module $M = Z \oplus (Z/2Z)$ does not satisfy qe-property, since if $x = (0,1)$, $N = 2Z \oplus (Z/2Z)$, then $Zx = 0 \oplus (Z/2Z)$ and M/N cannot be embedded in M/Zx, on the other hand M is not principally d-stable (it is not duo), see Lemma 2.18 below.

Other condition can be regarded to deduce full d-stability from principal d-stability.

Theorem 2.10. Let M be an R-module, with the property that any proper submodule of M is contained in a cyclic submodule. Then M is fully d-stable if it is principally d-stable**.**

Proof: Let N be a submodule of M contained in Rx (for some $x \in M$), then N is d-stable in Rx (since Rx is cyclic module and hence fully d-stable [2]), also Rx is d-stable in M (since M is principally d-stable). Then by transitive property of d-stability (see [2]), N is d-stable in M. Therefore M is fully d-stable. ◊

Note that the condition of Theorem 2.10 and the qe-property are independent (although the**y** have the same effect on principally d-stable modules) . In the next example a module satisfying the qe-property but not the other will be discussed , while in example 2.12 a module having the property of Theorem 2.10 will be given that does not satisfy qe-property.

In [2] we constructed an example of fully d-stable module which not quasi-projective, in the following , with the help of Corollary 2.9, an other example of a module which is not quasi-projective will be shown it is fully d-stable, first we prove it is principally d-stable and then it satisfies the qe-property. The direct proof of full d-stability is certainly more difficult.

Example 2.11. Let $M = \{a/b \in Q \mid b \text{ is square free}\}$, the following properties can be observed for M:

1. $M = \sum_{p \in PR} Z\frac{1}{p}$, where PR is the set of all prime numbers.(clear)
2. M is a torsion-free uniform (not finitely generated) Z-module.(clear)
3. M is duo. [10]
4. M is not quasi-projective.

Proof: Recall the following fact from [11], " Any torsion-free quasi-projective module over a Dedekind domain, which is not a complete discrete valuation ring, is torsionless" (Lemma 5.2, [11]). We will show that M is not torsionless. (Recall that an R-module M is torsionless if each non-zero element of M has non-zero image under some R-linear functional $f \in Hom_R(M,R)$. [8])

Let $f : M \to Z$ be a Z-homomorphism and $f(1) = n \neq 0$, let q be any prime not dividing n, then

$n = f(\frac{q}{q}) = qf(\frac{1}{q}) = qk$ a contradiction. Hence $f(1) = 0$ for each $f \in Hom_Z(M, Z)$. ◊

5. M is principally d-stable.

Proof: Any cyclic submodule of M is of the form $Z\frac{a}{b}$, $\frac{a}{b} \in M$. Since a cyclic submodule is fully d-stable module and if it is d-stable in M, then all its submodules are d-stable in M by transitive property of d-stability (see [2]), also, since $Z\frac{a}{b} \subset Z\frac{1}{b}$,it is enough to prove that $Z\frac{1}{b}$ are d-stable in M for $\frac{1}{b} \in M$. Let $N = Z\frac{1}{b}$, $\alpha : M \to M/N$, we will show first that $\alpha(1) = 0$ and then show that $\alpha(\frac{1}{b}) = 0$.

Assume that $\alpha(1) = \frac{m}{n} + N$ (note that $\frac{m}{n} \in N \leftrightarrow n|b$; also remember that both n and b are square free). Now $\alpha(1) = \alpha(\frac{n}{n}) = n\alpha(\frac{1}{n})$, if $\alpha(\frac{1}{n}) = \frac{k}{l} + N$, then $\frac{nk}{l} + N = \frac{m}{n} + N$, hence $\frac{m}{n} - \frac{nk}{l} = \frac{a}{b} \in N$ then $\frac{mb - na}{n^2} = b(\frac{k}{l})$. Since $\frac{k}{l} \in M$, so $\frac{mb - na}{n^2} \in M$ and we must have $n|mb$, and then $n|b$ which implies $\frac{m}{n} \in N$, that is $\alpha(1) = 0$. Next, let $\alpha(\frac{1}{b}) = \frac{p}{q} + N$, we have $0 = \alpha(\frac{b}{b}) = b(\frac{p}{q}) + N$, hence $\frac{bp}{q} \in N$, that is,

$\frac{bp}{q} = \frac{c}{b}$ which implies $\frac{p}{q} = \frac{c}{b^2}$, but $\frac{p}{q} \in M$, so $b|c$ and $\frac{p}{q} \in N$, that is, $\alpha(\frac{1}{b}) = 0$, in other words

$N \subset \ker(\alpha)$, hence N is d-stable. ◊

6. M has the qe-property and hence (by Corollary 2.7) is fully d-stable.

Proof: First note that if $y = \frac{a}{b}$ and $x = \frac{1}{b}$ are elements of M then M/Zx can be embedded in M/Zy by $m + Zx \mapsto am + Zy$. Let $x = \frac{1}{b}$ and $b = p_1 p_2 ... p_n$ for distinct primes $p_1, p_2, ..., p_n$, let A be a submodule of M containing y. Let $N = \{p_1, p_2, ..., p_n\}$, $J = \{p \in PR \mid \frac{1}{p} \text{ is in the set of generators of } A\}$, $K = PR - J$ and $L = PR - N$. It is clear that $N \subset J$ and $K \subset L$, also it is clear that $M = A + B$, where $B = \sum_{p \in K} Z\frac{1}{p}$, and note that $A \cap B = Z$.

Now $M/A \cong B/Z \cong \oplus_{p \in K} (Z/pZ)$. On the other hand $Zx = \sum_{p \in N} Z\frac{1}{p}$, Hence

$\mathrm{M}/\mathrm{Zx} \cong \sum_{p\in L} \mathrm{Z}\frac{1}{p}\Big/\mathrm{Z} \cong \oplus_{p\in L}(\mathrm{Z}/p\mathrm{Z})$, then we conclude that M/A can be embedded in M/Zx (hence in M/Zy, by the above note). ◊

Example 2.12. Let $\mathrm{M} = \mathrm{Z}[x]$, the ring of polynomials over Z, be considered as a module over itself, then M is a cyclic module and hence it satisfies the property of Theorem 2.8. Let $\mathrm{N} = \langle 2, x \rangle$ be the ideal of M generated 2 and x, it is known that N is a maximal (submodule) in M and hence M/N is simple, while $\mathrm{M}/\langle x \rangle \cong \mathrm{Z}$ which contain no simple submodule, that is, M/N cannot be embedded in $\mathrm{M}/\langle x \rangle$, so M does not satisfy the qe-property. Certainly, M is a fully d-stable module. ◊

In [3], two equivalent concepts were introduced and investigated, namely, fully pseudo d-stable and d-terse modules. The last one is: " a module is d-terse if it has no distinct isomorphic factors". An analogous necessary (but not sufficient)condition for principal d-stability is proved in the following.

Proposition 2.13. Let M be a principally d-stable module. If $x, y \in \mathrm{M}$ and $\mathrm{M}/Rx \cong \mathrm{M}/Ry$, then $Rx = Ry$.

Proof: Let $\varphi : \mathrm{M}/Rx \to \mathrm{M}/Ry$ be an isomorphism, π_x *and* π_y be the natural epimorphisms onto M/Rx *and* M/Ry resp. , let $\alpha = \varphi \circ \pi_x, \beta = \varphi^{-1} \circ \pi_y$, then (by hypothesis M is principally d-stable) we have $Ry \subset \ker \alpha = \pi_x^{-1}(\ker \varphi) = Rx$ and $Rx \subset \ker \beta = \pi_y^{-1}(\ker \varphi^{-1}) = Ry$. Therefore $Rx = Ry$. ◊

Remark 2.14. By the above Proposition we can deduce, simply, that the Z-module Q (which is not fully d-stable, see [2]) is not principally d-stable too. Note that $Q/\mathrm{Z} \cong Q/\mathrm{Z}x$, for each $x \in Q$. Similarly the Z-module $\mathrm{Z}_{(p^\infty)}$ is isomorphic to each of its factors, that is, any two factors of it are isomorphic, hence it is not principally d-stable.

In the following we will investigate the coincidence of principal d-stability with full d-stability over certain type of rings . First we need to recall some facts about duo and quasi-projective modules.

Proposition 2.15. [10] Let R be a Dedekind domain. Then the following statements are equivalent for a finitely generated R-module M:

(i) M is a duo module.

(ii) $\mathrm{M} \cong I$ for some ideal I of R or $\mathrm{M} \cong (R/P_1^{n_1}) \oplus \ldots \oplus (R/P_k^{n_k})$ for some positive integers

$k, n_1, ..., n_k$ and distinct maximal ideals P_i $(1 \le i \le k)$ of R. ◊

Note that the first possibility of statement (ii) means M is torsion free and the second is torsion.

Proposition 2.16. [11] A torsion module M over a Dedekind domain R is quasi-projective if and only if each P-primary component M_P is a direct sum copies of the same cyclic module R/P^k for some fixed positive integer k depending on P. ◊

Proposition 2.17. [11] A torsion module M over a Dedekind domain R is quasi-projective if and only if M is quasi-injective but not injective. ◊

Now we are ready to prove the following theorem which leads, further, to a link between the two dual concepts , full stability and full d-stability in certain conditions.

Theorem 2.18. Let R be a Dedekind domain. Then the following statements are equivalent for a finitely generated R-module M:
(i) M is duo .
(ii) M is fully d-stable.
(iii) M is principally d-stable.

Proof: (i)$\Rightarrow$ (ii). By Proposition 2.15, M is a duo module implies either $\mathrm{M} \cong I$ for some ideal I of R (which is projective, since every ideal of a Dedekind domain is projective [4], p.215) or $\mathrm{M} \cong (R/P_1^{n_1}) \oplus ... \oplus (R/P_k^{n_k})$ for some positive integers $k, n_1, ..., n_k$ and distinct maximal ideals P_i $(1 \le i \le k)$ of R (which is quasi-projective by Proposition 2.16). In any case M is fully d-stable ([2], Proposition 2.3).
(ii) $\Rightarrow$(iii). Clear by definitions.
(iii) $\Rightarrow$(i) . by Corollary 2.2. ◊

Corollary 2.19. For a finitely generated torsion module M over a Dedekind domain R, the following statements are equivalent:
(i) M is fully stable.
(ii) M is fully d-stable.

Proof: M is fully stable implies M is duo, then by Proposition 2.10 and the note after it, we have $\mathrm{M} \cong (R/P_1^{n_1}) \oplus ... \oplus (R/P_k^{n_k})$, which means that M is quasi-projective. Hence M is fully d-stable([2], Proposition 2.3).

Conversely, if M is fully d-stable, then it is duo and hence quasi-projective (see part one). Now by Proposition 2.17, M is quasi-injective. Therefore M is fully stable(see [1]). ◊

Remarks 2.20.

(i) $Z_{(p^\infty)}$ is a torsion module over a Dedekind domain, which is fully stable[1] but not fully d-stable[2]. Note that this module is not finitely generated.

(ii) Z is a Dedekind domain, it is finitely generated module over itself, fully d-stable[2] but not fully stable[1]. It is clear that Z is torsion free Z-module.

(iii) By the above theorem and a Corollary in [1], we can conclude the following statement: " A finitely generated torsion module M over a Dedekind domain R is fully d-stable if and only if, for each $x, y \in \mathrm{M}$, $ann_R(y) = ann_R(x)$ implies $Rx = Ry$ ".

We need to recall another fact about duo modules, in order to prove a next result.

Lemma 2.21. [10] Let R be a domain. An R-module $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_2$, with a non zero torsion free submodule M_1 and a non zero submodule M_2, is not duo. ◊

The proof of the following theorem can be found implicitly in the proof of Theorem 2.18, but we will give another proof.

Theorem 2.21. Let M be a finitely generated module over a P. I. D., R. Then M is principally d-stable if and only if it is fully d-stable.

Proof: Let M be a finitely generated module over a P. I. D., R. It is known that $\mathrm{M} = F \oplus T(\mathrm{M})$, where F is a free module and $T(\mathrm{M})$ is the torsion submodule of M (see, for example, [7]). We have the following cases:

(i) $T(\mathrm{M})=0$, then M is free, hence either $\mathrm{M} \cong R$ which is fully d-stable, or $\mathrm{M} \cong R \oplus \ldots \oplus R$, k times and k>1, which implies M is not duo, so neither fully nor principally d-stable.

(ii) $F \neq 0$ and $T(\mathrm{M}) \neq \mathbf{0}$, then by Lemma 2.21, M is not duo, so neither fully nor principally d-stable. (note: it is known that any free module over a P. I. D. is torsion free)

(iii) $F = 0$, then M is torsion, hence by the proof of Corollary 2.19 and that a principally d-stable module is duo, M is fully d-stable if and only if M is principally d-stable. (note that a P. I. D. is Dedekind domain) ◊

Now we collect the cases and conditions that leads to the equivalence of the two concepts, full and principal d-stability, that we get (till now) by the following:

1. quasi-projective modules.
2. modules with q-e property.
3. finitely generated modules over Dedekined domain.

The following statement about principally d-stable modules, has an analogous statement in the case of fully d-stable which is proved in [2], but we will give a proof for completeness.

Proposition 2.22. If M is a torsion free principally d-stable module over an integral domain R which is not a field, then M is not injective.

Proof: Assume M is injective, then it is divisible. Let $0 \neq r$ be a non invertible element of R, then for each $x \in M$, there exists $y \in M$ such that $x = ry$. Define $f : M \to M$ by $f(x) = y \leftrightarrow x = ry$, f is an endomorphism of M (since M is torsion free). M is principally d-stable implies M is duo (Corollary 2.2), hence for each $x \in M$, there exists $s \in R$ such that $f(x) = sx$ [10], so we have $rsx = x$ which implies $rs = 1$(since M is torsion free) and this contradicts the assumption that r is not invertible. Therefore M is not injective. ◊

Corollary 2.23. Let R be an integral domain, which is not a field , M an injective principally d-stable module over R, then M is not torsion free. ◊

In the following we have another result about torsion free modules over integral domain. Recall that, in case of torsion free module M the "rank" is the maximum number (cardinal number) of linearly independent elements in M (see [6]) .

Proposition 2.24. Let M be a torsion free module over an integral domain R. If M is quasi-injective of rank >1, then M is not duo, consequently neither fully d-stable nor principally d-stable.

Proof: Assume that x, y are two linearly independent elements in M , then $Rx \cap Ry = 0$. Let $f : Rx \to M$ be defined by $f(rx) = ry$, then f is an R-homomorphism, that can be extended to an endomorphism, say g, of M (since M is quasi-injective) and it is clear that $g(Rx) = Ry \not\subset Rx$, that is, M is not duo. ◊

In [3], we prove an equivalent statement to the definition of fully d-stable module which was "M is fully d-stable if and only if $\ker g \subset \ker f$ for each R- module A and any two R-homomorphisms $f, g : M \to A$ with g surjective". In the end of this section a similar statement for principally d-stable module can be stated, and the proof will be omitted .

Proposition 2.25. Let M be an R-module. M is principally d-stable if and only if for each R-module A and any two R-homomorphisms $f, g : M \to A$ with g surjective and $\ker g$ is cyclic in M, $\ker g \subset \ker f$. ◊

3- Full d-stability and Endomorphism ring

The endomorphism ring of a module, sometimes, gives additional information about the module itself , so it is natural to investigate the endomorphism ring of a fully d-stable module (and in particular principally d-stable module), to this aim we have the following results.

First recall the concept of "regular module", which is a generalization of the concept of Von Neumann's regular ring, "there have been considered three types of modules by Fieldhouse, Ware and Zelmanowitz each called regular. The Fieldhouse-regular module was defined to be a module whose submodules are pure submodules and the Ware-regular modules was defined as a projective module in which every submodule is a direct summand, while a left module M over a ring R is called a Zelmanowitz-regular module if for each $x \in \mathrm{M}$ there is a homomorphism $f : \mathrm{M} \to R$ such that $f(x)x = x$." [5]. Azumaya in [5], consider the following definition " a module M is *regular* if every cyclic submodule is a direct summand". This definition is more convenience for our aim since the projectivity condition leads to the equivalence of the duo, fully d-stability and principal d-stability concepts, but we need to investigate the last two separately. So we will consider the Azumaya-regular definition:

Definition 3.1.**[5]** An R -module is regular if each of its cyclic submodule is a direct summand.

Proposition 3.2. If M is a regular $R-$module and if $End_R(\mathrm{M})$ is commutative, then M is a duo module .

Proof: Let $f \in End_R(\mathrm{M})$ and $x \in \mathrm{M}$, since M is regular, we have $\mathrm{M} = Rx \oplus L$ For some submodule L of M. Assume that $f(x) = rx + l$, $r \in R$ and $l \in L$, let $\pi : \mathrm{M} \to \mathrm{M}$ defined by $\pi(sx+t) = sx$ for each $s \in R, t \in L$.

Now, $f(\pi(x)) = f(x) = rx + l$ and $\pi(f(x)) = rx$, but $End_R(\mathrm{M})$ is commutative ,so, $f(x) = rx$. Therefore M is a duo module.(lemma 1.1, [10]) ◊

Corollary 3.3. If M is a regular $R-$module and if $End_R(\mathrm{M})$ is commutative, then M is principally d-stable .

Proof: By proposition 3.2 M is duo and by ([2], proposition 3.1) any direct summand of M is d-stable, but M is regular , hence any cyclic submodule is d-stable. ◊

Corollary 3.4. If M is a regular quasi-projective $R-$module and if $End_R(\mathrm{M})$ is commutative, then M is fully d-stable. ◊

Lemma 3.5. If R is a commutative ring and M is a duo $R-$module, then $End_R(\mathrm{M})$ is commutative.

Proof: Let $f, g \in End_R(\mathrm{M})$ and $x \in \mathrm{M}$, then $f(x) = rx$ and $g(x) = sx$ for some $r, s \in R$ (lemma 1.1, [10]) . Hence $f(g(x)) = f(sx) = sf(x) = srx$ and $g(f(x)) = g(rx) = rg(x) = rsx$, since R is commutative, we have $f(g(x)) = g(f(x))$ .Therefore $End_R(\mathrm{M})$ is commutative. ◊

Recall that in [2], we show that "every quasi-projective duo $R-$module is fully d-stable. So we have the following result.

Corollary 3.6. If R is a commutative ring, and M is a regular quasi-projective $R-$module, then M is fully d-stable if and only if, $End_R(\mathrm{M})$ is commutative.

Proof: ($\Rightarrow$) by lemma (3.5) and ([2], proposition 2.3).

($\Leftarrow$) by proposition (3.2) and([2], proposition 2.3). ◊

Corollary 3.7. If R is a commutative ring, and M is a regular $R-$module, then M is principally d-stable if and only if, $End_R(\mathrm{M})$ is commutative.

Proof: ($\Rightarrow$) by lemma 3.5 and corollary 2.2. ($\Leftarrow$) by corollary3.3. ◊

Corollary 3.8. If R is a commutative ring and M is a regular quasi-projective $R-$module, then M is fully d-stable if and only if, $End_R(\mathrm{M})$ is fully d-stable. ◊

Lemma 3.9. If M is a regular $R-$module , $x \in \mathrm{M}$ and $\alpha : \mathrm{M} \to \mathrm{M}/Rx$, then α can be lifted to an endomorphism of M.

Proof: Since M is regular, $\mathrm{M} = Rx \oplus L$, for some submodule L of M, let $m \in \mathrm{M}$, and assume that $\alpha(m) = rx + l$, $r \in R$ and ,$l \in L$, then we can write $\alpha(m) = l + Rx$, $l \in L$, also l is unique for each $m \in \mathrm{M}$, for if $l_1 + Rx = l_2 + Rx$, then $l_1 - l_2 \in Rx \cap L = 0$. Hence we can define $f : \mathrm{M} \to \mathrm{M}$ by $f(m) = l \leftrightarrow \alpha(m) = l + Rx$, it clear that $\pi \circ f = \alpha$. ◊

We can summarize the previous results in the following Corollary.

Corollary 3.10. If R is a commutative ring, M is a regular $R-$module, then the following statements are equivalent :

1. M is principally d-stable.
2. $End_R(\mathrm{M})$ is a commutative ring.
3. $End_R(\mathrm{M})$ is fully d-stable. ◊

A similar result is found in [1] but in place of statement 1 there was " M is a fully stable module" , from which we get a link between full stability and principal d-stability, that is,

Corollary 3.11. If R is a commutative ring, M is a regular $R-$module, then the following statements are equivalent :

1. M is fully stable.
2. M is principally d-stable. ◊

Regularity of a module (in the mentioned sense) has other effect for d-stability (even stability) ,see the following .

Proposition 3.12. Let M be a torsion free module over an integral module R. If M is regular (but not simple) , then it is not duo and consequently neither fully d-stable nor principally d-stable and not fully stable.

Proof: Let $0 \neq \mathrm{x} \in \mathrm{M}$ such that $\mathrm{M} \neq \mathrm{Rx}$ then $\mathrm{M} = \mathrm{Rx} \oplus \mathrm{N}$ for some nonzero submodule N of M, but Rx is torsion free, so by Lemma 2.21 M is not duo. ◊

Other properties can be added for the endomorphism ring of a module, when it is hollow, (that is, the sum of any two proper submodules does not equal the module itself) . Recall that an $R-$module M is hopfian if every surjective endomorphism of M is an isomorphism .

Proposition 3.13. If M is a fully d-stable module over a commutative ring R, and if M is hollow, then $End_R(\mathrm{M})$ is a commutative local ring.

Proof: Since M is a fully d-stable, it is duo and hence by lemma 3.5 $End_R(\mathrm{M})$ is a commutative ring. Now M is hopfian (see [2]. Proposition 2.16), hence any non invertible element of $End_R(\mathrm{M})$ is not surjective. Let $L = \{ f \in End_R(\mathrm{M}) : \mathrm{Im} f \neq \mathrm{M} \}$, L is the subset of all non invertible elements of $End_R(\mathrm{M})$. If $f, g \in L$, then $\mathrm{Im}(f + g) \subset (\mathrm{Im} f) + (\mathrm{Im} g) \neq \mathrm{M}$ (since M is hollow), hence $f + g \in L$, that is, L is additively closed, and $End_R(\mathrm{M})$ is local (see [6], 7.1.1 and 7.1.2). ◊

Recall that, a module M has the exchange property if for any index set I, whenever $\mathrm{M} \oplus \mathrm{N} = \oplus_{i \in I} \mathrm{A}_i$ for modules N and A_i, then $\mathrm{M} \oplus \mathrm{N} = \mathrm{M} \oplus (\oplus_{i \in I} \mathrm{B}_i)$ for submodules B_i of A_i, $i \in I$.(see [9]). Also, it is known that " An indecomposable module has the exchange property if and only if its endomorphism ring is local"(see [12]). Using this remark, proposition 3.10 and the fact that hollow module is indecomposable, we have the following:

Corollary 3.14. A fully d-stable hollow module has the exchange property. ◊

R.B. Warfield proved the following : Let M be a module with a local endomorphism ring and suppose A and B are modules such that $\mathrm{A} \oplus \mathrm{M} \cong \mathrm{B} \oplus \mathrm{M}$, then $\mathrm{A} \cong \mathrm{B}$.(see [12])

Hence we can add the following corollary:

Corollary 3.15. A fully d-stable hollow module has the cancellation property. ◊

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